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Differential Invariance of Multiplicity on Analytic Varieties

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Contents

Introduction 165
A. Preliminaries 168
B. The Homology Tangent Cone 172
C. Multiplicity of the Homology Tangent Cone 177
D. The e_i are Differential Invariants 179
E. Appendix: Algebraic Definition of the e_i 183

Introduction

A (complex-)*analytic subvariety* A of an open set $U \subseteq \mathbb{C}^n$ is a (closed) subset of U whose intersection with some open neighborhood of each point of U is the set of common zeros of a finite number of functions defined and holomorphic in that neighborhood. The *smooth points* of A are those points around which A is a complex manifold, so the local geometry of A is interesting only near a non-smooth, or *singular*, point P. A basic measure of singularity is the *multiplicity* of A at P, which is defined to be the local covering number of a generic complex projection from A to $\mathbb{C}^d$ ($d = \dim A$) (cf. (B.2)). This is the same as the multiplicity of the local ring of germs of holomorphic functions on A at P (cf. [22, Chap. VIII, § 10], [18, Chap. III] for the definition of algebraic multiplicity). The question considered here is "*How basic an invariant is multiplicity?*" i.e. how does it behave under topological or differentiable deformation of the pair (A, P)?

Our main result is the following (O = origin).

Theorem. *Let $A \ni O$ (resp. $B \ni O$) be an analytic subvariety of the open set $U \subseteq \mathbb{C}^n$ (resp. $V \subseteq \mathbb{C}^n$). Suppose there is a homeomorphism α from U to V such that $\alpha(A) = B$, $\alpha(O) = O$ and both α and α^{-1} (as functions from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$) have a derivative (= linear approximation* [19, p. 16]) *at O. Then A and B have the same multiplicity at O.*

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In short, "weakly" diffeomorphic germs have the same multiplicity.

The above assertion is wrong if we drop the differentiability assumption. It is known, for example, that if A and B are any two curves in $\mathbb{C}^3$ irreducible at O then a (non-differentiable) homeomorphism α exists. More concretely, let $u: \mathbb{C}^2 \to \mathbb{C}$ be a continuous map such for $t \in \mathbb{C}$,

$$u(t^2, t^3) = t \qquad (|t| \leqq 1)$$
$$u(t^2, t^3) = t/|t| \qquad (|t| > 1)$$

(u exists by the Tietze extension theorem). Let $\alpha: \mathbb{C}^3 \to \mathbb{C}^3$ be the homeomorphism given by

$$\alpha(x, y, z) = (x - u(y + x^2, z + x^3), y + x^2, z + x^3).$$

Let $U = D^3$, where D is the open unit disc $|t| < 1$ in $\mathbb{C}$, and let $V = \alpha(U)$. Then α maps the manifold $U \cap (y = z = 0)$ onto the curve $\{(0, x^2, x^3) | x \in D\}$ which has a cusp (of multiplicity 2) at O.

However, Zariski asked [Bull. AMS 77 (1971), p. 483], whether for *hypersurfaces* ($\dim A = \dim B = n-1$) multiplicity is indeed invariant under embedded homeomorphisms. This is still open.

Known positive results (special cases) for Zariski's question are:

1. for plane curves (i.e. $n=2$) [21].
2. when one of A, B is smooth at O (follows from [1, p. 114, Th. 3] and [16, p. 261, Prop.]).
3. when the homeomorphism and its inverse both have a derivative at the origin [7].

Our result generalizes Ephraim's by removing the "hypersurface" condition.

We sketch the proof of the Theorem. We may assume that A, B are irreducible (B.2.1). The idea is to use the (Zariski) tangent cones of A and B as intermediaries. The *tangent cone* $C(A, O)$ of an analytic variety $A \ni O$ at O is the set of all limits of secants from O through points on A approaching O (cf. (B.1) for the precise definition).

Now the derivative of α at O takes $C(A, O)$ onto $C(B, O)$, hence (cf. (A.8)) takes each irreducible component A_i of $C(A, O)$ to an irreducible component B_i of $C(B, O)$. The derivative being real linear, we can deduce from a result of Ephraim [8, p. 310, Th. 4.6] that A_i *and* B_i *have the same multiplicity at* O (cf. (B.2.3)).

The multiplicity of A at O is, in general, not equal to the multiplicity of $C(A, O)$ which is the sum of the multiplicities of the A_i (e.g. a cusp). However (cf. Sect. C) it is equal to a linear combination of the multiplicities of the A_i, with coefficients e_i definable through a set $S \subset \mathbb{R} \times \mathbb{C}^n$ which is used by Draper [5, p. 197] in his analytic intersection theory. (In fact, the idea of this proof has its origin in Draper's paper.) S is a geometric avatar of the "Rees ring" often used in algebraic multiplicity theory (cf. (E.4)).

The set $S = S_A \subset \mathbb{R} \times \mathbb{C}^n$ has a natural projection onto $\mathbb{R}$ with $1 \times A$ and $0 \times C(A, O)$ as fibres (over 1 and 0 resp.). Thus $S \cap (0 \times \mathbb{C}^n) = \bigcup_i (0 \times A_i)$. The coefficient e_i is then defined by the relation $[S] \cdot [0 \times \mathbb{C}^n] = \sum e_i [0 \times A_i]$ where [ ] denotes "fundamental class in Borel-Moore homology" and "$\cdot$" is the homology intersection product (cf. (B.4.2)). If we let f_i be the corresponding coefficients for $C(B,O)$, then it follows from the existence of a fibre-preserving homeomorphism between S_A and S_B that $e_i = f_i$ (cf. Sect. D). Thus with mult = multiplicity)

$$\text{mult of } A = \sum e_i(\text{mult of } A_i) = \sum f_i(\text{mult of } B_i) = \text{mult of } B.$$

In Sect. E, we give an algebraic characterization (not used elsewhere) of the e_i: e_i is the length of the artin local ring G_{p_i}, where G is the graded ring $\bigoplus_{q \geq 0} m^q/m^{q+1}$ (m = maximal ideal of the local ring $\mathcal{O}_{A,O}$ of germs of analytic functions) – so that $C(A,O) = \operatorname{Spec}(G)$ – and p_i is the minimal prime ideal in G corresponding to A_i.

The tools used are the Borel-Moore homology [4] and its intersection theory developed in [3]. Hence all homology groups in this paper are Borel-Moore homology groups. In this regard, we would like to thank Professor Reinhard Schultz for helpful discussions.

We conclude this introduction with remarks on the notion of "differential equivalence".

Let (A,O), (B,O) be two germs of analytic spaces (cf. [11, p. 150] for the definition). We say that (A,O) and (B,O) are *differentially equivalent* if one of the following three equivalent conditions holds. (For simplicity we write $\mathbb{C}^k$, $\mathbb{R}^k$ for suitable open sets ($\ni O$) in $\mathbb{C}^k$, $\mathbb{R}^k$, and write A, B for suitable representatives of (A,O), (B,O) respectively):

1. *There exist analytic embeddings* $(A,O) \to (\mathbb{C}^n, O)$, $(B,O) \to (\mathbb{C}^n, O)$ *and a homeomorphism* α *with the same properties as in the Theorem.*

2. *There exist analytic embeddings* $i: (A,O) \to (\mathbb{C}^p, O)$, $j: (B,O) \to (\mathbb{C}^q, O)$ *and a homeomorphism* $h: A \to B$ *such that* i *and* $j \circ h$ *are "diffeo-equivalent" embeddings, i.e. there exist a pair of nonnegative integers* (r,s) *and a homeomorphism* $H: \mathbb{R}^{2p+r} \to \mathbb{R}^{2q+s}$ *such that both* H *and* H^{-1} *have a derivative at* $O \times O$ *and* $H \circ g_1 \circ i = g_2 \circ j \circ h$, *where* $g_1: \mathbb{R}^{2p} \to \mathbb{R}^{2p} \times \mathbb{R}^r$ (resp. $g_2: \mathbb{R}^{2q} \to \mathbb{R}^{2q} \times \mathbb{R}^s$) *is the inclusion* $x \to x \times O$ [12, p. 128].

A function $f: X \to \mathbb{R}^n$ (where (X,O) is an analytic germ) is *differentiable at* O if for some embedding $(X,O) \to (\mathbb{C}^N, O)$, f is the restriction (to X) of a function from $\mathbb{C}^N$ to $\mathbb{R}^n$ having a derivative at O. A map between analytic germs $f: (X,O) \to (Y,O)$ is *differentiable* if its composition with some embedding $(Y,O) \to (\mathbb{C}^M, O)$ is differentiable at O. (This property does not depend on the choice of the embeddings.) Condition 3 is:

3. *There exist differentiable maps* $f: (A,O) \to (B,O)$ *and* $g: (B,O) \to (A,O)$ *such that* $f \circ g$ *and* $g \circ f$ *are identity maps.*

[The implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial. For $3 \Rightarrow 1$, choose embeddings $A \to \mathbb{C}^N$, $B \to \mathbb{C}^M$; extend f (resp. g) to a map $F: \mathbb{C}^N \to \mathbb{C}^M$ (resp. $G: \mathbb{C}^M \to \mathbb{C}^N$) differentiable at O; and define a homeomorphism $\alpha: \mathbb{C}^N \times \mathbb{C}^M \to \mathbb{C}^N \times \mathbb{C}^M$ by

$$\alpha(x,y) = (x - G(y + F(x)),\ y + F(x)).]$$

The Theorem then states that *differentially equivalent germs have the same multiplicity.*

Example. In [2, § 4], Bloom shows that the curves $A \subseteq \mathbb{C}^2$, $B \subseteq \mathbb{C}^3$ given by

$$A = \{t^3, t^5) | t \in \mathbb{C}\},$$
$$B = \{(t^3, t^5, t^7) | t \in \mathbb{C}\}$$

are differentially equivalent at their respective origins (in fact t^7 is a C^1 function everywhere on A). The multiplicities at O are of course the same (namely 3). Incidentally, the Hilbert-Samuel polynomials are *different*, namely $3n-3$ for A at O and $3n-2$ for B at O.

A. Preliminaries

This section gives definitions, fixes notations and lists some properties of homology groups and intersection products. At the end we prove a useful Lemma (A.8). The main references are [4], [3].

(A.1) $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{C}$ denote as usual the integers, strictly positive integers, real numbers, complex numbers respectively. All spaces will be subsets of Euclidean spaces with the usual metric topology and hence will be paracompact. Unless otherwise indicated, *analytic* means *complex analytic* and *component* means *irreducible component* (of an analytic variety) (cf. [11] for these and other terminologies in Analytic Geometry).

(A.2) Recall that a *family of supports* Φ on a topological space X is a family of closed subsets of X having the following properties: If $A, B \in \Phi$, then $A \cup B \in \Phi$; if $A \in \Phi$ and if B is closed in A, then $B \in \Phi$.

We write $H_i^\Phi(X)$ (resp. $H_\Phi^i(X)$) to denote the i-th homology (resp. cohomology) group of X with coefficients in the constant sheaf $\mathbb{Z}$ and supports in Φ, as defined in [4] (resp. [9]). We omit i if its value is clear from the context.

In the preceding notations, we omit Φ if Φ is the family of all closed subsets of X, and we replace Φ by F (a closed subset of X) if Φ is the family of all closed subsets of F.

(A.2.1) In case F, $X \subset \mathbb{R}^n$ and $F \cap X$ is closed in X we write, by abuse of notation, $H^F(X)$ to mean $H^{F \cap X}(X)$.

(A.3) Let $f: X \to Y$ be a continuous map and Φ (resp. Ψ) a family of supports on X (resp. Y). If for every $\phi \in \Phi$, $f(\phi) \in \Psi$ (in symbol: $f(\Phi) \subset \Psi$) and the restriction of f to ϕ is proper, then f induces a homomorphism $f^{\Phi, \Psi}$ or f_* from $H_i^\Phi(X)$ into $H_i^\Psi(Y)$ (cf. [4, p. 143, Sect. 3] for the definition of f_* and its properties). As for *cohomology*, if $f^{-1}(\Psi) \subset \Phi$, then f induces a homomorphism $f^*: H_\Psi^i(Y) \to H_\Phi^i(X)$ [9, p. 199, 4.16].

If $\Phi \subset \Phi'$ are families of supports on X, the map $\mathrm{id}^{\Phi, \Phi'}: H_i^\Phi(X) \to H_i^{\Phi'}(X)$ is called "*enlargement of supports*". (It can be checked that this is equivalent to the definition given in [3, p. 464].)

For U open in X, there is a *restriction map* $j^{X, U}$: $H_i^\Phi(X) \to H_i^{\Phi \cap U}(U)$ where $\Phi \cap U = \{\phi \cap U | \phi \in \Phi\}$ is a family of supports on U [3, p. 464].

(A.3.1) f_* *is functorial:*
If $g: Y \to Z$ is continuous and $g^{\Psi,\Theta}$ is defined, then

$$(g \circ f)^{\Phi,\Theta} = g^{\Psi,\Theta} \circ f^{\Phi,\Psi}.$$

(A.3.2) f_* *is compatible with restriction maps:*

Let V be open in Y and $U = f^{-1}(V)$ (open in X). If $f^{\Phi,\Psi}$ is defined then so is $f^{\Phi \cap U, \Psi \cap V}: H^{\Phi \cap U}(U) \to H^{\Psi \cap V}(V)$, and we have

$$f^{\Phi \cap U, \Psi \cap V} \circ j^{X,U} = j^{Y,V} \circ f^{\Phi,\Psi}.$$

(A.3.3) *Inclusion maps* $i_{F,X}$ [4, p. 145, Th. 3.5(b)]:
$i_{F,X} = i^{\Phi \cap F, \Phi}: H^{\Phi \cap F}(F) \to H^{\Phi}(X)$ is the homomorphism induced by the inclusion $i: F \to X$ (A.3). We shall use $i_{F,X}$, $i^{\Phi \cap F, \Phi}$ interchangeably. If Φ is a family of supports on F, then $i_{F,X}$ is an isomorphism. In particular ($\Phi = E$, a closed subset of F; and $\Phi = F$)

$$H^E(F) \xrightarrow{\sim} H^E(X), \quad \text{and} \quad H(F) \xrightarrow{\sim} H^F(X).$$

(A.3.4) *Homology exact sequence* [4, p. 146, Th. 3.8]:
Let U be open in X and $F = X - U$. Then there exists an exact sequence:

$$\dots \to H_q(F) \xrightarrow[i_{F,X}]{} H_q(X) \xrightarrow[j^{X,U}]{} H_q(U) \to H_{q-1}(F) \to \dots$$

(A.4) *Fundamental classes:*

See [3, p. 469, 2.2] for the definition of fundamental classes. For a space X, $[X]$ will always denote a fundamental class of X (if it exists).

(A.4.1) For a d-dimensional analytic variety A, $H_k(A) = 0$ for $k > 2d$, and $H_{2d}(A) \simeq \Pi_\lambda \mathbb{Z}_\lambda$, one $\mathbb{Z}_\lambda = \mathbb{Z}$ for each irreducible component A_λ of A [3, pp. 476, 482]. In this case we write $[A] \in H_{2d}(A)$ for *the* fundamental class, i.e. the one which induces the canonical orientation at all smooth points. When A is irreducible, $[A]$ generates $H_{2d}(A) \simeq \mathbb{Z}$.

(A.4.2) Let M be a real n-dimensional oriented connected topological manifold. Then $H_{n+1}(M) = 0$ and $H_n(M) \simeq \mathbb{Z}$ [3, p. 467, 1.11]. (In particular, for a connected open subset U of $\mathbb{R}^n$, we have $H_n(U) \simeq \mathbb{Z}$.) We write $[M]$ for the generator of $H_n(M)$ which corresponds to the orientation.

(A.4.3) Let X be a space with fundamental class $[X]$. Then the *restriction* $j^{X,U}[X]$ of $[X]$ to an open subset U, is a fundamental class of U. If X is an analytic variety, then $j^{X,U}[X]$ is *the* fundamental class $[U]$. In particular, if M denotes the set of smooth points of an analytic variety A, then the restriction of $[A]$ to M gives the fundamental class $[M]$ [3, 3.2 and 2.3].

If X is closed in some space Y, we let $[X]^Y = i_{X,Y}[X]$ where $i_{X,Y}: H(X) \xrightarrow{\sim} H^X(Y)$. Note that if Y is closed in Z, then by (A.3.1)

$$[X]^Z = i_{X,Z}[X] = i_{Y,Z} \circ i_{X,Y}[X] = i_{Y,Z}[X]^Y.$$

(A.4.4) As in (A.2.1), $[F]^X$ will mean $[F \cap X]^X$ (notations as in (A.2.1)).

(A.5) *Poincaré duality:*

We shall write "$\cap$" for cap product [3, p. 504, Appendix]. (N.B. the cap product $H_i^{\Phi}(X)\otimes H_{\Psi}^j(X)\xrightarrow{\cap} H_{i-j}^{\Phi\cap\Psi}(X)$ is defined only if Φ and $\Phi\cap\Psi$ are paracompactifying. This is no restriction for us since we consider only metric spaces.)

For an n-dimensional oriented connected manifold X there is an isomorphism $\Delta: H_i^{\Phi}(X)\simeq H_{\Phi}^{n-i}(X)$ which is compatible with restrictions to open subsets [3, p. 467, 1.11]. Moreover, the isomorphism is characterized by $\Delta^{-1}\xi=[X]\cap\xi$ for $\xi\in H_{\Phi}^{n-i}(X)$ [3, p. 510, Th. 7.9 and p. 467–8, 1.11].

(A.6) *Intersection Product:*

Let X be an n-dimensional oriented manifold. For $a\in H_i^{\Phi}(X)$, $b\in H_j^{\Psi}(X)$ their intersection product, denoted by $a\cdot b$, is defined to be

$$a\cdot b=a\cap\Delta b\in H_{i+j-n}^{\Phi\cap\Psi}(X).$$

(A.6.1) The intersection product is bilinear, associative, anticommutative and compatible with both *enlargement of supports* and *restriction to open subsets.* (The last assertion follows from the definition and the corresponding properties for cap product and Δ, cf. (A.5) and [3, p. 565, Th. 7.2].)

(A.6.2) If $i+j=n$ and $\Phi\cap\Psi=$ a point Q, then $a\cdot b\in H_0^Q(X)\cong H_0(Q)\cong\mathbb{Z}$. (The last isomorphism comes out in a straightforward way from the definition of H_0.) We shall call the absolute value of $a\cdot b$ the *intersection number* of a and b, and denote it by $a\circ b$.

(A.6.3) *Criterion of unit multiplicity* [3, p. 484, 4.8].

Let A,B be two analytic subvarieties of an open subset U of $\mathbb{C}^n$. Suppose A (resp. B) is pure i-(resp. j-)dimensional and $A\cap B=C$, where C is irreducible and $\dim C=i+j-n$ (i.e. *A,B intersect properly at C*). Assume moreover that C contains a point which is smooth in both A and B, and at which A intersects B transversally. Then the conclusion is $[A]^U\cdot[B]^U=[C]^U$. (The special case when A,B are transversal complex linear spaces is useful for our purpose.)

(A.6.4) Let $V\subset W$ be connected oriented open subsets of $\mathbb{R}^s$ and $\mathbb{R}^t$ respectively such that V is closed in W. Let $i: V\to W$ be the inclusion map. Let $Y\subset W$ be either an analytic subvariety of W or the intersection of W with an oriented real-linear space, and let $[Y]$ be its fundamental class ((A.4.1), (A.4.2)).

Suppose that V is transversal to Y at all points of $V\cap Y_R$ where $Y_R\subset Y$ is an open subset consisting of smooth points. Assume also that $\dim_{\mathbb{R}}(Y-Y_R)<\dim_{\mathbb{R}} Y$ and that $\dim_{\mathbb{R}}(Y-Y_R)\cap V<s-t+\dim_{\mathbb{R}} Y$.

Then *$X=Y\cap V$ has a unique fundamental class $[X]\in H(X)$ such that*

$$\Delta i_{X,V}[X]=i^*\Delta i_{Y,W}[Y],\quad \text{where } i^*: H_Y(W)\to H_X(V);$$

or equivalently, by (A.4.3)

$$\Delta[X]^V=i^*\Delta[Y]^W.$$

Proof. This is a special case of [3, p. 475, 2.15].

(A.7) Let U (resp. U') be a connected oriented open set in $\mathbb{R}^n$ (resp. $\mathbb{R}^{n'}$), Φ, Ψ (resp. Φ',Ψ') families of supports on U (resp. U') and let $f: U\to U'$ be such that $f^{\Phi,\Phi'}$, $f^{\Phi\cap\Psi,\Phi'\cap\Psi'}$ and $f_{\Psi',\Psi}$ are defined, i.e. $f(\Phi)\subset\Phi'$, $f|_\phi$ is proper for $\phi\in\Phi$; $f(\Phi\cap\Psi)\subset\Phi'\cap\Psi'$; $f^{-1}(\Psi')\subset\Psi$. Under these conditions we have the *projection formula* [3, p. 466, 1.9]): in $H_{s-t}^{\Psi\cap\Psi'}(Y)$,

$$f_*(a\cap f^*b)=f_*a\cap b \qquad [a\in H_s^{\Phi}(X);\, b\in H_{\Psi'}^t(Y)].$$

(A.7.1) *Variant of projection formula:*
Let $\xi\in H_i^{\Phi}(U)$, $\eta\in H_j^{\Psi}(U)$. If $\eta'\in H_{j+n'-n}^{\Psi'}(U')$ is such that $\Delta\eta=f_{\Psi',\Psi}^{n-j}\Delta\eta'$, then

$$f_{i+j-n}^{\Phi\cap\Psi,\Phi'\cap\Psi'}(\xi\cdot\eta)=(f_i^{\Phi,\Phi'}\xi)\cdot\eta'.$$

Remark. In singular homology η is called the *transfer* $f_!\eta'$ of η', cf. [6, p. 310].

Proof. By Definition (A.6) and the hypothesis we have:

$$f_{i+j-n}(\xi\cdot\eta)=f_{i+j-n}(\xi\cap\Delta\eta)=f_{i+j-n}(\xi\cap f^{n-j}\Delta\eta').$$

By the projection formula the last term is:

$$f_i\xi\cap\Delta\eta'=f_i\xi\cdot\eta'. \quad \text{q.e.d.}$$

(A.7.2) We apply (A.7.1) to the situation in (A.6.4), i.e. take $U=V$, $U'=W$, $\Psi=X$, $\Psi'=Y$, $f=i\colon V\to W$ and Φ,Φ' such that $i^{\Phi,\Phi'}$ is defined ($\Rightarrow i^{\Phi\cap\Psi,\Phi'\cap\Psi'}$ is defined). By (A.6.4), $\eta=[X]^V$ and $\eta'=[Y]^W$ satisfy the condition in (A.7.1), hence:

$$i^{\Phi\cap X,\Phi'\cap Y}(\xi\cdot[X]^V)=i^{\Phi,\Phi'}(\xi)\cdot[Y]^W \qquad \text{for all } \xi\in H^{\Phi}(V).$$

In particular, letting $\Phi=\Phi'=V$ and $\xi=V$ (A.4.2), we have

$$i^{X,X}([V]\cdot[X]^V)=(i^{V,V}[V])\cdot[Y]^W=[V]^W\cdot[Y]^W.$$

But $[V]\cdot[X]^V\underset{\text{def}}{=}[V]\cap\Delta[X]^V=\Delta^{-1}\Delta[X]^V=[X]^V$ by (A.5). Thus by (A.6.3):

$$[X]^W\underset{(\text{A.4.3})}{=}i^{X,X}[X]^V=[V]^W\cdot[Y]^W.$$

(A.7.3) *Topological Invariance of Intersection*.
Let U,U' be as in (A.7). Assume $f\colon U\to U'$ is a homeomorphism with $\Phi'=f(\Phi)$, $\Psi'=f(\Psi)$. Write $f'_*=f_i^{\Phi,\Phi'}$, $f_*=f_j^{\Psi,\Psi'}$ and $\tilde{f}_*=f_{i+j-n}^{\Phi\cap\Psi,\Phi'\cap\Psi'}$. Then for $\xi'\in H_i^{\Phi}(U)$, $\xi\in H_j^{\Psi}(U)$ we have: $\tilde{f}_*(\xi'\cdot\xi)=\pm f'_*\xi'\cdot f_*\xi$.

Proof. By (A.7.1) it suffices to check:

$$\Delta\xi=\pm f^*\Delta f_*\xi.$$

Using (A.5) and (A.7) we have

$$f_*\Delta^{-1}(f^*\Delta\eta)=f_*([U]\cap f^*\Delta\eta)=f_*[U]\cap\Delta\eta \qquad \text{for } \eta\in H^{\Phi'}(U').$$

Since f_* is an isomorphism, $f_*[U]=\pm[U']$ by (A.4.2). By (A.5) again, we conclude that

$$f_*\Delta^{-1}f^*\Delta\eta=\pm\Delta^{-1}\Delta\eta=\pm\eta.$$

In particular, if $\eta = f_* \xi$ we get (since f_* is an isomorphism)

$$\Delta^{-1} f^* \Delta f_* \xi = \pm \xi$$

$$\text{i.e.} \quad f^* \Delta f_* = \pm \Delta \xi. \quad \text{q.e.d.}$$

(A.8) Lemma. *Let $f: A \to B$ be a homeomorphism of analytic spaces. Then for every component A_* of A, $f(A_*)$ is a component of B with the same dimension as A_*.*

Proof. Let $d = \dim A_*$ and $M = A_* - \text{Sing}(A)$ be the set of all smooth points of A which lie in A_*. M is a connected manifold which is open in A and with closure (in A) $\overline{M} = A_*$. Hence $f(M)$ is an open connected subset of B which is locally homeomorphic to $\mathbb{R}^{2d}$ and, regarding $f(M)$ as an open subvariety of B, $f(M) - \text{Sing}(f(M))$ is a pure d-dimensional complex manifold. It follows that $f(M)$ is also pure d-dimensional.

Now homology groups being topological invariants (A.3.1), we have $H_{2d}(f(M)) \simeq H_{2d}(M)$ which by (A.4.2) is isomorphic to $\mathbb{Z}$. By [3, p. 482, Lemma 4.3] $f(M)$ is irreducible, so is contained in one component of B, say B_*. This implies $f(A_*) = f(\overline{M}) \subset \overline{f(M)} \subset \bar{B}_* = B_*$. The same argument applied to f^{-1} and B_* gives that $f^{-1}(B_*)$ is contained in a component of A, say A_{**}.

Together we have: $A_* = f^{-1} f(A_*) \subseteq f^{-1}(B_*) \subseteq A_{**}$. But components are maximal irreducible analytic subspaces, hence $A_* = A_{**}$. In particular: $f(A_*) = B_*$. Also $\overline{f(M)} = B_*$, therefore B_* is d-dimensional. q.e.d.

B. The Homology Tangent Cone

(B.1) As mentioned in the introduction, the basic idea of the proof is to relate the analytic variety A to its (Zariski) tangent cone through a map $S \to \mathbb{R}$ which has A and the tangent cone as two of its fibres. Let $d = \dim A$.

Definition [20, p. 210, Def. 1G]: Let A be an analytic variety and $A \ni O$. The *tangent cone* of A at O, denoted by $C(A, O)$, is the set of all vectors v with the property that there are sequences of points $q_i \in A$ and of *complex* numbers a_i such that $q_i \to O$ and $a_i q_i \to v$, i.e. the direction of v (if $v \neq O$) is a limit of directions of secants from O to points of A converging to O.

(B.1.1) We may restrict the a_i in the above definition to be real and positive; this does not change $C(A, O)$ [20, p. 218, Remark 3E].

(B.1.2) If A is pure d-dimensional, then $C(A, O)$ is a pure d-dimensional algebraic variety which is a *cone* (i.e. $t \in \mathbb{C}$, $x \in C(A, O)$ implies $t x \in C(A, O)$) [20, p. 211, Lemma 1L and p. 214, Th. 2E].

(B.1.3) With α as in the Theorem, the derivative of α at O [19, p. 16], denoted by α_0, is a real-linear map inducing a homeomorphism between $C(A, O)$ and $C(B, O)$. The proof uses (B.1.1) but is otherwise straightforward, cf. [7, p. 802].

(B.2) Definition. The *multiplicity* of a d-dimensional analytic variety $A \subset \mathbb{C}^n$ at a point $O \in A$, *denoted by* $\mu(A, O)$, is defined to be the local covering number of the projection of a small neighborhood around O in A to a generic d-dimensional complex linear subspace (cf. [20, p. 233, Def. 7J]).

In other words, for "most" linear projections $\pi: \mathbb{C}^n \to \mathbb{C}^d$, π induces a $\mu(A,O)$-sheeted unramified covering $V - \pi^{-1}(D) \to \pi(V) - D$ where $V \subset A$ is some neighborhood of O and D is a nowhere dense analytic subvariety of the open set $\pi(V) \subset \mathbb{C}^d$, and where $\mu(A,O)$ does not depend on π.

Remark 1. $\mu(A,O)$ can be shown to be equal to the algebraic multiplicity of the analytic local ring of A at O (i.e. the local ring of germs at O of analytic functions on A) [5, p. 198, Th. 6.5], [17, p. 123, (A.15)]. Also we can replace "generic projection" in the above definition by "*transversal projection* π" which means that O is an isolated point of $\pi^{-1}\pi(O) \cap C(A,O)$ (cf. [5, p. 196, Th. 6.3], [20, p. 234, Th. 7P]).

Remark 2. If $A^1, A^2, \ldots$ are those components of A which have dimension d ($d = \dim A$) and pass through O, then $\mu(A,O) = \sum_i \mu(A^i, O)$. To see this, just use any projection which is sufficiently generic for A and for all the A_i (for example, any projection which is transversal for A (Remark 1) and hence for the A_i). An algebraic proof can also be given by using [18, p. 76, (23.5)].

(B.2.1) *To prove the Theorem, we can and shall assume that A and B are irreducible.* Let A^i be as in Remark 2. By (A.8) α maps each A^i onto some component of B, call it B^i. Also $\dim B = d$ and the B^i are all the irreducible d-dimensional components of B containing O. By Remark 2, it suffices to prove the Theorem for $\alpha: A^i \to B^i$.

(B.2.2) It is clear from the definition that $\mu(A,O) = \mu(A \cap U, O)$ where $U \subset \mathbb{C}^n$ is any open set in $\mathbb{C}^n$ containing O. Algebraically this follows from the fact that A and $A \cap U$ have the same local ring at O.

For simplicity, we omit from now on the reference point O. Thus $\mu(A) = \mu(A,O)$, $C(A) = C(A,O)$ etc.

(B.2.3) Let A^* be a component of $C(A)$. By (B.1.3) $\alpha_0(A^*)$ is an irreducible component, say B^*, of $C(B)$.

Claim. $\mu(A^*) = \mu(B^*)$.

Proof. We use a structure theorem proved by R. Ephraim [8, p. 310, Th. 4.6]: Any germ of analytic variety has a *decomposition*, i.e. it is complex-analytically isomorphic to a (finite) product of analytic germs each of which can't be further decomposed. If X, Y are two irreducible analytic germs, and X is C^∞-isomorphic to Y, then the decomposition of X and Y are the same up to permutations and complex conjugations. To be precise, if $X = X_1 \times \ldots \times X_k$, $Y = Y_1 \times \ldots \times Y_h$ are decompositions of X and Y (into indecomposables), then $k = h$ and, after a permutation of $Y_1 \ldots Y_k$, either $X_i \simeq Y_i$ or $X_i \simeq \overline{Y_i}$. ("$\simeq$" means complex-analytically isomorphic, $\overline{Y_i}$ = complex conjugate of Y_i.)

Applying the above to $\alpha_0: A^* \to B^*$, we have $A^* = A_1^* \times \ldots \times A_k^*$, $B^* = B_1^* \times \ldots \times B_k^*$ and A_i^* is analytically isomorphic either to B_i^* or to the complex conjugate of B_i^*. In either case, A_i^* and B_i^* have isomorphic (as $\mathbb{R}$-algebras) analytic local rings at the origin [8, p. 301, Remark 2.4], hence have the same multiplicity (B.2, Remark 1). But arguing as in [18, (42.6)], we see that

$$\mu(A^*)=\prod_i \mu(A_i^*), \qquad \mu(B^*)=\prod_i \mu(B_i^*).$$

Hence $\mu(A^*)=\mu(B^*)$. q.e.d.

Remark. The assertions $\mu(\bar{A})=\mu(A)$ and $\mu(A_1\times A_2)=\mu(A_1)\mu(A_2)$ in the above proof can also be proved topologically by observing that if π (resp. π_1, π_2) is transversal to A (resp. A_1, A_2) then the conjugate of π (resp. $\pi_1\times\pi_2$) is transversal to $\bar{A}$ (resp. $A_1\times A_2$). The results then follow from (B.2, Remark 1).

(B.3) As in (B.2.1), A is an irreducible analytic subvariety of the open set $U \subseteq \mathbb{C}^n$, and $O\in A$. We now define the sets S, $\bar{S}$ from A (resp. T, $\bar{T}$ from B) which provide the foundations for the proof of the Theorem. Let $\mathbb{C}^*=\mathbb{C}-\{0\}$, $\mathbb{R}^*=\mathbb{R}-\{0\}$. Consider the map ψ from $\mathbb{C}\times\mathbb{C}^n$ into itself defined by:

$$\psi(t, z_1, \dots, z_n)=(t, t\,z_1, \dots, t\,z_n).$$

Then $\mathscr{U}=\psi^{-1}(\mathbb{C}\times U)$ is an open set in $\mathbb{C}\times\mathbb{C}^n$. (The corresponding set $\psi^{-1}(\mathbb{C}\times V)$ for V will be denoted by $\mathscr{V}$.) Since $\psi^{-1}(t\times U)=\left(t\times\frac{1}{t}U\right)$ for $t\neq 0$, we see that $\mathscr{U}=\bigcup_{t\in\mathbb{C}^*}\left(t\times\frac{1}{t}U\right)\cup(0\times\mathbb{C}^n)$.

To define S (an analytic subvariety of $\mathscr{U}$) we consider $\psi^{-1}(\mathbb{C}\times A)$. ψ being analytic, $\psi^{-1}(\mathbb{C}\times A)$ is an analytic subvariety of $\mathscr{U}$, of dimension $<n+1$, and one of its components is $0\times\mathbb{C}^n$.

(B.3.1) **Definition.** S is the union of the other components.

It is clear that $S-(0\times\mathbb{C}^n)=\psi^{-1}(\mathbb{C}^*\times A)$. Also note that ψ: $\mathscr{U}-(0\times\mathbb{C}^n)\to\mathbb{C}^*\times U$ being an analytic isomorphism, $((t,z)\to(t,(1/t)\,z)$ is the inverse), the smooth points of $S-(0\times\mathbb{C}^n)$ are mapped (by ψ) to the smooth points of $\mathbb{C}^*\times A$, i.e. the points in $\mathbb{C}^*\times A^-$ where A^- denotes the set of smooth points in A. Now using the fact that the components of an analytic variety W are the closures (in W) of the connected components of the smooth part of W [20, p. 74, Th. 1G], we can conclude the following (B.3.2) and (B.3.3):

(B.3.2) $S-(0\times\mathbb{C}^n)=\psi^{-1}(\mathbb{C}^*\times A)$ is an irreducible analytic variety.

(B.3.3) $S\cap(0\times\mathbb{C}^n)=0\times C(A)$ (in particular, $0\times C(A)$ is closed in S).

Proofs. It is obvious that, as sets, $\psi^{-1}(\mathbb{C}\times A)=(0\times\mathbb{C}^n)\cup(S-(0\times\mathbb{C}^n))$.

Since A is assumed to be irreducible, A^-, and hence $\psi^{-1}(\mathbb{C}^*\times A^-)$, is connected; (B.3.2) follows. By [20, p. 79, Th. 2J] and (B.3.1), $(0\times\mathbb{C}^n)-S$ and $\psi^{-1}(\mathbb{C}^*\times A^-)$ are all the connected components of the smooth part of $\psi^{-1}(\mathbb{C}\times A)$. It follows that S is *irreducible* and is the closure (in $\psi^{-1}(\mathbb{C}\times A)$) of $\psi^{-1}(\mathbb{C}^*\times A^-)$. Since

$$S\supset S-(0\times\mathbb{C}^n)=\psi^{-1}(\mathbb{C}^*\times A)\supset\psi^{-1}(\mathbb{C}^*\times A^-),$$

S is also the closure of $S-(0\times\mathbb{C}^n)=\bigcup_{t\neq 0}\left(t,\frac{1}{t}A\right)$. From the definition of the tangent cone (B.1), we see then that $S\cap(0\times\mathbb{C}^n)=0\times C(A)$.

(B.3.4) Next we consider a real section $\bar{S}$ of S. This $\bar{S}$ will play an essential role in section D (cf. (D.1)).

Definition. $\bar{\mathscr{U}}=\mathscr{U}\cap(\mathbb{R}\times\mathbb{C}^n)$, $\bar{S}=S\cap(\mathbb{R}\times\mathbb{C}^n)=S\cap\bar{\mathscr{U}}$.

It is readily seen that $\bar{\mathscr{U}}=\bigcup_{t\in\mathbb{R}^*}\left(t\times\frac{1}{t}U\right)\cup(0\times\mathbb{C}^n)$ is a connected open set in $\mathbb{R}\times\mathbb{C}^n$, $\bar{\mathscr{U}}$ is closed in $\mathscr{U}$, $\bar{S}$ is a real-analytic variety which is closed in $\bar{\mathscr{U}}$, $\bar{S}\cap(t\times\mathbb{C}^n)=t\times\frac{1}{t}A$ $(t\in\mathbb{R}^*)$ and $\bar{S}\cap(0\times\mathbb{C}^n)=0\times C(A)$. The last statement follows from (B.3.3).

Fix an orientation in $\mathbb{R}\times\mathbb{C}^n$. Consider the inclusion $i:\bar{\mathscr{U}}\to\mathscr{U}$ and the analytic variety $S\subset\mathscr{U}$. Note that $\dim_{\mathbb{R}}\mathscr{U}=2n+2$, $\dim_{\mathbb{R}}\bar{\mathscr{U}}=2n+1$, $\dim_{\mathbb{R}}S=2d+2$, $\dim_{\mathbb{R}}\bar{S}=2d+1$. It can be checked easily that $\bar{\mathscr{U}}$ is transversal to S at smooth points of S of the form (t,x) with $t\in\mathbb{R}^*$ (cf. (C.2.2, Remark)). Also $\dim_{\mathbb{R}}(S-\psi^{-1}(\mathbb{C}^*\times A^-))\cap\bar{\mathscr{U}}=2d$ (B.3.3). Thus we can apply (A.6.4) (with $Y=S$, $Y_{\mathbb{R}}=\psi^{-1}(\mathbb{C}^*\times A^-)$) to conclude that $\bar{S}=S\cap\bar{\mathscr{U}}$ has a fundamental class $[\bar{S}]\in H_{2d+1}(\bar{S})$ and, by (A.7.2) that

(B.3.5) $$[\bar{S}]^{\mathscr{U}}=[\bar{\mathscr{U}}]^{\mathscr{U}}\cdot[\bar{S}]^{\mathscr{U}}.$$

(B.4) Through S (or $\bar{S}$) we can deform A to its tangent cone $C(A)$. However, to avoid loss of information we need something like the "algebraic" tangent cone $\sum e_i A_i$, $e_i\in\mathbb{Z}^+$, where A_i are the components of $C(A)$ (cf. [5, p. 198]). Actually, we work with the "homology tangent cone" $C_H(A)=C_H(A)^{\mathscr{U}}$ defined as the homology class:

$$C_H(A)=[S]^{\mathscr{U}}\cdot[0\times\mathbb{C}^n]^{\mathscr{U}}\in H_{2d}^{0\times C(A)}(\mathscr{U}).$$

Consider $\bigoplus_i H_{2d}(0\times A_i)\xrightarrow{\sim}H_{2d}(0\times C(A))\xrightarrow{\sim}H_{2d}^{0\times C(A)}(\mathscr{U})$. The first map is an isomorphism induced by the direct sum of inclusion maps: $\bigoplus i^{0\times A_i,\,0\times C(A)}$ [cf. 3, p. 482, Lemma 4.3]. The second map is also an isomorphism by (A.3.3).

(B.4.1) Let $[\![0\times A_i]\!]^{\mathscr{U}}$ denote the image of $[0\times A_i]\in H(0\times A_i)$ in $H_{2d}^{0\times C(A)}(\mathscr{U})$. By (A.3.1), (A.3), (A.4.3) we see:

$$[\![0\times A_i]\!]^{\mathscr{U}}=\mathrm{id}^{0\times A_i,\,0\times C(A)}\circ i_{0\times A_i,\,\mathscr{U}}[0\times A_i]=\mathrm{id}^{0\times A_i,\,0\times C(A)}[0\times A_i]^{\mathscr{U}}.$$

(B.4.2) **Definition.** e_i is the integer uniquely determined by

$$C_H(A)=[S]^{\mathscr{U}}\cdot[0\times\mathbb{C}^n]^{\mathscr{U}}=\sum e_i[\![0\times A_i]\!]^{\mathscr{U}}.$$

Thus e_i is the *intersection number* (in $\mathscr{U}$) [3, p. 483]

$$e_i=i(S\cdot(0\times\mathbb{C}^n),A_i).$$

e_i does not change when U is replaced by a smaller open set (A.6.1).

(B.4.3) Similarly, from $B \subset V$ we can define $\mathscr{V}$, $\bar{\mathscr{V}}$, T, $\bar{T}$, and integers f_i are then defined by

$$C_H(B) = [T]^{\mathscr{V}} \cdot [0 \times \mathbb{C}^n]^{\mathscr{V}} = \sum f_i [\![0 \times B_i]\!]^{\mathscr{V}}.$$

(B.4.4) It is a fact that all the e_i (and f_i) are *positive* (cf. [3, p. 488, 4.16]). This also follows from the fact that e_i and f_i can be defined algebraically (E.1.1).

(B.5) The integers e_i can also be derived from $\bar{S} \subset \bar{\mathscr{U}}$ (resp. f_i from $\bar{T} \subset \bar{\mathscr{V}}$).

Let $[\![0 \times A_i]\!]^{\bar{\mathscr{U}}}$ be the natural image of $[0 \times A_i]$ in $H_{2d}^{0 \times C(A)}(\bar{\mathscr{U}})$. Then by (A.4.3) and definitions:

(B.5.1)
$$[\![0 \times A_i]\!]^{\mathscr{U}} = i^{0 \times C(A),\, 0 \times C(A)} [\![0 \times A_i]\!]^{\bar{\mathscr{U}}},$$

where $i: \bar{\mathscr{U}} \to \mathscr{U}$ is the inclusion.

(B.5.2) **Lemma.**

$$[\bar{S}]^{\bar{\mathscr{U}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}} = \sum e_i [\![0 \times A_i]\!]^{\bar{\mathscr{U}}} \in H^{0 \times C(A)}(\bar{\mathscr{U}}).$$

Proof. We choose an orientation on $\mathbb{R} \times \mathbb{C}^n$ as before (B.3.4), and on $i\mathbb{R} \times \mathbb{C}^n$ in such a way that $[\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}} \cdot [i\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}} = [0 \times \mathbb{C}^n]^{\mathscr{U}}$ (cf. (A.6.3)). (We are using the convention $[X]^{\mathscr{U}} = [X \cap \mathscr{U}]^{\mathscr{U}}$ (A.2.1); and $i\mathbb{R}$ stands for the pure imaginary numbers in $\mathbb{C}$.) Then the lemma results from (B.4.2), (B.5.1) and the following two assertions:

(1)
$$i^{0 \times C(A),\, 0 \times C(A)}([\bar{S}]^{\bar{\mathscr{U}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}}) = [\bar{S}]^{\mathscr{U}} \cdot [i\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}}.$$

(2)
$$[\bar{S}]^{\mathscr{U}} \cdot [i\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}} = [S]^{\mathscr{U}} \cdot [0 \times \mathbb{C}^n]^{\mathscr{U}} \in H^{0 \times C(A)}(\mathscr{U}).$$

(1) follows immediately from (A.7.2) (let $V = \bar{\mathscr{U}}$, $W = \mathscr{U}$, $X = (0 \times \mathbb{C}^n) \cap \mathscr{U}$, $Y = (i\mathbb{R} \times \mathbb{C}^n) \cap \mathscr{U}$, $\Phi = \Phi' = \bar{S}$ and $\xi = [\bar{S}]^{\bar{\mathscr{U}}}$; by (A.4.3) $i[\bar{S}]^{\bar{\mathscr{U}}} = [\bar{S}]^{\mathscr{U}}$).

(2) follows from (B.3.5) and associativity of intersection (A.6.1):

$$\begin{aligned} [\bar{S}]^{\mathscr{U}} \cdot [i\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}} &= ([S]^{\mathscr{U}} \cdot [\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}}) \cdot [i\mathbb{R} \times \mathbb{C}^n]^{\mathscr{U}} \\ &= [S]^{\mathscr{U}} \cdot [0 \times \mathbb{C}^n]^{\mathscr{U}} \end{aligned}$$

q.e.d.

Similarly:

(B.5.3)
$$[\bar{T}]^{\bar{\mathscr{V}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{V}}} = \sum f_i [\![0 \times B_i]\!]^{\bar{\mathscr{V}}}.$$

(B.6) We can now outline the approach we shall take.

By (B.2.1) and (B.1.2) $C(A)$ is pure d-dimensional; let $A_1, \ldots, A_k$ be its components. Then $B_1 = \alpha_0(A_1), \ldots, B_k = \alpha_0(A_k)$ are the components of $C(B, O)$ (by (B.1.3), (A.8)).

Let $C_H(A) = \sum e_i [\![0 \times A_i]\!]^{\mathscr{U}}$ (resp. $C_H(B) = \sum f_i [\![0 \times A_i]\!]^{\mathscr{V}}$) be the homology tangent cone of A (resp. B).

In view of (B.2.3) ($\mu(A_i) = \mu(B_i)$), the theorem results from the following assertions:

(1)
$$\mu(A) = \sum_i e_i\, \mu(A_i), \qquad \mu(B) = \sum f_i\, \mu(B_i).$$

(2)
$$e_i = f_i.$$

These are proved in sections C and D respectively.

C. Multiplicity of the Homology Tangent Cone

In this section we prove assertion (1) of (B.6). Notation remains as in (B.3).

(C.1) Let $\pi: \mathbb{C}^n \to \mathbb{C}^d$ ($d = \dim A$) be a complex linear projection ($\pi^2 = \pi$) such that $\pi^{-1}(O) \cap C(A) = O$. (It is easy to see that such π do exist.) We write $L = \pi^{-1}(O) \cap U$.

The idea of the proof (quite similar to that of [5, Thm. 6.4]) is to interpret multiplicity as the intersection number $[A]^U \circ [L]^U$ between A and $\pi^{-1}(O)$. It is easy to see that $[A]^U \circ [L]^U = [1 \times A]^{\mathscr{U}} \circ [\mathbb{C} \times L]^{\mathscr{U}}$ which in turn can be shown to be the intersection number $[S]^{\mathscr{U}} \circ [1 \times L]^{\mathscr{U}}$ (using the associativity of intersection and the fact that S and $1 \times U$ intersect transversally). Similarly $\sum e_i \mu(A_i) = \sum e_i [A_i]^U \circ [L]^U$ can be shown to be the intersection number $[S]^{\mathscr{U}} \circ [0 \times L]^{\mathscr{U}}$. To complete the proof we need only show that $[S]^{\mathscr{U}} \circ [1 \times L]^{\mathscr{U}} = [S]^{\mathscr{U}} \circ [0 \times L]^{\mathscr{U}}$. This is done by showing that both numbers are equal to the value of $[S]^{\mathscr{U}} \cdot [\mathbb{C} \times L]^{\mathscr{U}} \in H_2^{\mathbb{C} \times O}(\mathscr{U}) \cong \mathbb{Z}$. Details are given in (C.2), (C.3).

(C.2) By shrinking U if necessary (this is O.K. by (B.2.2), (B.4.2), (B.2.1)) we can assume that $L \cap A = O$ (because near the origin A is "close" to $C(A)$, cf. [20, p. 215, 2.10]); and that $(A \cap U, \pi|_{A \cap U})$ is a branched covering over an open connected subset $W = \pi(A \cap U)$ of $\mathbb{C}^d$, i.e. $\pi|_{A \cap U}$ is a topological covering outside an analytic variety D in W [10, p. 101, Th. 16]. By (B.2, Remark 1) $\mu(A)$ = covering number of $\pi = \#(A \cap U \cap \pi^{-1}(a))$ for $a \in W - D$. Then for $a \in W - D$, $\pi^{-1}(a)$ intersects A transversally at smooth points of A [10, p. 106, Th. 19]. Moreover, joining a to O by an arc γ in W, we have that $\pi^{-1}(\gamma) \cap A$ is compact (because π is proper). Hence, by [3, p. 485, 4.10], $\#(A \cap \pi^{-1}(a)) = [A]^U \circ [L]^U$. Thus

(C.2.1) $$\mu(A) = [A]^U \circ [L]^U.$$

Next we apply (A.7.2) with $V = U$, $W = \mathscr{U}$, $Y = (\mathbb{C} \times L) \cap \mathscr{U}$, $X = Y \cap U = 1 \times L$, $\Phi = \Phi' = A$ and $\xi = [A]^U$. (Here we identify U with $1 \times U$ and A with $1 \times A$). Since $i^{A \cap 1 \times L, A \cap \mathbb{C} \times L} (= i^{1 \times O, 1 \times O})$ and $i^{A, A}$ are isomorphisms, (A.7.2) gives:

$$[A]^U \circ [L]^U = [1 \times A]^{\mathscr{U}} \circ [\mathbb{C} \times L]^{\mathscr{U}}.$$

(C.2.2) Together with (C.2.1) we have: $\mu(A) = [1 \times A]^{\mathscr{U}} \circ [\mathbb{C} \times L]^{\mathscr{U}}$. Similarly,

$$\mu(A_i) = [0 \times A_i]^{\mathscr{U}} \circ [\mathbb{C} \times L]^{\mathscr{U}}.$$

Remark. Recall that $S \cap (1 \times U) = 1 \times A$. In fact the intersection is transversal. Let $s = (1, z_0)$ be a smooth point of S. By (B.3.1) $\phi: (t, z) \to \left(t, \frac{1}{t} z\right)$ induces an isomorphism $d\phi$ between $\mathbb{C} \times T_{z_0}(A)$, (the tangent space of $\mathbb{C} \times A$ at s) and $T_s S$ (the tangent space of S at $(1, z_0)$). Straightforward computations show that $d\phi(1, O)_s = (1, -z_0)_s$. (Here $O \in \mathbb{C}^n$). Hence $T_s S$ and $T_s(1 \times U) = (0 \times \mathbb{C}^n)_s$ generate $(\mathbb{C} \times \mathbb{C}^n)_s$ i.e. S intersects $1 \times U$ transversally at smooth points of S which lie in $1 \times U$.

By the above remark and the criterion of unit multiplicity (A.6.3) we can conclude

$$[S]^{\mathscr{U}} \cdot [1 \times U]^{\mathscr{U}} = [1 \times A]^{\mathscr{U}}.$$

And the Eq. (C.2.2) becomes, by associativity (A.6.1),

$$\mu(A)=[S]^{\mathscr{U}}\circ([1\times U]^{\mathscr{U}}\cdot[\mathbb{C}\times L]^{\mathscr{U}}).$$

But $1\times U$ is transversal to $\mathbb{C}\times L$, hence, by (A.6.3),

(C.2.3) $$\mu(A)=[S]^{\mathscr{U}}\circ[1\times L]^{\mathscr{U}}.$$

Similar arguments also give

$$\begin{aligned}[S]^{\mathscr{U}}\circ[0\times L]^{\mathscr{U}}&=[S]^{\mathscr{U}}\circ([0\times\mathbb{C}^n]^{\mathscr{U}}\cdot[\mathbb{C}\times L]^{\mathscr{U}})\\&=([S]^{\mathscr{U}}\cdot[0\times\mathbb{C}^n]^{\mathscr{U}})\circ[\mathbb{C}\times L]^{\mathscr{U}}\\&=\sum e_i[\![0\times A_i]\!]^{\mathscr{U}}\circ[\mathbb{C}\times L]^{\mathscr{U}}\\&=\sum e_i([\![0\times A_i]\!]^{\mathscr{U}}\circ[\mathbb{C}\times L]^{\mathscr{U}}).\end{aligned}$$

(We have used (A.6.3), (A.6.1), (B.4.2), (B.4.4) here.)

In view of the definition of $[\![0\times A_i]\!]^{\mathscr{U}}$ (B.4.1), by the compatibility between intersections and enlargement of supports (A.6.1) we have

$$[\![0\times A_i]\!]^{\mathscr{U}}\circ[\mathbb{C}\times L]^{\mathscr{U}}=[0\times A_i]^{\mathscr{U}}\circ[\mathbb{C}\times L]^{\mathscr{U}},$$

which in turn is equal to $\mu(A_i)$ by (C.2.2).

Altogether we conclude

(C.2.4) $$[S]^{\mathscr{U}}\circ[0\times L]^{\mathscr{U}}=\sum e_i\,\mu(A_i).$$

(C.3) In view of (C.2.3) and (C.2.4), to complete the proof we are left with proving

$$[S]^{\mathscr{U}}\circ[1\times L]^{\mathscr{U}}=[S]^{\mathscr{U}}\circ[0\times L]^{\mathscr{U}}.$$

Proof. By associativity (A.6.1), for any $t\in\mathbb{C}$ we have

(C.3.1) $$[S]^{\mathscr{U}}\cdot([\mathbb{C}\times L]^{\mathscr{U}}\cdot[t\times\mathbb{C}^n]^{\mathscr{U}})=([S]^{\mathscr{U}}\cdot[\mathbb{C}\times L]^{\mathscr{U}})\cdot[t\times\mathbb{C}^n]^{\mathscr{U}}.$$

By the criterion of unit multiplicity (A.6.3), the left side of (C.3.1) is $[S]^{\mathscr{U}}\cdot[t\times L]^{\mathscr{U}}$. On the other hand, since $L\cap(1/t)A=O$ and $L\cap C(A)=O$, we see that $S\cap(\mathbb{C}\times L)=\mathbb{C}\times O$, and (since $H_2^{\mathbb{C}\times O}(\mathscr{U})\simeq\mathbb{Z}$ (A.3.3), (A.4.2)) therefore $[S]^{\mathscr{U}}\cdot[\mathbb{C}\times L]^{\mathscr{U}}=\lambda[\mathbb{C}\times O]^{\mathscr{U}}$ for some $\lambda\in\mathbb{Z}$; hence, by (A.6.3) again, the right side of (C.3.1) is $\lambda[t\times O]^{\mathscr{U}}$. Thus:

$$[S]^{\mathscr{U}}\circ[t\times L]^{\mathscr{U}}=|\lambda|\quad\text{for all } t\in\mathbb{C}.$$

In particular $[S]^{\mathscr{U}}\circ[1\times L]^{\mathscr{U}}=[S]^{\mathscr{U}}\circ[0\times L]^{\mathscr{U}}$. q.e.d.

Remark (not needed elsewhere). If we define the *local degree* of $\pi\colon A\to\mathbb{C}^d$ (denoted by $\deg\pi|_A$) to be $|\pi_*[A]|$ where $\pi_*\colon H_{2d}(A)\to H_{2d}(\mathbb{C}^d)\simeq\mathbb{Z}$, then it can be shown (using the projection formula (A.7)) that

$$\pm\deg\pi|_A=[A]^U\circ[L]^U.$$

(In fact we need only that π is continuous and $\pi^{-1}(O)\cap C(A)$ has an isolated point at O.)

Moreover, if π is real-linear (or even "close to" real linear, e.g. differentiable) then the argument in this section gives

$$\deg \pi|_A = \sum \pm e_i \deg \pi|_{A_i}.$$

D. The e_i are Differential Invariants

Consider the situation in Section B (especially (B.6)), i.e. we have homeomorphisms $\alpha: (U, A, O) \to (V, B, O)$, $\alpha_0: (C(A), A_i) \to (C(B), B_i)$ and sets $\mathscr{U}, S, \bar{\mathscr{U}}, \bar{S}$ (resp. $\mathscr{V}, T, \bar{\mathscr{V}}, \bar{T}$) which are derived from (U, A) (resp. (V, B)). Here α_0 is induced by a *real-linear* map $\alpha_0: \mathbb{C}^n \to \mathbb{C}^n$ (the derivative at O of α). Note that $\dim_{\mathbb{R}} U = 2n$, $\dim_{\mathbb{R}} \mathscr{U} = 2n+2$, $\dim_{\mathbb{R}} \bar{\mathscr{U}} = 2n+1$, $\dim_{\mathbb{R}} A = \dim_{\mathbb{R}} A_i = 2d$, $\dim_{\mathbb{R}} S = 2d+2$, $\dim_{\mathbb{R}} \bar{S} = 2d+1$. Also recall that the natural numbers e_i (resp. f_i) are uniquely determined by the equation (B.5.2) (resp. (B.5.3)):

$$[\bar{S}]^{\bar{\mathscr{U}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}} = \sum e_i [\![0 \times A_i]\!]^{\bar{\mathscr{U}}}$$
$$(\text{resp.}\ [\bar{T}]^{\bar{\mathscr{V}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{V}}} = \sum f_i [\![0 \times B_i]\!]^{\bar{\mathscr{V}}}).$$

(D.1) We show in this section that $e_i = f_i$.

The approach is the following. First we construct a homeomorphism h from $(\bar{\mathscr{U}}, \bar{S})$ to $(\bar{\mathscr{V}}, \bar{T})$ which respects the fibres, i.e. maps $\bar{\mathscr{U}} \cap (t \times \mathbb{C}^n)$ to $\bar{\mathscr{V}} \cap (t \times \mathbb{C}^n)$ for all $t \in \mathbb{R}$. (We can not do so for $(\mathscr{U}, S)$ and $(\mathscr{V}, T)$, and this is the sole reason for introducing $\bar{\mathscr{U}}, \bar{\mathscr{V}}$). Since homeomorphism preserves intersections (A.7.3), we have:

$$h_*[\bar{S}]^{\bar{\mathscr{U}}} \cdot h_*[0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}} = \pm h_*([\bar{S}]^{\bar{\mathscr{U}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}})$$
$$= \pm \sum e_i h_* [\![0 \times A_i]\!]^{\bar{\mathscr{U}}}.$$

On the other hand, we have (by definition)

$$[\bar{T}]^{\bar{\mathscr{V}}} \cdot [0 \times \mathbb{C}^n]^{\bar{\mathscr{V}}} = \sum f_i [\![0 \times B_i]\!]^{\bar{\mathscr{V}}}.$$

Now it is easy to see that: $h_*[0 \times \mathbb{C}^n]^{\bar{\mathscr{U}}} = \pm [0 \times \mathbb{C}^n]^{\bar{\mathscr{V}}}$, $h_*[\![0 \times A_i]\!]^{\bar{\mathscr{U}}} = \pm [\![0 \times B_i]\!]^{\bar{\mathscr{V}}}$ (cf. (D.3.2)). In (D.4) we show: $h_*[\bar{S}]^{\bar{\mathscr{U}}} = \pm [\bar{T}]^{\bar{\mathscr{V}}}$. It follows that $e_i = \pm f_i$. But the e_i and f_i are all positive (B.4.4), hence $e_i = f_i$.

Remark. It follows that the "$\pm$" in $h_*[\![0 \times A_i]\!]^{\bar{\mathscr{U}}} = \pm [\![0 \times B_i]\!]^{\bar{\mathscr{V}}}$ is independent of i.

(D.2) We now carry out the above scheme.

Definition. For $(t, x` \mathscr{U}$,

$$h(t,x) = \begin{cases} (t, t^{-1} \alpha(t x)) & t \neq 0 \\ (0, \alpha_0(x)) & t = 0. \end{cases}$$

Notation. For simplicity we also write h for restrictions of h to subsets of $\mathscr{U}$; and h_* will always mean $h_i^{\Phi, h(\Phi)}(x)$ for some subset X of $\mathscr{U}$ and family of supports Φ which will be clear from the context.

We check that h has the desired properties.

(D.2.1) *h maps $(\mathscr{U}, S)$ into $(\mathscr{V}, T)$ and $(\bar{\mathscr{U}}, \bar{S})$ into $(\bar{\mathscr{V}}, \bar{T})$.*

Proof. The first assertion follows easily from definitions and from $S \cap (0 \times \mathbb{C}^n) = 0 \times C(A)$ (B.3.3) and $\alpha_0(C(A)) = C(B)$ (B.1.3). The proof of the second assertion is similar.

(D.2.2) *The maps in* (D.2.1) *are bijective.* In fact the inverse is defined by

$$h^{-1}(t,x) = \begin{cases} (t, t^{-1}\alpha^{-1}(t x)) & t \neq 0 \\ (0, \alpha_0^{-1}(x)) & t = 0. \end{cases}$$

(D.2.3) *$h: \bar{\mathscr{U}} \to \bar{\mathscr{V}}$ is a homeomorphism* (hence so is $h: \bar{S} \to \bar{T}$).

Proof. By (D.2.2) it suffices to show that h is continuous (then by symmetry h^{-1} is continuous too). It is clear that h is continuous at all points $(t,x) \in \bar{\mathscr{U}}$ with $t \in \mathbb{R}^*$, so we consider points $(0,x) \in \bar{\mathscr{U}}$, i.e. $x \in \mathbb{C}^n$. Let $(t,y) \in \bar{\mathscr{U}}$ be a nearby point.

Case 1. If $t=0$, then

$$\begin{aligned} h(0,y) - h(0,x) &= (0, \alpha_0(y)) - (0, \alpha_0(x)) \\ &= (0, \alpha_0(y-x)). \end{aligned}$$

Case 2. If $t \in \mathbb{R}^*$, then

$$\begin{aligned} h(t,y) - h(0,x) &= (t, t^{-1}\alpha(t y)) - (0, \alpha_0(x)) \\ &= (t, t^{-1}\alpha(t y) - \alpha_0(x)). \end{aligned}$$

We estimate the second coordinate:

$$\begin{aligned} \left|\frac{1}{t}\alpha(t y) - \alpha_0(x)\right| &\leqq \left|\frac{1}{t}\alpha(t y) - \frac{1}{t}\alpha_0(t y)\right| + \left|\frac{1}{t}\alpha_0(t y) - \alpha_0(x)\right| \\ &= \left|\frac{\alpha(t y) - \alpha(0) - \alpha_0(t y)}{t}\right| + |\alpha_0(y) - \alpha_0(x)| \\ &= |y - x + x| \frac{|\alpha(t y) - \alpha(0) - \alpha_0(t y)|}{|t y|} + |\alpha_0(y-x)|. \end{aligned}$$

The first term tends to 0 with t and $|y-x|$ (by the definition of derivative). Hence, in either case, $h(t,y) - h(0,x)$ can be made arbitrarily small if (t,y) is close enough to $(0,x)$, i.e. h is continuous at $(0,x)$.

(D.2.4) The same proof breaks down when we attempt to prove the continuity of $h: \mathscr{U} \to \mathscr{V}$ (because α_0 is only *real*-linear). However, h clearly induces a homeomorphism from $\mathscr{U} - (0 \times \mathbb{C}^n)$ to $\mathscr{V} - (0 \times \mathbb{C}^n)$; and this fact will be needed in (D.4.1).

(D.3) In this subsection (D.3) everything happens in $\bar{\mathscr{U}} = \mathscr{U} \cap (\mathbb{R} \times \mathbb{C}^n)$.

With the homeomorphism h defined above we can conclude, using (A.7.3) and (B.5.2), that

(D.3.1)
$$\begin{aligned} h_*[\bar S]^{\bar{\mathscr U}}\cdot h_*[0\times\mathbb C^n]^{\bar{\mathscr U}} &= \pm h_*([\bar S]^{\bar{\mathscr U}}\cdot[0\times\mathbb C^n]^{\bar{\mathscr U}})\\ &= \pm h_*(\textstyle\sum e_i[\![0\times A_i]\!]^{\bar{\mathscr U}})\\ &= \pm\textstyle\sum e_i h_*[\![0\times A_i]\!]^{\bar{\mathscr U}}),\end{aligned}$$

where $h_*[\![0\times A_i]\!]^{\bar{\mathscr U}}\in H_{2d}^{0\times C(B)}(\bar{\mathscr V})$.

(D.3.2) *Remark.* It is clear that $h_*[0\times\mathbb C^n]^{\bar{\mathscr U}}=\pm[0\times\mathbb C^n]^{\bar{\mathscr V}}$ (h_* induces an isomorphism between $H_{2n}^{0\times\mathbb C^n}(\bar{\mathscr U})\simeq\mathbb Z$ and $H_{2n}^{0\times\mathbb C^n}(\bar{\mathscr V})\simeq\mathbb Z$ (A.3.3), (A.4.1)).

We show that $h_*[\![0\times A_i]\!]^{\bar{\mathscr U}}=\pm[\![0\times B_i]\!]^{\bar{\mathscr V}}$. Recall the Definition (B.5)

$$[\![0\times A_i]\!]^{\bar{\mathscr U}}=i^{0\times A_i,0\times C(A)}[0\times A_i],$$
$$[\![0\times B_i]\!]^{\bar{\mathscr V}}=i^{0\times B_i,0\times C(B)}[0\times B_i].$$

Since $h=h|_{0\times A_i}=0\times(\alpha_0|_{0\times A_i})$ induces an isomorphism $\mathbb Z\simeq H_{2d}(0\times A_i)\to H_{2d}(0\times B_i)\simeq\mathbb Z$ (A.4.1), we have $h_*[0\times A_i]=\pm[0\times B_i]$. The result then follows from the commutativity of the following diagram (A.3.1):

$$\begin{array}{ccc} H(0\times A_i) & \xrightarrow{i^{0\times A_i,0\times C(A)}} & H^{0\times C(A)}(\bar{\mathscr U})\\ \big\downarrow h_* & & \big\downarrow h_*\\ H(0\times B_i) & \xrightarrow[i^{0\times B_i,0\times C(B)}]{} & H^{0\times C(B)}(\bar{\mathscr V})\end{array}$$

(D.4) As in (D.1), it remains to show that $h_*[\bar S]^{\bar{\mathscr U}}=\pm[\bar T]^{\bar{\mathscr V}}$; and for this *it suffices to prove:* $h_*[\bar S]=\pm[\bar T]$ (because then using $h_*\circ i_{S,\bar{\mathscr U}}=i_{T,\bar{\mathscr V}}\circ h_*$ (A.3.1) we have

$$h_*[\bar S]^{\bar{\mathscr U}}=h_*\,i_{S,\bar{\mathscr U}}[\bar S]=i_{T,\bar{\mathscr V}}h_*[\bar S]=\pm i_{T,\bar{\mathscr V}}[\bar T]=\pm[\bar T]^{\bar{\mathscr V}}).$$

Idea of Proof. By definition $[\bar S]$ and $[\bar T]$ come from intersections that occur in $\mathscr U$ and $\mathscr V$ respectively. However h is continuous only on $\mathscr U-(0\times\mathbb C^n)$ (D.2.4). Hence we restrict the intersection (between S and $\mathbb R\times\mathbb C^n$) to $\mathscr U-(0\times\mathbb C^n)$ in order to obtain

$$h_*[\bar S-(0\times\mathbb C^n)]=\pm[\bar T-(0\times\mathbb C^n)].$$

This will then imply: $h_*[\bar S]=\pm[\bar T]$. We now give the details.

Notation. "Superscript *" means "leave out the part in $0\times\mathbb C^n$". Thus $\mathscr U^*=\mathscr U-(0\times\mathbb C^n)$, $\bar{\mathscr U}^*=\bar{\mathscr U}-(0\times\mathbb C^n)$, $S^*=S-(0\times\mathbb C^n)$ etc.

(D.4.1) *Remark 1.* If X is a closed subset of $\mathscr U$ having a fundamental class $[X]$, we write $[X^*]=j^{X,X^*}[X]$, cf. (A.3). By (A.4.3), $[X^*]$ is a fundamental class of X^*. It follows from (A.3.2) that

$$\begin{aligned} j^{\mathscr U,\mathscr U^*}[X]^{\mathscr U} &= j^{\mathscr U,\mathscr U^*}i_{X,\mathscr U}[X]=i_{X^*,\mathscr U^*}j^{X,X^*}[X]\\ &= i_{X^*,\mathscr U^*}[X^*]=[X^*]^{\mathscr U^*}.\end{aligned}$$

In particular, the restriction of $[S]^{\mathscr U}$ (resp. $[\bar S]^{\mathscr U}$, $[\bar{\mathscr U}]^{\mathscr U}$) to $\mathscr U^*$ is $[S^*]^{\mathscr U^*}$ (resp. $[\bar S^*]^{\mathscr U^*}$, $[\bar{\mathscr U}^*]^{\mathscr U^*}$). (N.B.: $\bar S^*$ and $\bar{\mathscr U}^*$ are not connected.)

Remark 2. Since $[\bar{S}]^{\mathscr{U}}=[\bar{\mathscr{U}}]^{\mathscr{U}}\cdot[S]^{\mathscr{U}}$ (B.3.5), it follows from the above remark and the compatibility of intersection and restriction (A.6.1) that

$$[\bar{S}^*]^{\mathscr{U}^*}=[\bar{\mathscr{U}}^*]^{\mathscr{U}^*}\cdot[S^*]^{\mathscr{U}^*}.$$

Similarly

$$[\bar{T}^*]^{\mathscr{V}^*}=[\bar{\mathscr{V}}^*]^{\mathscr{V}^*}\cdot[T^*]^{\mathscr{V}}.$$

(D.4.2) *Claim.* $h_*[\bar{S}^*]^{\mathscr{U}^*}=\pm[\bar{T}^*]^{\mathscr{V}^*}$.

Proof. In view of (D.4.1, Remark 2) the assertion will result from (A.7.3) if we can show

(a) $h_*[S^*]^{\mathscr{U}^*}=\pm[T^*]^{\mathscr{V}^*}$ and
(b) $h_*[\bar{\mathscr{U}}^*]^{\mathscr{U}^*}=\pm[\bar{\mathscr{V}}^*]^{\mathscr{V}^*}$.

Proof of (a). Recall that S^* (resp. T^*) is an irreducible analytic variety (B.3.2). By (A.4.1), $H^{S^*}_{2d+2}(\mathscr{U}^*)\simeq H_{2d+2}(S^*)\simeq\mathbb{Z}$ (resp. $H^{T^*}_{2d+2}(\mathscr{V}^*)\simeq\mathbb{Z}$) and has the fundamental class $[S^*]^{\mathscr{U}^*}$ (resp. $[T^*]^{\mathscr{V}^*}$) as a generator. Since h_* is an isomorphism, (a) is proved.

Proof of (b). By definition we have (cf. (D.4.1, Remark 1))

$$[\bar{\mathscr{U}}^*]^{\mathscr{U}^*}=i_{\bar{\mathscr{U}}^*,\mathscr{U}^*}\circ j^{\bar{\mathscr{U}},\bar{\mathscr{U}}^*}[\bar{\mathscr{U}}],$$
$$[\bar{\mathscr{V}}^*]^{\mathscr{V}^*}=i_{\bar{\mathscr{V}}^*,\mathscr{V}_*}\circ j^{\bar{\mathscr{V}},\bar{\mathscr{V}}^*}[\bar{\mathscr{V}}].$$

By (A.4.2), $[\bar{\mathscr{U}}]$ (resp. $[\bar{\mathscr{V}}]$) generates $H_{2n+1}(\bar{\mathscr{U}})\simeq\mathbb{Z}$ (resp. $H_{2n+1}(\bar{\mathscr{V}})\simeq\mathbb{Z}$). Thus $h_*[\bar{\mathscr{U}}]=\pm[\bar{\mathscr{V}}]$ (h_* is an isomorphism). By (A.3.2), (A.3.1),

$$h_*[\bar{\mathscr{U}}^*]^{\mathscr{U}^*}=\pm[\bar{\mathscr{V}}^*]^{\mathscr{V}^*}.\quad \text{q.e.d.}$$

(D.4.3) We are now ready to prove $h_*[\bar{S}]=\pm[\bar{T}]$ (D.4). Let $i_S=i_{S^*,\mathscr{U}^*}$, $i_T=i_{T^*,\mathscr{V}^*}$, $j_S=j^{S,S^*}$, $j_T=j^{T,T^*}$. Consider the following commutative diagram (A.3.1), (A.3.2):

$$\begin{array}{ccccc}
[\bar{S}]\in H_{2d+1}(\bar{S}) & \xrightarrow{j_S} & H_{2d+1}(\bar{S}^*) & \xrightarrow{i_S} & H^{S^*}_{2d+1}(\mathscr{U}^*)\\
\downarrow ?\quad \downarrow h_* & & \downarrow h_* & & \downarrow h_*\\
\pm[\bar{T}]\in H_{2d+1}(\bar{T}) & \xrightarrow{j_T} & H_{2d+1}(\bar{T}^*) & \xrightarrow{i_T} & H^{T^*}_{2d+1}(\mathscr{V}^*)
\end{array}$$

As in the proof of (b) (D.4.2) we have (by definition)

$$i_S j_S[\bar{S}]=[\bar{S}^*]^{\mathscr{U}^*},$$

and similarly

(D.4.4) $$i_T j_T[\bar{T}]=[\bar{T}^*]^{\mathscr{V}^*}.$$

By the commutativity of the above diagram and (D.4.2) we see that

$$i_T j_T h_*[\bar{S}]=h_*\, i_S j_S[\bar{S}]=h_*[\bar{S}^*]^{\mathscr{U}^*}=\pm[\bar{T}^*]^{\mathscr{V}^*}.$$

By (D.4.4) it remains to show that $i_T \circ j_T$ is injective. But $i_T: H(\bar{T}^*) \to H^{\bar{T}^*}(\mathscr{V}^*)$ is an isomorphism (A.3.3), and the fact that j_T is injective can be seen from the following exact sequence (A.3.4) (keeping in mind $\bar{T} - \bar{T}^* = \bar{T} \cap (0 \times \mathbb{C}^n) = 0 \times C(B)$ (B.3.3)):

$$0 = H_{2d+1}(0 \times C(B)) \to H_{2d+1}(\bar{T}) \xrightarrow{j_T} H_{2d+1}(\bar{T}^*),$$

where $\dim_{\mathbb{R}} C(B) = 2d$, so by (A.4.1), $H_{2d+1}\, C(B) = 0$.

This completes the proof of (D.1), hence of the Theorem.

E. Appendix: Algebraic Definition of the e_i

(E.1) In this section we prove the following algebraic characterization of e_i. (It is an adaptation of folklore, cf. e.g. [14].)

Proposition. *Let $O \in A$, an analytic subvariety of $U \subseteq \mathbb{C}^n$ all of whose components have the same dimension, and let $\mathcal{O}_{A,O}$ be the local ring of germs at O of analytic functions on A. Let $\mathfrak{m} = \mathfrak{m}_{A,O}$ be the maximal ideal of $\mathcal{O}_{A,O}$, and let G be the graded ring $\bigoplus_{q \geq 0} \mathfrak{m}^q/\mathfrak{m}^{q+1}$. Then there is a natural one-one correspondence $A_i \leftrightarrow p_i$ between components of the tangent cone $C(A) = C(A, O)$ and minimal prime ideals in G, and e_i is the length of the artin local ring G_{p_i}.*

Before discussing the proof we note:

(E.1.1) **Corollary.** $e_i > 0$.

(E.1.2) From [18, (23.5)], applied to the localization of G at its maximal ideal $\bigoplus_{q>0} \mathfrak{m}^q/\mathfrak{m}^{q+1}$, we now obtain the main result of Sect. C:

$$\mu(A) = \sum_i e_i\, \mu(A_i).$$

However the proof in Sect. C is more in keeping with the spirit of the rest of the paper; and it gives a more general result - cf. the remark at the end of Sect. C.

(E.2) We begin the proof of the Proposition with some preliminary observations.

Let $f = f(z_1, \ldots, z_n)$ be a convergent power series, and write

$$f = f_d + f_{d+1} + \ldots + f_m + \ldots$$

where $f_m = f_m(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree m, and $f_d \neq 0$.

Recall [20, p. 221, Theorem 4D], that $C(A)$ is the algebraic subvariety of $\mathbb{C}^n$ defined by the vanishing of all the initial forms f_d as f runs through the power series vanishing in a neighborhood N of O on A (here both d and N depend on f). Accordingly, we consider $C(A)$ to be the (not necessarily reduced!) affine algebraic variety with coordinate ring $\mathbb{C}[z_1, \ldots, z_n]/\{f_d(z_1, \ldots, z_n)\}$ (factoring out all f_d as above), which is nothing but the graded ring G in the Proposition. So the minimal primes of G correspond naturally to the components of $C(A)$. ("Components" are the same in the algebraic or analytic categories, since irreducible algebraic varieties have connected smooth part, so are irreducible as analytic varieties.) These minimal primes are homogeneous, so each component of $C(A)$ is mapped into itself by multiplication by any complex number, and hence passes through O. Let us show that the map from the set of irreducible components of the *germ* of $C(A)$ at O to the set of algebraic components of $C(A)$, taking an irreducible germ to the unique algebraic component containing it, is *bijective* (whence each algebraic component is irreducible at 0).

Algebraically this map is realized by considering the algebraic local ring R of $C(A)$ at O (i.e. R is the localization of G at the maximal ideal $\bigoplus_{q>0} \mathfrak{m}^q/\mathfrak{m}^{q+1}$) as a subring of the analytic local ring

$$\tilde{R} = \mathbb{C}\langle z_1, \ldots, z_n \rangle / \{f_d(z_1, \ldots, z_n)\}$$

(where $\mathbb{C}\langle\ldots\rangle$ denotes "convergent power series") and associating to each minimal prime ideal $\tilde{p}$ of $\tilde{R}$ its contraction $p=\tilde{p}\cap R$. Now R and $\tilde{R}$ clearly have the same completion, say $\hat{R}$. $\hat{R}/p\hat{R}$ is the completion of R/p, and if $\hat{m}$ is the maximal ideal of $\hat{R}/p\hat{R}$, then the graded ring $\bigoplus_{q\geq 0} \hat{m}^q/\hat{m}^{q+1}$ is isomorphic to the graded *integral domain* $G/p\cap G$; it follows that $\hat{R}/p\hat{R}$ is also an integral domain, i.e. *$p\hat{R}$ is a prime ideal.* So

$$p\tilde{R}=[(p\tilde{R})\hat{R}]\cap\tilde{R}=p\hat{R}\cap\tilde{R}$$

is a prime ideal contained in $\tilde{p}$, and since $\tilde{p}$ is minimal, therefore $p\tilde{R}=\tilde{p}$, so that our map is injective. Also, since *every* minimal prime in R is the contraction of a prime ideal in $\tilde{R}$, our map is surjective (this is also geometrically clear), hence bijective.

In summary, we have:

Lemma. *Let A_i be an algebraic component of $C(A)$, and let p_i be the corresponding minimal prime ideal in G. Then A_i is analytically irreducible at 0, and if $\tilde{p}_i$ is the corresponding minimal prime ideal in the analytic local ring $\tilde{R}$, then $p_i=\tilde{p}_i\cap G$, $\tilde{p}_i=p_i\tilde{R}$,* and the artin local rings *G_{p_i}, $\tilde{R}_{\tilde{p}_i}$ have the same length.*

Proof. Everything has been proved except the last assertion, which follows from $\tilde{p}_i=p_i\tilde{R}$ and the flatness of $\tilde{R}_{\tilde{p}_i}$ over G_{p_i}. (Note that $\tilde{R}$ is flat over R since $\hat{R}$ is faithfully flat over both $\tilde{R}$ and R.)

(E.3) Lemma. *Let $S\subseteq\mathbb{C}\times\mathbb{C}^n$ be the analytic set defined in* (B.3), *and let $t:\mathbb{C}\times\mathbb{C}^n\to\mathbb{C}$ be the projection. Let $\tilde{R}$ be as in* (E.2) *above. Then*

$$\tilde{R}\cong\mathcal{O}_{S,O}/t\mathcal{O}_{S,O}.$$

After proving this Lemma, and in view of the preceding Lemma (E.2), we can get the Proposition from the following fact, which expresses the *equality of algebraic and topological intersection numbers.* (This fact is "well-known", cf. [3, p. 488, 4.16] and for zero-dimensional intersections [5, Theorem 6.5] or [17, p. 118, (A.8)]; but we don't know of an adequate reference, so a proof will be given.)

Fact. *Let L, S be pure-dimensional closed analytic subvarieties of an open set $U\subseteq\mathbb{C}^m$, and let B be an irreducible component of $L\cap S$ such that*

$$\operatorname{codim} B=\operatorname{codim} L+\operatorname{codim} S$$

(where "codim" denotes "codimension in $\mathbb{C}^m$"). The topological intersection number $i(L.S,B)$ is defined as in [3, p. 482, 4.4]. *Assume that B contains a point where L is smooth, and that $O\in B$. Let B^* be a component of the germ of B at O, set $R=\mathcal{O}_{S,O}$, and let p be the prime ideal in R consisting of germs of functions vanishing on B^*. Let I be the $\mathcal{O}_{\mathbb{C}^m,O}$-ideal consisting of germs of functions vanishing on L, so that (by standard local analytic geometry) IR_p is primary for the maximal ideal of R_p. Then the multiplicity of the ideal IR_p is $i(L.S,B)$.*

In (E.4) below we will prove the Lemma, and in (E.5) we will prove the Fact.

Finally, to get the Proposition, apply the Fact with L given by $t=0$ (notation as in Lemma), so that $I=t\mathcal{O}_{S,O}$ and $L\cap S=0\times C(A)$; and with B the component $0\times A_i$ of $L\cap S$. Then

$$\begin{aligned}
e_i&=i(L.S,B) && \text{(Definition (B.4.2))}\\
&=\text{multiplicity of } t(\mathcal{O}_{S,O})_p && \text{(by Fact)}\\
&=\text{length of } (\mathcal{O}_{S,O})_p/(t) && \\
&=\text{length of } \tilde{R}_{\tilde{p}_i} && \text{(by Lemma)}\\
&=\text{length of } G_{p_i} && \text{(cf. (E.2)).}
\end{aligned}$$

(Note: from the proof of the Lemma, it will be clear that if $p\subset\mathcal{O}_{S,O}$ is the ideal of germs vanishing on $0\times A_i$, then $p/(t)=\tilde{p}_i$.)

(E.4) (Proof of Lemma). As before, represent a convergent power series f as a sum of homogeneous polynomials:

$$f=f_d+f_{d+1}+f_{d+2}+\ldots\qquad(f_d\neq 0)$$

and set

$$f^*(t,z_1,\ldots,z_n)=t^{-d}f(tz_1,\ldots,tz_n)=f_d+tf_{d+1}+t^2f_{d+2}+\ldots.$$

We call f_d the "initial form" of f.

Let $I\subseteq \mathcal{O}_{\mathbb{C}^n,O}$ be the ideal of germs of functions vanishing on A, and let $F\subseteq I$ be a finite set of generators of I whose initial forms generate the kernel of the graded $\mathbb{C}$-algebra homomorphism $\mathbb{C}[Z_1,\ldots,Z_n]\to G$ taking the indeterminate Z_i to the natural image of the corresponding coordinate function z_i in $\mathfrak{m}/\mathfrak{m}^2\subseteq G$. We will show that *the ideal of germs in* $\mathcal{O}_{\mathbb{C}^{n+1},O}$ *vanishing on* S *is generated by* $\{f^*|f\in F\}$; the Lemma is then obvious.

With notations as in (B.3), let $N\subseteq U$ be a neighborhood of O such that all the $f\in F$ converge in N, and such that

$$(x_1,\ldots,x_n)\in A\cap N\Leftrightarrow f(x_1,\ldots,x_n)=0 \qquad \forall f\in F.$$

Then if x $\mathbb{C}-\{0\}$, we see that

$$\begin{aligned}
&(x,x_1,\ldots,x_n)\in S\cap\psi^{-1}(\mathbb{C}\times N)\\
&\quad\Leftrightarrow(xx_1,\ldots,xx_n)\in A\cap N\\
&\quad\Leftrightarrow f(xx_1,\ldots,xx_n)=0 \qquad \forall f\in F\\
&\quad\Leftrightarrow f^*(x,x_1,\ldots,x_n)=0 \qquad \forall f\in F;
\end{aligned}$$

while if $x=0$, then

$$\begin{aligned}
&(x,x_1,\ldots,x_n)\in S\cap\psi^{-1}(\mathbb{C}\times N)\\
&\quad\Leftrightarrow(x_1,\ldots,x_n)\in C(A) \qquad \text{(B.3.3)}\\
&\quad\Leftrightarrow f_d(x_1,\ldots,x_n)=0 \qquad \forall f\in F\ (d \text{ depends on } f)\\
&\quad\Leftrightarrow f^*(x,x_1,\ldots,x_n)=0 \qquad \forall f\in F.
\end{aligned}$$

Thus the vanishing of the $f^*(f\in F)$ defines the germ of S at O *set-theoretically*, and it remains to be shown that the ideal I^* generated by the f^* is a *radical ideal*, i.e. is equal to its own radical. Roughly speaking, this is true because I^* is a "proper transform" of the radical ideal $I\mathbb{C}\langle t,z,\ldots,z_n\rangle$ under blowing up. Here are the technical details.

Let H_0 be the convergent power series ring $\mathbb{C}\langle t,y_1,\ldots,y_n\rangle$, and let $I_y\subseteq H_0$ be the ideal generated by $\{f(y_1,\ldots,y_n)|f\in F\}$, so that $I_y=\sqrt{I_y}$. Define a $\mathbb{C}$-algebra homomorphism $\gamma_0\colon H_0\to\mathbb{C}\langle t,z_1,\ldots,z_n\rangle$ by

$$\gamma_0(y_i)=tz_i \qquad (1\leqq i\leqq n)$$
$$\gamma_0(t)=t.$$

Set

$$H=H_0[t^{-1}y_1,\ldots,t^{-1}y_n].$$

Then $y_i\in tH$ $(1\leqq i\leqq n)$, and so for $f=f_d+\ldots$ as above, since $f\in(y_1,\ldots,y_n)^dH_0$, therefore

$$t^{-d}f(y_1,\ldots,y_d)\in H.$$

The ideal J in H generated by all such $t^{-d}f$ $(f\in F)$ is the "proper transform" of I_y, i.e.

$$J=(I_yH_0[t^{-1}])\cap H$$

(cf. [13, p. 216, Lemma 6]). Clearly $J=\sqrt{J}$.

Now γ_0 extends to a map

$$\gamma\colon H\to\mathbb{C}\langle t,z_1,\ldots,z_n\rangle$$

with

$$\gamma(t^{-1}y_i)=z_i \qquad 1\leqq i\leqq n,$$

and the ideal I^* is the extension of J:

$$I^*=\gamma(J)\,\mathbb{C}\langle t,z_1,\ldots,z_n\rangle.$$

To show that $I^* = \sqrt{I^*}$, let M be the maximal ideal

$$M = (t, t^{-1} y_1, \ldots, t^{-1} y_n) H.$$

Then γ extends to a local homomorphism of $(n+1)$-dimensional regular local rings

$$\gamma_M : H_M \to \mathbb{C}\langle t, z_1, \ldots, z_n \rangle.$$

Since $M \mathbb{C}\langle t, z_1, \ldots, z_n \rangle$ is the unique maximal ideal, we see that γ_M extends to an *isomorphism of completions*

$$\hat{\gamma}_M : (H_M)\hat{} \simeq \mathbb{C}[[t, z_1, \ldots, z_n]].$$

Hence

$$I^* = (\hat{\gamma}_M(J)\, \mathbb{C}[[\cdot]]) \cap \mathbb{C}\langle \cdot \rangle;$$

and since H_M is excellent and $J = \sqrt{J}$, therefore $\hat{\gamma}_M(J)\, \mathbb{C}[[\cdot]]$ is a radical ideal, so the same is true of I^*. Q.E.D.

(E.5) (Proof of Fact). Suppose first that O is smooth both on L and on $L \cap S$. Then we can carry out, as follows, the standard reduction to the case $B = \{O\}$:

There is a coordinate system $(t_1, t_2, \ldots, t_m)$ in a neighborhood $V \subseteq U$ of O in $\mathbb{C}^m$ such that $L \cap V$ is given (ideal theoretically) by

$$t_1 = t_2 = \ldots = t_r = 0 \quad (r = \operatorname{codim}. L)$$

and $L \cap S \cap V = B \cap V$ is given (set-theoretically) by

$$t_1 = \ldots = t_r = t_{r+1} = \ldots = t_{r+e} = 0 \quad (e = \operatorname{codim}. S).$$

By [3, p. 483, Proposition 4.5]

$$i(L.S, B) = i((L \cap V).(S \cap V), B \cap V);$$

so we may assume that $V = U$.

Algebraically, we have $I = (t_1, \ldots, t_r)\mathcal{O}_{\mathbb{C}^m, O}$, $p = \sqrt{IR}$, and R/p is a regular local ring of dimension $m - e - r$, with maximal ideal $(t_{r+e+1}, \ldots, t_m)(R/p)$. Algebraic associativity for multiplicity (denoted "μ") [18, p. 81, (24.7)] gives

$$\begin{aligned}\mu((t_1, \ldots, t_r, t_{r+e+1}, \ldots, t_m) R) &= \mu((t_1, \ldots, t_r) R_p)\, \mu((t_{r+e+1}, \ldots, t_m)(R/p)) \\ &= \mu(IR_p).\end{aligned}$$

If M is the submanifold of U given by $t_{r+e+1} = \ldots = t_m = 0$, then topological associativity [3, p. 484, 4.7] gives

$$i(M.L, M \cap L)\, i((M \cap L).S, O) = i(L.S, B)\, i(M.B, O)$$

that is (cf. (A.6.3))

$$i((M \cap L).S, O) = i(L.S, B).$$

So it suffices to show that

$$\mu((t_1, \ldots, t_r, t_{r+e+1}, \ldots, t_m) R) = i((M \cap L.S, O).$$

Thus we may indeed assume that $B = \{O\}$; and this case can be settled by [3, p. 485, 4.10] and the references mentioned in (E.3) above.

To reduce the general case to the preceding one, note that the points of B which are not smooth on both L and $L \cap S$ form an analytic subvariety of B, which is nowhere dense in B (since B is irreducible and contains, by assumption, a smooth point of L). So we can choose $s \in B$ at which L and $L \cap S$ are smooth, and set $R' = \mathcal{O}_{S,s}$. Let $\mathcal{I}$ (resp. $\mathcal{P}$) be the $\mathcal{O}_{\mathbb{C}^m}$-ideal of functions vanishing on L (resp. B). Then $p' = \mathcal{P} R'$ is a prime ideal in R' (since B is smooth at s), and it will suffice to show that the ideals IR_p, $\mathcal{I} R'_{p'}$ have the same multiplicity. (The multiplicity of $\mathcal{I} R'_{p'}$ is $i(L.S, B)$ by the preceding case.)

In fact we will show more, namely that the Hilbert-Samuel functions

$$\lambda(n) = \text{length of the } R_p\text{-module } I^n R_p / I^{n+1} R_p$$

and

$$\lambda'(n) = \text{length of the } R'_{p'}\text{-module } \mathscr{I}^n R'_{p'} / \mathscr{I}^{n+1} R'_{p'}$$

coincide.

For this purpose, observe that if $\bar{I} = IR$ and

$$E^{a,n} = [p^a(\bar{I}^n/I^{n+1})]/[p^{a+1}(\bar{I}^n/\bar{I}^{n+1})]$$

then

$$\lambda(n) = \sum_{a=0}^{\infty} \dim_{k(p)}(E^{a,n} \otimes_R R_p) = (\text{say}) \sum_{a=0}^{\infty} e^{a,n}$$

where "$\dim_{k(p)}$" denotes vector-space dimension over the field $k(p) = R_p/pR_p$. (Note that $e^{a,n} = 0$ if $p^a R_p \subseteq I R_p$, so the sum makes sense). Now if $\bar{\mathscr{I}} = \mathscr{I} \mathscr{O}_S$ and $\mathscr{E}^{a,n}$ is the coherent $\mathscr{O}_B$-module

$$\mathscr{E}^{a,n} = [\mathscr{P}^a(\bar{\mathscr{I}}^n/\bar{\mathscr{I}}^{n+1})]/[\mathscr{P}^{a+1}(\bar{\mathscr{I}}^n/\bar{\mathscr{I}}^{n+1})]$$

then $E^{a,n}$ is the stalk $\mathscr{E}^{a,n}_0$ and from [15, p. 20-10, Prop. 6] we see that outside a nowhere dense subvariety of B, $\mathscr{E}^{a,n}$ is locally free of rank $e^{a,n}$. Thus for fixed a, n, $e^{a,n}$ depends only on $\mathscr{P}$ and $\bar{\mathscr{I}}$; and a similar argument with s in place of O gives

$$\lambda'(n) = \sum_{a=0}^{\infty} e^{a,n}.$$

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