DIFFERENTIAL INVARIANCE OF MULTIPLICITY

YIH-NAN GAU AND JOSEPH LIPMAN[1]

An *analytic variety* A in an open set $U \subset C^n$ is a (closed) subset of U whose intersection with some open neighborhood of each point of U is the set of common zeros of a finite number of functions defined and holomorphic in that neighborhood. The *multiplicity* of A at $P \in A$ is defined to be the local covering number of a generic complex projection from A to C^d ($d = \dim A$). (This is the same as the multiplicity of the local ring $\mathcal{O}_{A,P}$ of germs of holomorphic functions on A at P.)

Our result is the following (O = origin).

THEOREM. *Let U, V be open subsets of C^n, and $A \ni O$ (resp. $B \ni O$) be an analytic variety in U (resp. V). Suppose there is a homeomorphism α from U to V such that $\alpha(A) = B$, $\alpha(O) = O$ and both α and α^{-1} (as functions from subsets of R^{2n} to R^{2n}) have a derivative (= linear approximation) at O. Then A and B have the same multiplicity at O:* $\operatorname{mult}(A) = \operatorname{mult}(B)$. *In short, "weakly" diffeomorphic germs have the same multiplicity.*

REMARKS. 1. The above assertion is wrong if we drop the differentiability assumption. (E.g. if u: $C^2 \to C$ is a continuous map with $u(t^2, t^3) = t$ for all $t \in C$, then α: $C^3 \to C^3$ defined by

$$(x, y, z) \to \left(x - u\left(y + x^2, z + x^3\right), y + x^2, z + x^3\right)$$

is a homeomorphism which takes the line $\{(t, 0, 0)\}$ to the curve $\{(0, t^2, t^3)\}$ which has a cusp at O.)

2. The hypersurface case of the theorem was proved by Ephraim [**2**].

3. If we add the "hypersurface" condition but drop the "derivative" condition then we have the Zariski conjecture: multiplicity is an embedded topological invariant [**4**].

1980 *Mathematics Subject Classification*. Primary 14B05, 32C40.

[1]Supported by NSF grant MCS-7903444.

0082-0717/82/0000-0631/$02.00

4. In the proof we can shrink U, A (and V, B). This is because multiplicity is a local invariant. Also we can take A, B to be irreducible because of the following useful lemma.

LEMMA. *Let $f\colon A \to B$ be a homeomorphism between analytic varieties (or, more generally, analytic spaces). Then for every component A_* of A, $f(A_*)$ is a component of B with the same dimension ("component" always means "irreducible component").*

The idea of the proof of the theorem is to use the Zariski tangent cones $C(A, O), C(B, O)$ of A and B at O as intermediaries. First we note that the derivative of α at O, denoted by α_0, maps $C(A, O)$ homeomorphically onto $C(B, O)$ and so by the lemma takes each component A_i of $C(A, O)$ to some component, say B_i, of $C(B, O)$. Now since α_0 is real analytic (being real linear), it follows from a structure theorem proved by Ephraim [3] that $\operatorname{mult}(A_i) = \operatorname{mult}(B_i)$.

The strategy is to relate $\operatorname{mult}(A)$ to the $\operatorname{mult}(A_i)$'s. It turns out that $\operatorname{mult}(A)$ is a linear combination of the $\operatorname{mult}(A_i)$: $\operatorname{mult}(A) = \Sigma_i e_i \operatorname{mult}(A_i)$ for some positive integers e_i. The theorem is then proved by showing $e_i = f_i$, where the f_i are defined similarly for B so that $\operatorname{mult}(B) = \Sigma f_i \operatorname{mult}(B_i)$. (In the hypersurface case, the e_i are the exponents that appear in the factorization of the initial form of the defining equation. In this case, multiplicity is the degree of the initial form and $\operatorname{mult}(A) = \Sigma e_i \operatorname{mult}(A_i)$ follows immediately.)

In general, the coefficients e_i are defined through a set $S \subset C \times C^n$ which is used by Draper in his analytic intersection theory. S is constructed as follows. Consider the union $\bigcup_{t \in C^*}(t \times (1/t)A)$, an analytic variety in $\bigcup_{t \in C^*}(t \times (1/t)U)$. The set S is then defined to be the closure of $\bigcup_{t \in C^*}(t \times (1/t)A)$ in

$$\mathbf{U} := \bigcup_{t \in C^*} (t \times (1/t)U) \cup (0 \times C^n).$$

It can be checked that S is an analytic variety (which is irreducible if A is) in $\mathbf{U}$ and that $S \cap (1 \times C^n) = 1 \times A$, $S \cap (0 \times C^n) = 0 \times C(A, O)$. (Sketch of proof: if A is defined by $\{g_i = 0\}$, then $t \times (1/t)A$ is defined by $\{g_i(tx) = 0\}$ or equivalently $\{t^{-d_i} g_i(tx) = 0\}$. Hence $S \cap (0 \times C^n)$ is defined by $\{\lim_{t \to 0} t^{-d_i} g_i(tx) = \text{leading form of } g_i = 0\}$.) Let T denote the corresponding set for B.

We then apply the intersection theory in Borel-Moore homology [1] to define e_i (and f_i). The following properties are needed (among other things):

1. For a d-dim (complex) analytic variety A, $H_{2d}(A) \simeq \Pi_i H_{2d}(A_i) \simeq \Pi_i Z$, where the A_i are the irreducible components of A with $\dim A_i = d$. For A irreducible the fundamental class of A, denoted by $[A]$, is a generator of $H_{2d}(A) \simeq Z$.

2. If L is an open subset of a linear space of real dimension d, then $H_d(L) \simeq Z$ and is generated by the fundamental class which is denoted by $[L]$.

3. The intersection theory for homology classes has the usual associativity, anticommutativity, unit multiplicity and projection properties.

4. Topological invariance of intersection: If h is a homeomorphism, then $h_*(\xi \cdot \eta) = \pm h_*\xi \cdot h_*\eta$, where "$\cdot$" is the intersection product.

Consider now $[S] \cdot [0 \times C^n] \in H_{2d}(0 \times C(A, O))$. (Recall that $S \cap (0 \times C^n) = 0 \times C(A, O)$.) Since $H_{2d}(0 \times C(A, O)) \simeq \Pi_i H_{2d}(0 \times A_i)$ and $H_{2d}(0 \times A_i) \simeq Z$ is generated by $[0 \times A_i]$, we can write $[S] \cdot [0 \times C^n] = \Sigma_i e_i [0 \times A_i]$ for some unique integers e_i, and similarly

$$[T] \cdot [0 \times C^n] = \sum_i f_i [0 \times B_i].$$

It is a fact that both e_i and f_i are positive. As mentioned earlier, to complete the proof it suffices to show:

1°. $\text{mult}(A) = \Sigma_i e_i \,\text{mult}(A_i)$ and $\text{mult}(B) = \Sigma_i f_i \,\text{mult}(B_i)$,

2°. $e_i = f_i$ for all i.

To prove 1° we take a complex linear space L of $\dim(n - d)$ which is transversal to A (meaning $L \cap C(A, O) = \{O\}$). Then it is well known that $\text{mult}(A) = [L] \circ [A]$ (the intersection number at 0). Applying the usual properties of intersection product, it can be shown that $[S] \circ [t \times L]$ is independent of t, and that $[S] \circ [1 \times L] = \text{mult}(A)$, $[S] \circ [0 \times L] = \Sigma e_i \,\text{mult}(A_i)$. (Here we need the fact that S intersects $1 \times C^n$ transversally along $1 \times A$.)

As for 2° we can construct a homeomorphism between

$$\bar{\mathbf{U}} := \mathbf{U} \cap (R \times C^n) = \bigcup_{t \in R^*} (t \times (1/t)U) \cup (0 \times C^n)$$

and

$$\bar{\mathbf{V}} := \mathbf{V} \cap (R \times C^n) = \bigcup_{t \in R^*} (t \times (1/t)V) \cup (0 \times C^n)$$

which takes $1 \times A$ to $1 \times B$ and $0 \times A_i$ to $0 \times B_i$. (We cannot construct a homeomorphism between $\mathbf{U}$ and $\mathbf{V}$; however, it is easy to see that $\mathbf{U}^* = \mathbf{U} - (0 \times C^n)$ and $\mathbf{V}^* = \mathbf{V} - (0 \times C^n)$ are homeomorphic. This is needed later.) So we define $[\bar{S}] = [S] \cdot [R \times C^n]$, $[\bar{T}] = [T] \cdot [R \times C^n]$. Then on the one hand

$$h_*([\bar{S}] \cdot [0 \times C^n]) = h_*\left(\sum e_i [0 \times A_i]\right) = \sum e_i h_* [0 \times A_i] = \sum \pm e_i [0 \times B_i],$$

and on the other hand (using topological invariance of intersection),

$$\begin{aligned} h_*([\bar{S}] \cdot [0 \times C^n]) &= h_*[\bar{S}] \cdot h_*[0 \times C^n] \\ &= \pm [\bar{T}] \cdot [0 \times C^n] = \pm \sum f_i [0 \times B_i]. \end{aligned}$$

(We need "$\mathbf{U}^*$ is homeomorphic to $\mathbf{V}^*$" to prove the second equality.) Hence $e_i = \pm f_i$. Q.E.D.

REMARK. It can be shown that e_i is the length of the Artin local ring G_{P_i} where G is the graded ring $\oplus_{n \geq 0} \mathfrak{m}^n_{A,O} / \mathfrak{m}^{n+1}_{A,O}$ ($\mathfrak{m}_{A,O}$ = maximal ideal of $\mathcal{O}_{A,O}$), and P_i is the minimal prime ideal in G corresponding to A_i. (Note that $C(A, O) = \text{spec}(G)$.) From this we see that $e_i > 0$; and by local algebra, we get another proof of $\text{mult}(A) = \Sigma e_i \,\text{mult}(A_i)$.

REFERENCES

1. A. Borel and A. Haefliger, *La class d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513.
2. R. Ephraim, *C^1 preservation of multiplicity*, Duke Math. J. **43** (1976), 797–803.
3. ______, *The cartesian product structure and C^∞ equivalence of singularities*, Trans. Amer. Math. Soc. **224** (1976), 299–311.
4. O. Zariski, *Some open questions in the theory of singularities*, Bull. Amer. Math. Soc., **77** (1971), 483–491.

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