EQUIMULTIPLICITY, REDUCTION, AND BLOWING UP

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INTRODUCTION

This is a largely expository account of some recent developments in multiplicity theory. Emphasis is placed throughout on the benefits of interaction between two rather different ways of looking at the subject - the approaches of local analytic geometry and of commutative algebra. So if this paper does not exactly fit in with the theme of the present conference - the use of analytic methods to prove algebraic theorems - it is at least meant to support the underlying philosophy that algebraists cannot afford to ignore analysts (and vice-versa).

Here is a brief summary of the contents. In Section 1, we introduce the basic condition $\ell(I) = h(I)$ on an ideal I in a local ring R; here $\ell(I)$ is the "analytic spread" of I and h(I) is the height of I; thus $\ell(I) = h(I)$ means that I has a reduction of the principal class, and also that the fibers over Spec (R/I) in the blowup of I all have the same dimension. (All these terms are explained in Section 1). We also discuss a fundamental theorem of Rees, extended by Böger and Ratliff, relating reductions of ideals and equality of multiplicities.

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Section 2 treats some criteria of Dade, in terms of multiplicity, for $\ell(I) = h(I)$. In particular, when I is prime and R/I is regular, one has that Spec(R) is normally pseudo-flat along Spec(R/I) if and only if the local rings R and R_I have the same multiplicity. The geometry of this situation is discussed in Section 4 and further in Section 5, where an examination of tangent and normal cones along lines suggested by Schickhoff shows how close normal-pseudo flatness - i.e. equimultiplicity - is to normal flatness (cf. Theorem 5 in Section 5).

In Section 3, we generalize some results of Teissier relating the condition $\ell(I) = h(I)$ to equimultiplicity in a family of zero-dimensional ideals. Teissier's results, developed for the theory of equisingularity, are technically somewhat similar to those in Section 2; both can be viewed as elaborations of the theorem of Rees in Section 1.

CONTENTS

§1. Reduction and multiplicity (Rees, Böger, Ratliff)

In [14] Northcott and Rees introduced the notion of reduction of ideals: if $I \subseteq J$ are ideals in a noetherian commutative ring R, then I is called a reduction of J if $IJ^n = J^{n+1}$ for some integer n; or, equivalently, if J is contained in the integral closure $\overline{I}$ of I, $\overline{I}$ being the ideal consisting of all elements x which satisfy an "integral dependence" relation of the form

$$x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0, \; a_i \in I^i \text{ for } 1 \leq i \leq m$$

(This equivalence is stated in [13, p. 34, Exercise 4]; it is proved in [14, p. 156] in case J contains a regular element; for the general case use e.g. [11, p. 792, Lemma (1.1)]. For more details on integral dependence cf. [10, p. 659].)

An important aspect of this notion is its relation with "multiplicity". If R is a local ring, and $I \subseteq R$ is an ideal primary for the maximal ideal of R, then for large m the length of the R-module R/I^m is given by a polynomial of degree d = (dimension of R):

$$\lambda_I(m) = e(I)m^d/d! + \text{terms of degree} < d$$

where e(I) is a positive integer, the <u>multiplicity</u> of I (cf. [21, p. 294]). Now if I is a reduction of some ideal $J \supseteq I$, say $IJ^n = J^{n+1}$, then for every $m > 0$ we have $I^m J^n = J^{m+n}$, whence, for large m,

$$\lambda_J(m+n) \geq \lambda_I(m) \geq \lambda_J(m)$$

and it follows that e(I) = e(J).

A basic result for our purposes is the following <u>converse</u>, which is much more difficult. Recall that a noetherian local ring R is <u>formally equidimensional</u> (or, in Nagata's terminology, <u>quasi-unmixed</u>) if each minimal prime ideal p in the completion $\hat{R}$ satisfies

$$\dim(\hat{R}/p) = \dim(\hat{R})(=\dim(R)) \quad \text{"dim"} = \text{"dimension"}$$

THEOREM 1 (Rees [16]) . If $I \subseteq J$ are ideals which are primary for the maximall ideal in a formally equidimensional local ring R, and if e(I) = e(J), then I is a reduction of J.

We will not describe here the proof, but rather the geometric meaning of this result in the special case when R is the local ring of germs of holomorphic functions at a point v on a pure d-dimensional complex analytic variety V, and J = M is the maximal ideal of R. In this case, any d-tuple of elements $(x_1, \dots, x_d)$ in R defines a map-germ $\phi: (V,v) \to (\mathbb{C}^d, 0)$; and $(x_1, \dots, x_d)R$ is primary for M if and only if there is a holomorphic map f from a neighborhood of v in V onto a neighborhood N of 0 in $\mathbb{C}^d$, whose germ at v is ϕ, and such that f is proper, with finite fibers. Such an f gives a <u>branched covering</u> of N, i.e. there is a nowhere dense subvariety W of N such that f makes $f^{-1}(N - W)$ into a covering space of N - W. The number of points in the fibre $f^{-1}(x)$ is independent of the choice of $x \in N - W$; this number can be shown to be equal to the $\mathbb{C}\langle\langle x\rangle\rangle$-vector space dimension of $R\otimes_{\mathbb{C}\langle x\rangle}\mathbb{C}\langle\langle x\rangle\rangle$ (where $\mathbb{C}\langle x\rangle = \mathbb{C}\langle x_1,\ldots,x_d\rangle \subseteq R$ is the ring of convergent power series in $x_1,\ldots,x_d$; and $\mathbb{C}\langle\langle x\rangle\rangle$ is its fraction field); and this is the same as the multiplicity of the ideal $(x_1,\ldots,x_d)R$. We shall refer to this number as the <u>degree</u> of ϕ. Thus if $(x_1,\ldots,x_d)R$ is M-primary then the degree of ϕ is $\geq$ the multiplicity of R (i.e. of M), and we have equality for some (in fact "almost any") choice of $(x_1,\ldots,x_d)$. Rees' theorem gives a necessary and sufficient algebraic condition for <u>equality</u>, namely <u>that</u> $(x_1,\ldots,x_d)R$ <u>be a reduction of</u> M. Let us interpret this condition geometrically.

Using Nakayama's lemma, one checks that $(x_1,x_2,\ldots,x_d)R$ is a reduction of M if and only if, in the graded $\mathbb{C}$-algebra $G = \bigoplus_{n\geq 0} M^n/M^{n+1}$, the images $\bar{x}_i$ of the x_i in M/M^2 generate an irrelevant ideal (i.e. an ideal containing all elements of sufficiently large degree). <u>What this last condition means</u> is that first of all the $\bar{x}_i$ are linearly independent over $\mathbb{C}$, so that there is an embedding of the germ (V,v) into $(\mathbb{C}^m,0)$ for some m (for example m = embedding dimension of R, i.e. the dimension of the $\mathbb{C}$-vector space M/M^2), and a linear projection $\pi\colon (\mathbb{C}^m,0)\to(\mathbb{C}^d,0)$ such that the restriction of π to V is -germwise- the ϕ associated as above to $(x_1,\ldots,x_d)$; and secondly, <u>the linear space</u> $\pi^{-1}(0)$ <u>has no line in common with the Zariski tangent cone of V at the origin</u>. (Recall that the Zariski tangent cone - sometimes denoted by C_3 - is the union of all limits of lines joining $y\in V\subseteq\mathbb{C}^m$ to the origin, the limits being taken as y approaches the origin within V; and G is the coordinate ring of this cone.) In other words, $\|w\|/\|\pi(w)\|$ is <u>bounded</u> for w in some punctured neighborhood of the origin on V.

This geometric condition for ϕ to have minimal degree is well-known [4, p. 196, Theorem 6.3], [20, p. 234, Theorem 7P]; but the connection with Rees' theorem does not seem to have been pointed out.

Rees' Theorem has been generalized by Böger [2] as follows. First of all, in [14, p. 149] Northcott and Rees defined the "analytic spread" $\ell(I)$ of an ideal $I\neq R$ in a local ring (R,M) to be $\delta+1$, where δ is the dimension of the closed fiber $f^{-1}(\{M\})$ for the map $f\colon X\to \mathrm{Spec}(R)$ obtained by blowing up I ($\delta = -1$ if I is nilpotent).

Next, recall that h(I), the <u>height</u> of I, is defined to be $\min(\dim(R_p))$ as p runs through all prime ideals containing I. If

$$f_0\colon f^{-1}(\mathrm{Spec}(R/I))\to \mathrm{Spec}(R/I)$$

is the restriction of the above f, then we have:

$\ell(I) \geq h(I)$, with equality if and only if all the fibers of f_0 have the same dimension

This follows from the upper semicontinuity of fiber dimension [6, (13.1.5)], since at a minimal prime p of I, the dimension of the fiber $f_0^{-1}(\{p\})$ is $\dim(R_p) - 1$ (IR_p being primary for the maximal ideal of R_p). In fact we see that

$\ell(I) \geq \mathrm{alt}(I)$

where alt(I), the "altitude" of I, is $\max(\dim(R_p))$ as p runs through the minimal prime divisors of I.

In case the residue field R/M is infinite, Northcott and Rees show that $\ell(I)$ is the least possible number of elements which can generate a reduction of I. Since I has the same radical as any of its reductions, we see again that $\ell(I) \geq \mathrm{alt}(I)$ (and similarly $\ell(I) \geq \dim(R) - \dim(R/I) \geq h(I)$); and - if R/M is infinite - $\ell(I) = h(I)$ if and only if I has a reduction I_0 of the "principal class" (i.e. I_0 is generated by $h(I_0)$ elements).

The condition $\ell(I) = h(I)$ plays an important role in the sequel. It is satisfied, for example, by any ideal I which is primary for the maximal ideal M.

EXERCISE $\ell(I) = h(I)$ iff $\dim(R) = \dim(R/I) + h(I)$ and there is an integer $m > 0$ and a system of parameters $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ in R such that $(x_1, \ldots, x_r)R$ is a reduction of I^m. (We can take $m = 1$ if R/M is infinite.)

Here is Böger's theorem:

THEOREM 2 Let $I \subseteq J \subseteq \sqrt{I}$ be ideals in a formally equidimensional local ring R, such that $\ell(I) = h(I)$. Then I is a reduction of J if (hard!) and only if (easy!) the R_p-ideals IR_p and JR_p have the same multiplicity for every minimal prime divisor p of I.

As observed by Ratliff, Theorem 2 is readily reduced to Theorem 1 by means of the following remarkable result:

THEOREM 3 ([3, Theorem 4], [15, Theorem 2.12]) In a formally equidimensional local ring R, for every ideal I such that $\ell(I) = h(I)$, the integral closure $\overline{I}$ has no embedded prime divisors.

(Ratliff also shows that this property characterizes formally equidimensional local rings [ibid., Theorem 2.29].)

In outline, Dade proves Theorem 3 by considering the above map $f: X \to \mathrm{Spec}(R)$ obtained by blowing up I, and its restriction

$$f_0: f^{-1}(\mathrm{Spec}(R/I)) \to \mathrm{Spec}(R/I)$$

The hypothesis that R is formally equidimensional (and hence satisfies the "altitude formula") ensures that every component E of $f^{-1}(\mathrm{Spec}(R/I))$ has dimension $\dim(R) - 1$ at each of its closed points (at least if $h(I) > 0$, which we can assume...). From this it follows, since the fibers of f_0 all have dimension $\ell(I) - 1 = h(I) - 1$, that

$$\dim(f(E)) \geq \dim(R) - h(I) = \dim(R/I)$$

so f(E) is a component of Spec(R/I) (and in fact every component of every fiber of f_0 has dimension $h(I) - 1$, i.e. the map f_0 is equidimensional [6, (13.3.2)]). Now the integral closure $\overline{I}$ is a finite intersection of ideals of the form

$$I_v = \{x \in R \mid v(x) \geq \min_{y \in I} (v(y))\}$$

where v is a discrete valuation centered on X at some E as above. I_v is a primary ideal whose radical is the generic point of f(E); thus every associated prime ideal of $\overline{I}$ is a minimal prime divisor of I. QED

Actually Ratliff did not use blow ups, but rather the closely related - and technically much simpler - notion of "Rees ring" (also used by Rees in his proof of Theorem 1!). The more picturesque geometric approach was given independently by Teissier [18, Appendix 1] in the context of analytic geometry.

We end this section with a reformulation of Theorem 2, in terms which will be technically useful later on.

For any ideals L, I in a local ring (R,M) such that L + I is M-primary, we set

$$e(L,I) = \sum_p e(L(R/p))e(IR_p)$$

where: p runs through the prime ideals containing I and such that

$$\dim(R) - \dim(R_p) = \dim(R/p) = \dim(R/I)$$

e(L(R/p)) is the multiplicity of the ideal L(R/p) in R/p; and $e(IR_p)$ is the multiplicity of IR_p.

THEOREM 2' Let $I \subseteq J$ be ideals in a formally equidimensional local ring (R,M) such that $\ell(I) = h(I) = h(J)$. If there exists an ideal L such L + I is M-primary and such that

$$e(L,I) = e(L,J)$$

then I is a reduction of J. Conversely, if I is a reduction of J, then e(L,I) = e(L,J) for every ideal L such that L + I is M-primary.

REMARKS In a formally equidimensional local ring (R,M), every ideal $I \subseteq M$ satisfies

$$h(I) + \dim(R/I) = \dim(R)$$

(Reduce to the case where I is prime, which is treated in [13, p. 125, (34.5)].) So h(I) = h(J) implies dim(R/I) = dim(R/J). Moreover $\ell(I) = h(I)$ implies that <u>every</u> minimal prime divisor p of I satisfies $\dim(R_p) = h(I)$ (since $h(I) = \ell(I) \geq \mathrm{alt}(I)$), whence dim(R/p) = dim(R/I).

With these remarks, the equivalence of Theorems 2 and 2' is a simple exercise.

§2. Equimultiplicity criteria for $\ell(I) = h(I)$ (Dade)

Let I be an ideal in a local ring (R,M) (M is the maximal ideal of R), let $r = \dim(R/I)$, and let $\underline{x} = (x_1,\ldots,x_r)$ be a sequence in R such that

(a) $\underline{x}R + I$ is an M-primary ideal, and

(b) $\dim(R/\underline{x}R) = \dim(R) - r = \dim(R) - \dim(R/I)$.

(In other words: $\underline{x}$ is part of a system of parameters in R, and the image of $\underline{x}$ in R/I is a system of parameters.)

We define $e(\underline{x}R,I)$ as in Section 1 (just before Theorem 2'), and let $e(\underline{x}R + I)$ [respectively $e(I(R/\underline{x}R))$] be the multiplicity of the M-primary ideal $\underline{x}R + I$ [respectively the $M/\underline{x}R$-primary ideal $I(R/\underline{x}R)$]. Then [8, pp. 119-122] we have the inequalities

$$e(\underline{x}R,I) \le e(\underline{x}R + I) \le e(I(R/\underline{x}R)) \qquad (2.1)$$

REMARK (2.1) gives the inequality

$$e(\underline{x}R,I) \le e(I(R/\underline{x}R)) \qquad (2.2)$$

which can also be proved more directly, cf. [8, p. 115, Lemma 3.4]. For a geometric interpretation of (2.2) cf. Section 3 below ((3.1) and (3.9)).

THEOREM 4 ([3], [1], [7]) Let (R,M) be a formally equidimensional local ring, with R/M infinite. For an ideal I in R, the following conditions are equivalent:

(i) $\ell(I) = h(I)$.

(ii) For every ideal L such that L + I is M-primary and $\ell(L) \le \dim(R/I)$ we have

$$e(L,I) = e(L + I)$$

(iii) There exists a sequence $\underline{y} = (y_1,\ldots,y_r)$ whose image in R/I is a system of parameters, and such that

$$e(\underline{y}R,I) = e(\underline{y}R + I)$$

(iv) There exists a sequence $\underline{x} = (x_1,\dots,x_r)$ satisfying (a) and (b) above, and such that

- if $r < \dim(R)$ then $e(\underline{x}R, I) = e(I(R/\underline{x}R))$
- if $r = \dim(R)$ then $e(\underline{x}R, I) = e(\underline{x}R)$.

Proof (Indications) (i) ⇒ (ii). There is a sequence $\underline{x} = (x_1, x_2, \dots, x_t)$ generating a reduction of L, where $t = \ell(L) \le \dim(R/I)$. (Actually, since $\underline{x}(R/I)$ is M/I-primary, we must have $t = \dim(R/I)$.) Also, if $s = \ell(I) = h(I)$, there is a sequence $\underline{z} = (z_1,\dots,z_s)$ generating a reduction of I. Since $L + I$ is M-primary, so therefore is $\underline{x}R + \underline{z}R$; but

$$t + s \le \dim(R/I) + h(I) \le \dim(R)$$

and hence $(x_1,\dots,x_t, z_1,\dots,z_s)$ is a system of parameters in R. Moreover it is clear that both L and I - and hence L + I - are contained in the integral closure of $\underline{x}R + \underline{z}R$, so $\underline{x}R + \underline{z}R$ is a reduction of L + I. Applying the "associativity formula" for multiplicities [13, (24.7)], and the fact that "reduction preserves multiplicity" (cf. beginning of Section 1), we conclude that

$$e(L, I) = e(\underline{x}R, \underline{z}R) = e(\underline{x}R + \underline{z}R) = e(L + I)$$

(ii) ⇒ (iii). Let $\underline{y}$ be <u>any</u> sequence whose image in R/I is a system of parameters; and set $L = \underline{y}R$. Clearly $\ell(L) \le \dim(R/I)$.

(iii) ⇒ (iv). Choose $\underline{y}$ as in (iii). In Lemma 1 of [7] it is shown, R/M being infinite, that for a "generic" sequence $\underline{x} = (x_1,\dots,x_r)$ $(r = \dim(R/I))$ of elements in $\underline{y}R + I$, we have:

- $\underline{x}$ is part of a system of parameters in R;
- $\underline{x}R + I = \underline{y}R + I$

[whence $e(\underline{x}R, I) = e(\underline{y}R, I)$];

- if $r < \dim(R)$ then $e(I(R/\underline{x}R)) = e(\underline{y}R + I)$ and if $r = \dim(R)$ then $e(\underline{x}R) = e(\underline{y}R + I)$

[The proof is an elaboration of that of [13, (24.1)]: set $J = \underline{y}R + I$, and inductively choose

$$x_i \in J - (I + MJ + (x_1,\dots,x_{i-1})R)$$

such that the image of x_i in $R_i = R/(x_1,\dots,x_{i-1})R$ is superficial of order 1 for the ideal JR_i.]. So

$$e(\underline{x}R,I) = e(\underline{y}R,I) = e(\underline{y}R + I)$$

and (iv) follows.

(iv) ⇒ (i). With

$$s = \dim(R/\underline{x}R) = \dim(R) - r$$

choose a sequence $\underline{z} = (z_1,\dots,z_s)$ in I whose image in $R/\underline{x}R$ generates a reduction of $I(R/\underline{x}R)$. (This is possible because R/M is infinite.) Then $(z_1,\dots,z_s, x_1,\dots,x_r)$ is a system of parameters in R. Setting $I_0 = \underline{z}R \subseteq I$, we have

$$\dim(R/I_0) = \dim(R) - s = r = \dim(R/I)$$

and using (2.2) (and the definition of e(L,I) in Section 1) we get

$$e(\underline{x}R,I) \le e(\underline{x}R,I_0) \le e(I_0(R/\underline{x}R)) = e(I(R/\underline{x}R))$$

By assumption, then, if $r < \dim(R)$, we have equality throughout, so that

$$e(\underline{x}R,I_0) = e(\underline{x}R,I) \tag{2.3}$$

If $r = \dim(R)$, then $I_0 = (0)$ and $e(\underline{x}R,I_0) = e(\underline{x}R)$ (cf. [13, (23.5)]); so (2.3) still holds.

Now by the remarks at the end of Section 1, we have

$$s = \dim(R) - \dim(R/I_0) = h(I_0)$$

$$s = \dim(R) - \dim(R/I) = h(I)$$

$$s \ge \ell(I_0) \ge h(I_0)$$

whence

$$s = \ell(I_0) = h(I_0) = h(I) \tag{2.4}$$

By Theorem 2', (2.3) and (2.4) imply that I_0 is a reduction of I; hence

$$h(I) = s \geq \ell(I) \geq h(I)$$

and so $\ell(I) = h(I)$. QED

REMARK (2.5) The implication (i) ⇒ (ii) is given essentially in [14, p. 158]. It does not require R/M to be infinite (since one can easily reduce to to this case), nor R to be formally equidimensional. Similarly (ii) ⇒ (iii) ⇒ ⇒ (i) and (iv) ⇒ (i) without R/M being infinite; and (ii) ⇒ (iii) ⇒ (iv) without R being formally equidimensional. Finally, for (iii) ⇒ (i) the assumption "R formally equidimensional" can be weakened to "R equidimensional and Catenary", cf. [1, Satz 1].

COROLLARY [3]. Let p be a prime ideal in a local ring (R,M) and assume that R/p is regular. If $\ell(p) = h(p)$ then the local R and R_p have the same multiplicity; and the converse is true if R is formally equidimensional.

Proof Let $\underline{y} = (y_1,\ldots,y_r)$ be a sequence whose image in R/p is a regular system of parameters. As in Section 1, $\ell(p) \geq \dim(R) - \dim(R/p) \geq h(p)$, so if $\ell(p) = h(p)$ then $\dim(R) - \dim(R_p) = \dim(R/p)$; and this last equality also holds for any prime p if R is formally equidimensional. In any case we have

$$e(\underline{y}R,p) = e(pR_p) = \text{multiplicity of } R_p$$
$$e(\underline{y}R + p) = e(M) = \text{multiplicity of } R$$

So the corollary follows from the implications (iii) ⇒ (i) ⇒ (ii) of Theorem 4. (As in Remark (2.5), (i) ⇒ (ii) requires no assumption on R; and in proving (iii) ⇒ (i) one can always reduce to the case when R/M is infinite.)

REMARK (2.6) The condition "R normally pseudo-flat along p" introduced by Hironaka in [9] amounts to "R/p regular and $\ell(p) = h(p)$". Hironaka proved the first part of the above Corollary, and Schickhoff [17] proved the converse; both these proofs were given in the context of analytic geometry. We will elaborate geometrically and algebraically on this situation in Sections 4 and 5.

REMARK (2.7) If R/M is infinite, R/p is regular, and $\ell(p) = h(p) > 0$, then the proof of (iii) ⇒ (iv) in Theorem 4 yields that for "almost all" sequences $\underline{x}$ whose image in R/p is a regular system of parameters, the multiplicity of R_p (or of R) is equal to the multiplicity of $R/\underline{x}R$, which is the local ring of the origin on a "section of Spec(R) transversal to Spec(R/p)". (Here the word "almost" can be omitted if R is Cohen-Macaulay, cf. the next remark.)

REMARK (2.8) (In connection with (iv) of Theorem 4.) Let I be an ideal in a local ring (R,M), with $\ell(I) = h(I)$. Let $\underline{x} = (x_1,\ldots,x_r)$ be any sequence whose image in R/I is a system of parameters. Then $\underline{x}$ is part of a system of parameters in R; and we have

$$e(\underline{x}R, I) = e(I(R/\underline{x}R))$$

<u>if and only if every minimal prime divisor</u> q <u>of</u> $\underline{x}R$ <u>with</u> $\dim(R/q) = \dim(R/\underline{x}R)$ <u>is such that</u> R_q <u>is a Cohen-Macaulay local ring of dimension</u> r (a condition which is automatically satisfied if R itself is Cohen-Macaulay).

Proof Assuming, as we may, that R/M is infinite, we choose a reduction $(z_1,\ldots,z_s)R = \underline{z}R$ of I ($s = \ell(I) = h(I)$). Then $(z_1,\ldots,z_s, x_1,\ldots,x_r)$ is a system of parameters in R; and by (2.2) we have

$$e(\underline{x}R, I) = e(\underline{x}R, \underline{z}R) \le e(\underline{z}(R/\underline{x}R)) = e(I(R/\underline{x}R))$$

By [13, (23.5)] we have

$$e(\underline{z}(R/\underline{x}R)) = \sum_{q \supset \underline{x}R} e(\underline{z}(R/q))\,\lambda(R_q/\underline{x}R_q)$$

$$\dim(R/q) = \dim(R/\underline{x}R)$$

where "λ" denotes length. Furthermore, every minimal prime divisor p of $\underline{z}R$ satisfies $\dim(R_p) = s$ (since $h(\underline{z}R) = h(I) = s$), and so the "associativity formula" [13, (24.7)] gives

$$e(\underline{x}, \underline{z}R) = e(\underline{x}R + \underline{z}R)$$
$$= \sum_q{}' e(\underline{z}(R/q))e(\underline{x}R_q)$$

where $\sum'$ is the sum over all q as above which also satisfy $\dim(R_q) = r$. Since $\lambda \geq e$, with equality if and only if R_q is Cohen-Macaulay, the assertion follows.

§3 Equimultiple families of ideals (Teissier)

Let $\phi: S \to R$ be a homomorphism of commutative noetherian rings, and let I be an ideal in R such that R/I is a finite S-module (via ϕ). For every prime ideal p in S, let k(p) be the fraction field of S/p, set

$$R(p) = R \otimes_S k(p) \text{ and } I(p) = IR(p)$$

so that

$$R(p)/I(p) = (R/I) \otimes_S k(p)$$

is a finite k(p)-module. This setup represents a "family of zero-dimensional ideals parametrized by Spec(S)". We have $I(p) \neq R(p)$ if and only if $p \supseteq \phi^{-1}(I)$; so we get essentially the same family if we replace ϕ by the induced map $S/\phi^{-1}(I) \to R/\phi^{-1}(I)R$. <u>We will assume therefore that</u> $\phi^{-1}(I) = (0)$, so that $I(p) \neq R(p)$ for any p.

By [13, (22.7)] the vector-space dimension

$$\dim_{k(p)}(R(p)/I(p)^n) = \dim_{k(p)}((R/I^n) \otimes_S k(p))$$

is given for large integers n by $\lambda_p(n)$, where λ_p is a polynomial whose degree $d_p = d_p(I)$ is the altitude of the ideal I(p), i.e. the dimension of the (semi-local) ring of fractions $R(p)_{1+I(p)}$ (or of its completion, which is also the I-adic completion of R(p)). We have

$$\lambda_p(n) = e_p(I)\frac{n^{d_p}}{d_p!} + \text{lower degree terms}$$

where $e_p(I)$ is an integer, the "<u>multiplicity</u> <u>of</u> I <u>at</u> p".

By induction on n, we see that R/I^n is a finite S-module; Nakayama's lemma implies then that $\dim_{k(p)}((R/I^n)\otimes_S k(p))$ is the least possible number of generators of the S_p-module $(R/I^n)\otimes_S S_p$. From this it is immediate that if $p \subseteq P$ are two prime ideals in S, then, for all $n > 0$,

$$\dim_{k(P)}((R/I^n)\otimes_S k(P)) \geq \dim_{k(p)}((R/I^n)\otimes_S k(p))$$

Consequently:

PROPOSITION (3.1) (Semicontinuity) With preceding notation, if $p \subset P$ are two prime ideals in S, then <u>either</u>: $d_P(I) > d_p(I)$, <u>or</u>: $d_P(I) = d_p(I)$ and $e_P(I) \geq e_p(I)$.

We will be interested only in the case when $d_p(I)$ is independent of p, and when moreover the dimension of the localization $R(p)_m$ is the same for all maximal ideals $m \supseteq I(p)$ in $R(p)$. To get a grasp on this condition, observe first that $R(p)_m = R_q/pR_q$, where $q \supseteq I$ is a prime ideal in R and $p = \phi^{-1}(q)$; note here that R/q is finite over its subring S/p (since R/I is finite over S), so that $\dim(R/q) = \dim(S/p)$. We assume that $\dim(R) < \infty$. Then we have:

$$\begin{aligned}\dim(R/pR) - \dim(S/p) &= \dim(R/pR) - \dim(R/q) \\ &\geq \dim(R_q/pR_q) \\ &\geq \dim(R_q) - \dim(S_p) \\ &\geq \dim(R_q) + \dim(S/p) - \dim(S) \\ &= \dim(R_q) + \dim(R/q) - \dim(S)\end{aligned} \tag{3.2}$$

Also, by (3.1), and since $\dim(R) < \infty$,

$$\sup_p\{d_p(I)\} = d_P(I) \text{ for some } \underline{\text{maximal}} \text{ ideal } P \text{ in } S.$$

Thus we are led to:

PROPOSITION (3.3) Assume that

every prime ideal $q \supseteq I$ in R satisfies (3.3.1)

$$\dim(R_q) + \dim(R/q) = \dim(R) < \infty$$

and that

every maximal ideal P in S satisfies (3.3.2)

$$\dim(R/PR) = \dim(R) - \dim(S)$$

Then for every prime ideal p in S we have

$$d_p(I) = \dim(R) - \dim(S) = h(I)$$

and

$$e_p(I) = \sum_{\substack{\phi^{-1}(q)=p \\ q \supseteq I}} [k(q): k(p)] e(I(R_q/pR_q)) \qquad (3.3.3)$$

where for every prime ideal $q \supseteq I$ in R with $\phi^{-1}(q) = p$, the fraction field $k(q)$ of R/q is a finite field extension of the fraction field $k(p)$ of S/p, and the multiplicity $e(I(R_q/pR_q))$ is well-defined because $I(R_q/pR_q)$ is primary for the maximal ideal of the local ring R_q/pR_q (which is a localization of $R(p)$ at a maximal ideal containing $I(p)$).

Proof Let $q \supseteq I$ be a prime ideal in R, and $p = \phi^{-1}(q)$. From the remarks preceding (3.3), we get, for some maximal ideal P in S,

$$\dim(R/PR) \geq d_P(I) \geq d_p(I) \geq \dim(R_q/pR_q)$$
$$\geq \dim(R_q) + \dim(R/q) - \dim(S)$$

From (3.3.1) and (3.3.2) we then conclude that

$$d_p(I) = \dim(R_q/pR_q) = \dim(R) - \dim(S)$$

Now (3.3.3) is an easy consequence of [13, (21.2)]. Moreover, since R/I is finite over its subring $S/\phi^{-1}(I) = S$, we have $\dim(S) = \dim(R/I)$; and (3.3.1) implies that the height $h(I) = \inf_{q \supseteq I}\{\dim(R_q)\}$ satisfies

$$h(I) = \dim(R) - \dim(R/I) = \dim(R) - \dim(S)$$

QED

Before continuing the algebraic discussion, we describe the geometric avatar of the above situation.

EXAMPLE (3.4) Let $F: X \to Y$ be a morphism of complex analytic varieties. Let $x_0 \in X$, $y_0 = F(x_0)$, let $S = \mathcal{O}_{Y,y_0}$ and $R = \mathcal{O}_{X,x_0}$ be the corresponding local rings of germs of holomorphic functions, and let $\phi: S \to R$ be the ring homomorphism induced by F. Let $\mathcal{I}$ be a coherent $\mathcal{O}_X$-ideal such that R/I is a finite S-module and $\phi^{-1}(I) = 0$, where $I = \mathcal{I}_{x_0} \subsetneq R$. This means that if Z is the subspace of X on which the functions in $\mathcal{I}$ vanish, then - after X and Y are replaced by suitable neighborhoods of x_0 and y_0 - $Z \cap F^{-1}(\{y_0\}) = \{x_0\}$, and F induces a proper surjective map $Z \to Y$ with finite fibers.

Corresponding to (3.3.1) and (3.3.2) we assume that:

Every irreducible component X' of X satisfies (3.3.1)*

$$\dim(X') = \dim_{y_0}(Y_0) + \dim_{x_0}(F^{-1}\{y_0\})$$

It follows - after shrinking Y - that every component of Y has the same dimension, and - after shrinking X - that the local fiber dimension $\dim_x(F^{-1}\{F(x)\})$ is constant on X (and equal to $\dim(X) - \dim(Y)$).

For any $y \in Y$, we set $X_y = F^{-1}(\{y\})$, and

$$e_y(\mathcal{I}) = \sum_{x \in X_y \cap Z} e(\mathcal{I}\mathcal{O}_{X_y,x}) \quad (e = \text{multiplicity}) \qquad (3.3.3)^*$$

Here is the geometric meaning of $e_p(I)$ for a prime ideal p in S. Let Y_p be the irreducible subvariety of Y whose germ at y_0 is defined by p. (Again we may need to shrink Y.)

PROPOSITION (3.5) In a suitably small neighborhood of y_0, we have

$$e_y(\mathcal{I}) \geq e_p(I) \text{ for all } y \in Y_p$$

with equality for $y \in Y_p - W$, where W is a nowhere dense subvariety of Y_p.

The proof involves techniques of analytic geometry which can be found in [12, §8]: one shows that, $i_p: Y_p \to Y$ being the inclusion map, the graded $\mathcal{O}_{Y_p}$ - algebra

$i_p^*F_*(\bigoplus_{n\geq 0} I^n/I^{n+1})$ is finitely presented, hence flat over $Y_p - W$ for some W as above; then for each $n \geq 0$

$$F_*(I^n/I^{n+1}) \otimes_{\mathcal{O}_{Y,y}} \mathbb{C} \qquad (y \in Y_p - W)$$

is a $\mathbb{C}$-vector space whose dimension $\lambda_y(n)$ does not depend on y; in particular, $e_y(I)$ - which is determined by the function $\lambda_y(n)$ - is constant on $Y_p - W$. The transition from geometry to local algebra can then be made by means of the following observation, applied to $M = F_*(I^n/I^{n+1})$ $(n = 0,1,2,\ldots)$:

Let M be a coherent $\mathcal{O}_Y$-module, presented in a neighborhood U of y_0 by an $m \times n$ matrix of functions (a_{ij}) (i.e. M is the cokernel of the map $\mathcal{O}_U^m \to \mathcal{O}_U^n$ given by this matrix). Then for $y \in U$, the vector-space dimension of $M(y) = M \otimes_{\mathcal{O}_{Y,y}} \mathbb{C}$ is n-(rank of $(a_{ij}(y))$). Hence if Y_p is a subvariety of Y, irreducible at y_0, and represented by the prime ideal p in $S = \mathcal{O}_{Y,y_0}$; and if M is the S-module M_{y_0}; then, with k(p) the fraction field of S/p, we have, in some neighborhood of y_0, that for all $y \in Y_p$

$$\dim_{\mathbb{C}} M(y) \geq \dim_{k(p)}(M \otimes_S k(p))$$

with equality outside some nowhere dense subvariety of Y_p.

(Another possibility at this point would be to use techniques suggested by [5, §4].)

This Section is related to Section 2 via the following Corollary of (3.3):

COROLLARY (3.6) With assumptions as in (3.3), suppose further that R is local, with maximal ideal M, and that S is a domain. Note that since R/I is finite over its subring $S/\phi^{-1}(I) = S$, therefore S is local with maximal ideal $m = \phi^{-1}(M)$, and $mR + I$ is M-primary. Let L be any m-primary ideal in S. Then, with the notation of Section 1, we have

$$e_m(I) = [(R/M):(S/m)]e(I(R/mR)) \tag{3.6.1}$$

$$e(L)e_{(0)}(I) = [(R/M):(S/m)]e(LR,I) \tag{3.6.2}$$

If, moreover, S is Cohen-Macaulay and R_q is flat over S for every minimal prime divisor q of mR, then, for any system of parameters $\underline{x} = (x_1,\ldots,x_r)$ in S, we have

$$e(\underline{x}S)e_m(I) = [(R/M): (S/m)]e(I(R/\underline{x}R)) \qquad (3.6.3)$$

Proof (3.6.1) is immediate from definitions (or from (3.3.3)). If S is Cohen-Macaulay, then the multiplicity $e(\underline{x}S)$ is also the length of the S-module $S/\underline{x}S$; so if q is a minimal prime divisor of mR, and R_q is flat over S, then denoting length of an R_q-module E by $\lambda(E)$ we see that

$$\lambda(R_q/\underline{x}R_q) = e(\underline{x}S)\lambda(R_q/mR_q)$$

and since $\underline{x}R \subseteq mR \subseteq \sqrt{\underline{x}R}$, (3.6.3) follows from (3.6.1) and [13, (23.5)].

To get (3.6.2), apply (3.3.3), noting first of all for any prime ideal $q \supseteq I$, since R/I is finite over S and R/q is finite over $S/\phi^{-1}(q)$, and by (3.3.1):

$$\phi^{-1}(q) = (0) \Leftrightarrow \dim(R/q) = \dim(S) = \dim(R/I)$$
$$\dim(R/q) = \dim(R) - \dim(R_q)$$

and secondly that if $\phi^{-1}(q) = (0)$, the "projection formula" [21, p. 299, Cor. 1] gives

$$e(L)[k(q): k((0))] = [(R/M): (S/m)]e(L(R/q))$$

QED

The main results in this Section are contained in the following Theorems 4a and 4b, which give another equimultiplicity criterion for $\ell(I) = h(I)$.

THEOREM 4a With assumptions as in (3.3), assume further that R is local, with maximal ideal M (whence, as in (3.6) S is local, with maximal ideal $m = \phi^{-1}(M)$), and that R is flat over S (which implies (3.3.2)). If $\ell(I) = h(I)$, then

$$e_p(I) = e_m(I)$$

for every prime ideal p in S.

Proof Assuming, as we may, that $p \neq m$, we first construct a local homomorphism $\psi: S \to S^*$ with kernel p, where S^* is a one-dimensional local noetherian domain, essentially of finite type over S, with residue field purely transcendental over that of S, and with fraction field equal to that of S/p: to do this, let $f: X \to \mathrm{Spec}(S/p)$ be obtained by blowing up an ideal in S/p generated by a system of parameters $(y_1, y_2, \ldots, y_d)$; then the closed fiber $f^{-1}(\{m/p\})$ is a projective space $\mathbb{P}^{d-1}$ over the residue field of S; and we can take S^* to be the local ring on X of the generic point of this fiber (i.e. S^* is the localization of $(S/p)[y_2/y_1, \ldots, y_d/y_1]$ at the prime ideal generated by m/p).

Now in $R \otimes_S S^*$ there is just one maximal ideal N containing $I(R \otimes_S S^*)$; in other words $(R/I) \otimes_S S^*$ is a local ring. To see this, note that since $(R/I) \otimes_S S^*$ is finite over S^*, every maximal ideal in $(R/I) \otimes_S S^*$ contains $m^*((R/I) \otimes_S S^*)$, where m^* is the maximal ideal of S^*; so we need only check that

$$(R/I) \otimes_S (S^*/m^*) = (R/I + mR) \otimes_{S/m} (S^*/m^*)$$

is local, which is clear because $R/I + mR$ is a local Artin ring finite over S/m, and S^*/m^* is purely transcendental over S/m. We set

$$R^* = (R \otimes_S S^*)_N \qquad \text{(localization at N)}$$

We have a natural map $\phi^*: S^* \to R^*$, via which the noetherian local ring R^* is flat over the Cohen-Macaulay local domain S^* (we are preparing to apply (3.6) to ϕ^*!). With $I^* = IR^*$, we have that

$$R^*/I^* = (R/I) \otimes_S S^*$$

is finite over S^*. Moreover

$$(\phi^*)^{-1}(I^*) = (0)$$

because otherwise every prime ideal in $R/I \otimes_S S^*$ would contract to the maximal ideal in S^*, hence to m in S; but if $q \supseteq I$ is a prime in R with $\phi^{-1}(q) = p$ (such q exist since R/I is finite over $S = S/\phi^{-1}(I)$), then the natural commutative diagram

$$\begin{array}{ccccc} & & S & \longrightarrow & R/I \\ & \swarrow & \downarrow & & \downarrow \\ S^* & \longrightarrow & k(p) & \longrightarrow & k(q) \end{array}$$

defines a homomorphism $(R/I)\otimes_S S^* \to k(q)$ whose kernel is a prime ideal contracting to p in S, contradiction.

So ϕ^* defines a family of zero-dimensional ideals. We have, for every integer n,

$$(R^*/(I^*)^n)\otimes_{S^*}k(p) = ((R/I^n)\otimes_S S^*)\otimes_{S^*}k(p) = (R/I^n)\otimes_S k(p) \tag{3.7}$$

$$\begin{aligned}(R^*/(I^*)^n)\otimes_{S^*}(S^*/m^*) &= ((R/I^n)\otimes_S S^*)\otimes_{S^*}(S^*/m^*) \qquad (3.7)' \\ &= ((R/I^n)\otimes_S (S/m))\otimes_{S/m}(S^*/m^*)\end{aligned}$$

From this it follows that

$$e_{(0)}(I^*) = e_p(I)$$

$$e_{m^*}(I^*) = e_m(I)$$

It will therefore suffice to show that

$$e_{(0)}(I^*) = e_{m^*}(I^*)$$

<u>If we can verify</u> the hypotheses (3.3.1) and (3.3.2) for $\phi^*: S^* \to R^*$ and I^*, and that $\ell(I^*) = h(I^*)$, then (3.6.2) and (3.6.3) apply with $L = xS^*$, x being any non-zero element of S^*. This reduces us to showing that

$$e(xR^*,I^*) = e(I^*(R^*/xR^*)) \tag{3.8}$$

But (3.8) follows easily from Remark (2.8) because R^* is flat over S^*, so that for every minimal prime divisor q of xR^* we have that

$$\dim(R^*_q) = \dim(S^*) + \dim(R^*_q/xR^*_q) = 1$$

and x is a regular element of R^*_q, i.e. R^*_q is Cohen-Macaulay.

Let us then verify what we need to (see above). To begin with we check that (3.3.1) holds for ϕ^*. Since R^*/I^* is finite over its subring S^*, therefore $\dim(R^*/I^*) = 1$, and it will suffice to show that for every <u>non-maximal</u> prime ideal $q \supseteq I^*$, we have

$$\dim(R^*_q) = \dim(R^*) - 1 \qquad (3.9)$$

But these q correspond precisely to the maximal ideals in $R^*_{(0)} = R^* \otimes_{S^*} k(p)$ containing $I^*R^*_{(0)}$, hence to the maximal ideals in $R(p) = R \otimes_S k(p)$ containing $I(p)$ (cf. (3.7)), and so by (3.3)

$$\dim(R^*_q) = h(I)$$

On the other hand, by flatness,

$$\begin{aligned}\dim(R^*) &= \dim(S^*) + \dim(R^*/m^*R^*) \\ &= 1 + \dim(R/mR) && \text{(cf. (3.7)')} \\ &= 1 + \dim(R) - \dim(S) && \text{(cf. (3.3.2))} \\ &= 1 + h(I) && \text{(cf. (3.3))}\end{aligned}$$

This gives us (3.9); and furthermore, it follows that

$$h(I^*) = \dim(R^*) - 1 = h(I)$$

But clearly

$$\ell(I^*) \le \ell(I) = h(I) = h(I^*) \le \ell(I^*)$$

and so

$$\ell(I^*) = h(I^*)$$

Finally (3.3.2) holds for ϕ^* because R^* is flat over S^*. This completes the proof.

For the next Theorem 4b, we need the following analogue of Theorem 2:

THEOREM 2'' With assumptions as in (3.3), suppose further that R is a formally equidimensional local ring (which implies (3.3.1)), and that

$\ell(I) = h(I)$. Let $J \supseteq I$ be an ideal in R, with $\phi^{-1}(J)$ nilpotent in S. If $e_p(I) = e_p(J)$ for every minimal prime ideal p of S, then I is a reduction of J. Conversely, if I is a reduction of J, then $e_p(I) = e_p(J)$ for every prime ideal p in S.

Proof By Theorem 3 of Section 1, it will suffice to show that IR_q is a reduction of JR_q for every minimal prime divisor q of I. Since $\ell(I) = h(I)$ and R is formally equidimensional, every such q satisfies

$$\begin{array}{ccc} \dim(R/q) & = & \dim(R/I) \\ \| & & \| \\ \dim(S/\phi^{-1}(q)) & & \dim(S) \end{array}$$

(cf. end of Section 1). So the q we are interested in are precisely those which contain I, and for which $p = \phi^{-1}(q)$ satisfies $\dim(S/p) = \dim(S)$. For such a p, S_p is artinian, so pR_q is a nilpotent ideal, and it is enough to show that $I(R_q/pR_q)$ is a reduction of $J(R_q/pR_q)$ (cf. [16, p. 9, Lemma 1.2] or [11, p. 792, Lemma (1.1)]).

Now $p \supseteq \phi^{-1}(J)$ (which is nilpotent); and replacing ϕ by the induced map $S/\phi^{-1}(J) \to R/\phi^{-1}(J)R$ we see that (3.3.3) holds with J in place of I. Since $e_p(I) = e_p(J)$ and $J \supseteq I$, (3.3.3) gives

$$e(I(R_q/pR_q)) = e(J(R_q/pR_q))$$

and the conclusion follows from Theorem 1 (R_q/pR_q is formally equidimensional by [6, (7.1.8)]). Conversely, if I is a reduction of J, then (3.3.3) gives $e_p(I) = e_p(J)$ for every prime p is S.

QED

The underlying idea of the following Theorem is due essentially to Teissier [18, Appendix 1]. It is similar to the proof of (iv) ⇒ (i) in Section 2.

THEOREM 4b Let ϕ: (S,m) → (R,M) be a homomorphism of noetherian local rings, and let $I \subseteq M$ be an R-ideal such that R/I is a finite S-module (via ϕ) and $\phi^{-1}(I) = (0)$. Assume that R is formally equidimensional and that

$$\dim(R) = \dim(S) + \dim(R/mR)$$

If for every prime ideal p in S we have

$$e_p(I) = e_m(I)$$

then $\ell(I) = h(I)$.

Proof As usual we may assume that S/m and R/M are infinite. Since R/I is finite over S, therefore mR + I is M-primary. So, with

$$s = \dim(R/mR) = \dim(R) - \dim(S)$$

we can choose a sequence $\underline{z} = (z_1,\ldots,z_s)$ in I whose image in R/mR generates a reduction of I(R/mR)(*). Let $I_0 = \underline{z}R$. Then, I claim,

$$\ell(I_0) \geq h(I_0) = s \geq \ell(I_0)$$

The only thing to check here is that $h(I_0) = s$: but if $\underline{x} = (x_1,\ldots,x_r)$ is a system of parameters in S, then any prime ideal in R containing $\underline{z}R + \underline{x}R$ contains $\underline{z}R + mR$, hence must be M; and since

$$r = \dim(S) = \dim(R) - s$$

we see that $(z_1,\ldots z_s, x_1,\ldots x_r)$ is a system of parameters in R, so that

$$\dim(R/I_0) = \dim(R) - s$$

Finally, since R is formally equidimensional we have (cf. end of Section 1)

$$h(I_0) = \dim(R) - \dim(R/I_0) = s$$

We will show that I_0 <u>is a reduction of</u> I, whence

$$s \geq \ell(I) \geq h(I) = h(I_0) = s$$

* This is where we need R/M to be infinite. To avoid the reduction to this case, we could replace I by some power I^n.

so that indeed $\ell(I) = h(I)$.

If we knew that R/I_0 was finite over S (which is <u>necessary</u> if I_0 is to be a reduction of I) then by semicontinuity (3.1) we would have, for any prime p in S (since $d_p(I) = d_m(I) = h(I)$, cf. (3.3)):

$$e_p(I) \le e_p(I_0) \le e_m(I_0) = e_m(I)$$

so that $e_p(I) = e_p(I_0)$, and by Theorem 2", I_0 would be a reduction of I.

Now we do know that R/I_0 is <u>quasi-finite</u> over S, i.e. $R/(I_0 + mR)$ is finite over S; so if the local rings R and S were <u>complete</u>, then R/I_0 would indeed be finite over S [21, p. 259, Corollary 2]. Let us therefore reduce the general case to the case where R and S are complete.

First of all, if $\hat{I} = I\hat{R}$ and $\hat{m} = m\hat{S}$ (the maximal ideal of $\hat{S}$), then clearly the image of $\underline{z}$ in $\hat{R}/\hat{m}\hat{R}$ generates a reduction of $\hat{I}(\hat{R}/\hat{m}\hat{R})$.

Moreover, if $I_0\hat{R}$ is a reduction of $I\hat{R}$, then for some n we have

$$\begin{aligned} I^{n+1} = (I\hat{R})^{n+1} \cap R &= [(I_0\hat{R})(I\hat{R})^n] \cap R \\ &= (I_0 I^n)\hat{R} \cap R \\ &= I_0 I^n \end{aligned}$$

so that I_0 is a reduction of I. So we need only check that the hypotheses of Theorem 4b hold for the completion $\hat{\phi}\colon \hat{S} \to \hat{R}$ and for $\hat{I}$.

Since R/I is finite over S, therefore

$$\hat{R}/\hat{I} = (R/I)^{\wedge} = (R/I) \otimes_S \hat{S}$$

is finite over $\hat{S}$, and $\hat{S} \to \hat{R}/\hat{I}$ is injective (because $S \to R/I$ is injective and $\hat{S}$ is flat over S) i.e. $\hat{\phi}^{-1}(\hat{I}) = 0$. $\hat{R}$ is formally equidimensional (since R is), and

$$\begin{array}{ccc} \dim(\hat{R}) & \dim(\hat{S}) + \dim(\hat{R}/\hat{m}\hat{R}) \\ \| & \| \\ \dim(R) = & \dim(S) + \dim(R/mR) \end{array}$$

Finally, if P is a prime ideal in $\hat{S}$, and $p = P \cap S$, then for any integer n, we have (since R/I^n is finite over S):

$$(R/I^n) \otimes_{\hat{S}} k(P) = (R/I^n) \otimes_S k(P)$$
$$= (R/I^n) \otimes_S \hat{S} \otimes_{\hat{S}} k(P)$$
$$= (R/I^n) \otimes_S k(p) \otimes_{k(p)} k(P)$$

and we conclude that

$$e_P(\hat{I}) = e_p(I) = e_m(I) = e_{\hat{m}}(\hat{I})$$

QED

From (3.5) and the preceding three theorems, we obtain the following geometric statements, which constitute an expansion of Teissier's "principle of specialization of integral dependence" [18, Appendix 1]. This principle has some beautiful applications to equisingularity theory, cf. [18, p. 128], [19, p. 603, Theorem 2.18].

SCHOLIUM In Example (3.4), assume that $e_y(I)$ is independent of y. Then $\ell(I) = h(I)$. [In fact if $I_0 \subseteq I$ is any coherent ideal such that $I_0 \mathcal{O}_{X_{y_0}, x_0}$ is a reduction of $I\mathcal{O}_{X_{y_0}, x_0}$, then I_0 is a reduction of I in some neighborhood U of x_0 i.e. $I_0 I^n = I^{n+1}$ in U, for some integer n.] Conversely, if F is flat at x_0 and $\ell(I) = h(I)$, then there is a neighborhood V of y_0 such that $e_y(I) = e_{y_0}(I)$ for all y in V.

Now suppose that $\ell(I) = h(I)$, and let $J \supseteq I$ be a coherent $\mathcal{O}_X$-ideal. Assume that there is a set $W \subseteq Y$ whose closure is a neighborhood of y_0, and such that for every $y \in W$, $I\mathcal{O}_{X_y}$ is a reduction of $J\mathcal{O}_{X_y}$ [equivalently: $e_y(I) = e_y(J)$, where we set $e_y(J) = 0$ if $y \notin F$(zero-set of J)]. Then I is a reduction of J in some neighborhood of x_0 [and hence (since $Z \cap F^{-1}(\{y_0\}) = \{x_0\}$) $e_{y_0}(I) = e_{y_0}(J)$].

§4 Equimultiplicity along subvarieties

In this Section, we give another approach to the Corollary in Section 2. Though this approach is technically less simple, nevertheless it is inspired by geometric considerations which make it, I think, more illuminating.

Let $v \in V$, a complex analytic variety, and let W be a subvariety of V passing through v and irreducible near v. Let $R = \mathcal{O}_{V,v}$ be the local ring of germs of holomorphic functions at v, let $p \subseteq R$ be the (prime) ideal of germs of functions vanishing on W, and let $e(R_p) = e(pR_p)$ be the multiplicity of the localization R_p. For any $y \in V$, let $e(V,y)$ be the multiplicity of V at y (i.e. the multiplicity of the maximal ideal of $\mathcal{O}_{V,y}$). By methods similar to those indicated in the proof of (3.5), one proves:

PROPOSITION (4.1) In a suitably small neighborhood of v, we have

$$e(V,y) \geq e(R_p) \text{ for all } y \in W$$

with equality for $y \in W-Z$, where Z is a nowhere dense subvariety of W.

This Proposition may also be interpreted as follows: there is a sequence of closed subvarieties

$$V = V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots$$

such that

$$y \in V_i \Leftrightarrow e(V,y) \geq i$$

(cf. [12, §8] or [20, p. 237]). After shrinking V, we may assume that each V_i passes through v, and correspondingly we get a sequence of ideals in R

$$(0) = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_e \qquad e = e(V,v)$$

Proposition (4.1) says then that:

$$p \supseteq I_c \Leftrightarrow e(R_p) \geq c$$

In other words, <u>the</u> (<u>global</u>, <u>analytic</u>) <u>multiplicity stratification on</u> V <u>induces the</u> (<u>local</u>, <u>algebraic</u>) <u>multiplicity stratification on</u> Spec(R).

DEFINITION (4.2) V is equimultiple along W at v if $e(V,y)$ is constant in some neighborhood of v in W, i.e. (by (4.1)), if R and R_p have the same multiplicity.

The discussion in Section 1 leads to the following criterion for equimultiplicity:

PROPOSITION (4.3) Assume that every component of V has dimension d, and that (V,v) is embedded germwise in $(\mathbb{C}^m,0)$. Then V is equimultiple along W if and only if there exists a linear projection π: $(\mathbb{C}^m,0) \to (\mathbb{C}^d,0)$ such that, near v, each $y \in W$ satisfies

$$\pi^{-1}\pi(y) \cap V = \{y\} \quad \text{(non-splitting), and}$$

$$\pi^{-1}\pi(y) \cap C_3(V,y) = \{y\}$$

where $C_3(V,y)$ is the Zariski tangent cone of V at y.

For example, if W is smooth at v (i.e. the local ring $R/p = \mathcal{O}_{W,v}$ is regular, and if $\ell(p) = h(p)$, then we get such a π by using $(x_1,\dots,x_t,z_1,\dots,z_s)$ as coordinates, where $s = \ell(p)$, $(z_1,\dots,z_s)R$ is a reduction of p, and the image of $(x_1,\dots,x_t)$ in R/p is a regular system of parameters. (This is basically how Hironaka first proved the easier half of the Corollary in Section 2, cf. [9, §3].) Conversely, if V is equimulitple along W, then the idea behind what we are about to do is that every π as in (4.3) can be obtained in the manner just described, whence $\ell(p) = h(p)$.

We next give an algebraic generalization of (4.3); and then show how it leads to another proof - and perhaps a better understanding - of the Corollary of Theorem 4 in Section 2.

LEMMA (4.4) Let (S,m) be a formally equidimensional local domain; and let $(R,M) \supseteq (S,m)$ be a local ring which is a finite torsion-free S-module, such that the natural map $S/m \to R/M$ is an isomorphism, and such that mR is a reduction of M. Let P be a prime ideal in R such that the completion $(R/P)^\wedge$ is reduced (i.e. without nonzero nilpotents). Then the following conditions are equivalent:

(i) $e(R) = e(R_P)$

(where $e(R) = e(M)$ is the multiplicity of R, and similarly $e(R_P) = e(PR_P)$ is the multiplicity of R_P).

(ii) With $p = P \cap S$, we have

(a) $e(S) = e(S_p)$

(b) The natural map $R \otimes_S S_p \to R_p$ is an isomorphism.

(c) The fraction fields of S/p and R/P are naturally isomorphic and

(d) $e(pR_p) = e(R_p)$.

[To relate (4.4) to (4.3), note that (b) and (c) express "generic non-splitting" and (d) expresses "generic transversality" for $\mathrm{Spec}(R/P) \subseteq \mathrm{Spec}(R)$ with respect to the projection $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$.]

Proof Let $P = P_1, P_2, \ldots, P_t$ be the prime ideals in R lying over p; the fraction field of R/P_i is a finite extension of that of S/p, of degree, say, f_i. Every minimal prime Q in R consists entirely of zero-divisors; so $Q \cap S = (0)$ and therefore $\dim(R/Q) = \dim(S)$; consequently [6, (7.1.8)] R is formally equidimensional and so for each i

$$\begin{aligned}\dim(R_{P_i}) &= \dim(R) - \dim(R/P_i)\\ &= \dim(S) - \dim(S/p)\\ &= \dim(S_p)\end{aligned}$$

So we can apply the "projection formula" [21, p. 299] to get

$$[R:S]e(S) = e(mR) = e(M) = e(R) \tag{4.4.1}$$

$$[R:S]e(S_p) = [R\otimes_S S_p : S_p]e(S_p) = \sum_{i=1}^{t} f_i e(pR_{P_i}) \geq e(R_p) \tag{4.4.2}$$

Now S/p is a subspace of R/P (since R/P is finite over S/p), so that $(S/p)^\wedge \subseteq (R/P)^\wedge$ is reduced, and therefore [13, (40.1)] $e(S) \geq e(S_p)$. If $e(R) = e(R_p)$, then (4.4.1) and (4.4.2) give $e(S) \leq e(S_p)$, whence $e(S) = e(S_p)$ and equality holds throughout (4.4.2); thus (i) ⇒ (ii). Conversely if (ii) holds then we have equality throughout (4.4.2) and (i) follows.

LEMMA (4.5) With assumptions as in (4.4), suppose that conditions (b) and (d) of (ii) are satisfied, and that $\ell(p) = h(p)$. Then $\ell(P) = h(P)$.

Proof If $Q \supseteq pR$ is a prime ideal in R, and $q = Q \cap S \supseteq p$, then, as in the proof of (4.4), we have

$$\dim(R_Q) = \dim(S_q) \geq h(p)$$

with equality when $Q = P$. Then $h(pR) = h(p)$.

Clearly, then,

$$\ell(p) \geq \ell(pR) \geq h(pR) = h(p)$$

so that $\ell(pR) = h(pR)$. Consequently (cf. end of Section 1) every minimal prime divisor Q of pR satisfies

$$\dim(S/Q \cap S) = \dim(R/Q) = \dim(R/pR) = \dim(S/p)$$

i.e. $Q \cap S = p$; and since $R \otimes_S S_p = R_p$, we conclude that $Q = P$; thus $\sqrt{pR} = P$. Now Theorem 2 of Section 1 shows that pR is a reduction of P, so that

$$\ell(P) = \ell(pR) = h(pR) = h(P)$$

QED

Now here is the promised proof of the Corollary in Section 2. More precisely, we prove the implication $[e(R) = e(R_P)] \Rightarrow [\ell(P) = h(P)]$, the converse being, as in Section 2, an easy consequence of "associativity."

COROLLARY (4.6) Let R be a formally equidimensional local ring, and let P be a prime ideal in R such that R/P is regular and $e(R) = e(R_P)$. Then $\ell(P) = h(P)$.

Proof (Sketch) Imitating the proof of (40.1) in [13] (to which, incidentally, (4.4) is clearly related), and using (4.5), we reduce to the case where R is of the form T/fT, with T a complete regular local ring and $f \in T$. In fact, if R is equicharacteristic, we can take $f = 0$, and then P itself is generated by h(P) elements, so that $\ell(P) = h(P)$.

If $f \neq 0$, let Q be the inverse image of P in T, so that T/Q is regular, and the graded ring $G = \oplus_{n\geq 0} Q^n/M_TQ^n$ (M_T = maximal ideal of T) is a polynomial ring in $h(Q) = h(P) + 1$ variables. Now it is easy to see that the graded ring $\overline{G} = \oplus P^n/MP^n$ is just $G/\overline{f}G$, where $\overline{f} \neq 0$ is the image of f in

$Q^e/M_T Q^e$ ($e = e(R) = e(R_p)$). Hence the Krull dimension of $\overline{G}$ is $h(P)$, i.e. $\ell(P) = h(P)$.

REMARK In the context of algebraic varieties over a field, the method of reducing (4.6) to the hypersurface case by using a generic projection was shown to me by Zariski in 1972.

§5. Tangent cones and equimultiple subvarieties (Schickhoff)

Let $v \in W \subseteq V$, $p \subseteq R$ be as at the beginning of Section 4. We assume further that V is pure-dimensional, i.e. all its components have the same dimension (so that R is formally equidimensional [13, (45.1), (45.6)]); that W is smooth near v (i.e. R/p is a regular local ring); and that V is equimultiple along W at v (i.e. $e(R) = e(R_p)$). By (4.6), the blowup $f: X \to \mathrm{Spec}(R)$ of P is equidimensional over $\mathrm{Spec}(R/P)$. Correspondingly, the analytic blowup $F: Y \to V$ of W has all its fibers over W - near v - pure dimensional, of dimension equal to [codimension of W in V] -1 (which is the same as $\dim(R_p)-1$). We can, however, say even more about the fibers.

The fiber $F^{-1}(v)$ can be interpreted as follows (details omitted). Embed (V,v) into $(\mathbb{C}^m,0)$ in such a way that W becomes a linear subspace of (a neighborhood of 0 in) $\mathbb{C}^m$. Let E be a linear subspace of complementary dimension, in general position, intersecting W at 0. Let $\underline{v} = (v_i)$ be a sequence of points of V-W approaching 0, let $v_i + W$ be the linear space spanned by v_i and W, and let L_i be the line $(v_i + W) \cap E$. Assume that as $v_i \to 0$, the line L_i has a limit $L = L_{\underline{v}}$ (think of lines in E through 0 as points of a projective space). <u>There is a natural one-one correspondence between the set of all such $L_{\underline{v}}$ and the fiber</u> $F^{-1}(0)$. (To avoid choosing E, one can use the set of all limiting positions of the spaces $v_i + W$, in some Grassmannian). In particular, if we consider sequences (v_i) with $v_i \in V \cap E$, we see that <u>if</u> $\overline{F}: \overline{Y} \to V \cap E$ <u>is the blowup of the origin</u> 0, <u>then the fiber</u> $\overline{F}^{-1}(0)$ <u>is naturally contained in</u> $F^{-1}(0)$. (In fact $\overline{F}^{-1}(0)$ is the projectivized Zariski tangent cone of $V \cap E$ at 0, i.e. it is the set of all $L_{\underline{v}}$ with $\underline{v} \subseteq V \cap E$.) Note that

$$\dim(\bar{F}^{-1}(0)) = \dim(V \cap E) - 1 = \dim(R_p) - 1$$

even if V is not equimultiple along W.

Now the main result of this section is that, under the above assumptions (V equimultiple along W which is smooth at v) the fibers $\bar{F}^{-1}(0)$ and $F^{-1}(0)$ are equal (as sets).

From this we will deduce that - as a reduced variety - the tangent cone of V at 0 is a product

$$C_3(V,0)_{red} = C_3(V \cap E,\ 0)_{red} \times W \tag{5.1}$$

This should be compared with the condition of normal flatness of V along W, which says that (5.1) holds even with non-reduced structures.

The above result - and more - is due to Schickhoff [17, §3; in particular Theorem 3.27]. We will give here an algebraic generalization.

Algebraically, the space E gives an ideal $I = (x_1,\ldots,x_r)R$ with $r = \dim(R/p)$ such that $I + p = M$, the maximal ideal of R; and the inclusion $\bar{F}^{-1}(0) \subseteq F^{-1}(0)$ corresponds to the natural surjective homomorphism of graded rings

$$\begin{aligned}\bigoplus_{n\geq 0} p^n/Mp^n = \bigoplus_{n\geq 0} p^n/(p^{n+1} + Ip^n) &\xrightarrow{\ \theta\ } \bigoplus_{n\geq 0} p^n/(p^{n+1} + I \cap p^n) \\ &= \bigoplus_{n\geq 0} (p^n + I)/(p^{n+1} + I) \\ &= \bigoplus_{n\geq 0} (M/I)^n/(M/I)^{n+1}\end{aligned}$$

The equality $\bar{F}^{-1}(0) = F^{-1}(0)$, and (5.1), become:

THEOREM 5 Let (R,M) be a formally equidimensional local ring, and let p be a prime ideal in R such that R/p is regular. Let $\underline{x} = (x_1,\ldots,x_r)$ be a sequence in R whose image in R/p is a regular system of parameters, and let $I = \underline{x}R$ (so that $I + p = M$). Consider the commutative diagram (with indeterminates $X_1,\ldots,X_r$):

$$
\begin{array}{ccc}
H & & G \\
\| & & \| \\
\Big(\bigoplus_{n\geq 0} p^n/Mp^n\Big)[X_1,\ldots,X_r] & \xrightarrow{\psi} & \bigoplus_{n\geq 0} M^n/M^{n+1} \\
j \uparrow \downarrow \pi & & \downarrow \\
\bigoplus_{n\geq 0} (p^n/Mp^n) & \xrightarrow{\theta} & \bigoplus_{n\geq 0} (M/I)^n/(M/I)^{n+1} \\
\| & & \| \\
\overline{H} & & \overline{G}
\end{array}
$$

where the vertical arrows are the natural maps (j = inclusion, $\pi \circ j$ = identity, $\pi(X_i) = 0$); and ψ and θ are induced by the inclusions $p^n \subseteq M^n$, and the condition that $\psi(X_i)$ is the image of x_i in M/M^2 (so that ψ and θ are surjective). The following conditions are equivalent:

(i) $e(R) = e(R_p)$.

(ii) $\dim(R/I) = \dim(R) - r$, and the kernel of θ, $\ker(\theta)$, is nilpotent.

(iii) $\ker(\psi)$ is nilpotent.

Proof (iii) $\Rightarrow$ (i). The Krull dimension $\dim(G)$ is the same as $\dim(R)$; so if $\ker(\psi)$ is nilpotent, then

$$
\begin{aligned}
\dim(\overline{H}) = \dim(H) - r &= \dim(G) - r \\
&= \dim(R) - \dim(R/p) \\
&= \dim(R_p)
\end{aligned}
$$

(the last equality holds because R is formally equidimensional), i.e. $\ell(p) = h(p)$, and therefore (Corollary, Section 2) $e(R) = e(R_p)$.

(ii) $\Rightarrow$ (iii). As in the preceding proof, if (ii) holds then

$$
\begin{aligned}
\dim(\overline{H}) = \dim(\overline{G}) &= \dim(R/I) \\
&= \dim(R) - r
\end{aligned}
$$

Hence

$$\dim(H) = \dim(\bar{H}) + r = \dim(R) = \dim(G)$$

Now R, being formally equidimensional, is universally catenary; and Proj(G) is a divisor on the scheme obtained from R by blowing up M; it follows that every minimal prime ideal Q in G satisfies

$$\dim(G/Q) = \dim(G) = \dim(H)$$

and so every minimal prime divisor of $\ker(\psi)$ is a minimal prime ideal in H, i.e. it is of the form qH with q a minimal prime ideal in $\bar{H}$.

What we need to show is that <u>every</u> minimal prime qH in H contains $\ker(\psi)$. Let X be the ideal $(X_1,\ldots,X_r)H$. Then

$$\ker(\psi) \subseteq \pi^{-1}(\ker(\theta)) \subseteq \pi^{-1}(q) = qH + X$$

and so qH + X contains a minimal prime divisor of $\ker(\psi)$, say $qH + X \supseteq q^*H$ (see preceding paragraph). But then

$$q = (qH + X) \cap \bar{H} \supseteq q^*H \cap \bar{H} = q^*$$

whence

$$qH \supseteq q^*H \supseteq \ker(\psi)$$

as desired.

(i) ⇒ (ii). If $e(R) = e(R_p)$ then $\ell(p) = h(p)$ (Corollary, Section 2), and $\dim(R/I) = \dim(R) - r$ (Remark (2.8)).

We also have $I + p = M$ and

$$h(I + p) = \dim(R) = \dim(R/p) + \dim(R_p) = r + h(p) = \ell(I) + \ell(p)$$

So the fact that $\ker(\theta)$ is nilpotent is contained in Corollary (5.3) below.

LEMMA (5.2) Let (R,M) be a formally equidimensional local ring, and let I,J be ideals in R such that

$$h(I + J) = \ell(I) + \ell(J)$$

Then, with "$\overline{\quad}$" denoting "integral closure" (of an ideal) we have

$$\overline{IJ} = \overline{I} \cap \overline{J}$$

REMARK To understand better the condition $h(I + J) = \ell(I) + \ell(J)$, note that if I is integral over $\underline{x}R = (x_1,\ldots,x_s)R$, $(s = \ell(I))$, and J is integral over $\underline{z}R = (z_1,\ldots,z_r)R$, $(r = \ell(J))$, then $I + J$ is integral over $\underline{x}R + \underline{z}R$ and

$$\dim(R/\underline{x}R + \underline{z}R) = \dim(R/I + J) \leq \dim(R)-h(I + J) = \dim(R)-r-s$$

So $(x_1,\ldots,x_s, z_1,\ldots,z_r)$ is part of a system of parameters in R.

COROLLARY (5.3) With assumptions as in (5.2), the canonical surjective map of graded rings

$$\bigoplus_{n\geq 0} J^n/(J^{n+1} + IJ^n) \longrightarrow \bigoplus_{n\geq 0} J^n/(J^{n+1} + I \cap J^n)$$

has nilpotent kernel.

Proof of (5.3) We must show that if $x \in J^{n+1} + I \cap J^n$, then for some integer m,

$$x^m \in J^{mn+1} + IJ^{mn} = J^{mn}(I + J) \tag{5.4}$$

Now blowing up J^n gives the same result as blowing up J, and therefore $\ell(J^n) = \ell(J)$. So we have

$$h(I + J^n) = h(I + J) = \ell(I) + \ell(J) = \ell(I) + \ell(J^n)$$

whence, by (5.2), $I \cap J^n \subseteq \overline{IJ^n}$. Hence x is integral over $J^{n+1} + IJ^n$, i.e. there is an equation

$$x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0$$

where, for $i = 1,2,\ldots,m$, we have

$$a_i \in (J^{n+1} + IJ^n)^i = J^{ni}(I + J)^i$$

Since

$$x^{m-i} \in (J^{n+1} + I \cap J^n)^{m-i} \subseteq J^{nm-ni}$$

(5.4) follows immediately.

Proof of (5.2) Clearly $\overline{IJ} \subseteq \overline{I} \cap \overline{J}$; so we have to show that <u>if</u> $z \in R$ <u>is integral over both</u> I <u>and</u> J, <u>then</u> z <u>is integral over</u> IJ.

We may assume that R is complete (cf. [11, top of p. 796]). Now z is integral over an ideal K if and only if it is so modulo every minimal prime ideal p in R [11, p. 792, Lemma (1.1)]. Using the remark following (5.2) (assuming, as we may, that R/M is infinite) we see that the condition $h(I + J) = \ell(I) + \ell(J)$ continues to hold modulo each such p. Thus we may assume that R is a complete local domain.

Let $f: X \longrightarrow \mathrm{Spec}(R)$ be obtained by blowing up IJ and normalizing. Since R is complete, f is proper. Let $\nu_1,\ldots,\nu_t$ be the discrete valuations corresonding to the components of the divisor on X defined by the invertible ideal $IJ\mathcal{O}_X$. It will suffice to show, for each i = 1,2,...,t, that if $\nu_i(z) \geq \nu_i(I)$ and $\nu_i(z) \geq \nu_i(J)$ then $\nu_i(z) \geq \nu_i(IJ)$. This is clearly so if either $\nu_i(I) = 0$ or $\nu_i(J) = 0$, i.e. if $\nu_i(I + J) = 0$. But if $\nu_i(I + J) > 0$, then we would have

$$\dim(f^{-1}(\mathrm{Spec}(R/I + J))) = \dim(X)-1 = \dim(R)-1 = \dim(R) -1$$

and consequently

$$\begin{aligned}\ell(IJ)-1 = \dim(f^{-1}\{M\}) &\geq \dim(R)-1-\dim(R/I + J)\\ &= h(I + J)-1\\ &= \ell(I) + \ell(J)-1\end{aligned}$$

So, finally it is enough to prove:

PROPOSITION (5.5) For any two ideals I,J in a local ring (R,M), we have

$$\ell(IJ) < \ell(I) + \ell(J)$$

Proof For any ideal K in R, let $f_K: X_K \longrightarrow \mathrm{Spec}(R)$ be obtained by blowing up K, so that (by definition)

$$\dim(f_K^{-1}\{M\}) = \ell(K)-1$$

There is a natural R-morphism

$$X_{IJ} \longrightarrow X_I \otimes_R X_J$$

which is in fact a closed immersion. So $f_{IJ}^{-1}\{M\}$ is a closed subscheme of $f_I^{-1}\{M\} \otimes_{R/M} f_J^{-1}\{M\}$, and comparison of dimensions gives

$$\ell(IJ)-1 \leq (\ell(I)-1) + (\ell(J)-1)$$

QED

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ADDED IN PROOF (a) For an interesting treatment of Theorem 1 and related matters cf:

22. D. Schaub, Propriété topologique du gradué associé d'un anneau (S_r), *Comm. in Algebra 5* (1977), 1223-1239.

23. D. Schaub, Multiplicité et dépendance intégrale sur un idéal, *Colloque d' Algèbre Commutative,* Rennes, 1976, 1-20.

(b) In [3] Dade proves that for f: X ⟶ Spec(R) as in the first paragraph of Section 5 above, <u>every point</u> $x \in X$ <u>has multiplicity ≤ that of</u> R. A simpler proof is given in:

24. U. Orbanz, Multiplicities and Hilbert functions under blowing up, preprint.