QUASI-ORDINARY SINGULARITIES OF SURFACES IN $\mathbf{C}^3$

JOSEPH LIPMAN[1]

Contents

§1. Overview
§2. Saturation and local topology
§3. Quadratic and monoidal transforms

1. Overview. Let me begin by saying what quasi-ordinary singularities are, and what they may be good for.

A singular point P on a surface $F \subset \mathbf{C}^3$ is *quasi-ordinary* if there is a finite map of analytic germs $\pi\colon (F, P) \to (\mathbf{C}^2, 0)$ whose discriminant locus Δ (the curve in $\mathbf{C}^2$ over which π ramifies) has a normal crossing at 0 (i.e. 0 is either a smooth point or an ordinary double point of Δ). Given π, we can choose local coordinates x, y, z such that $\pi(x, y, z) = (x, y)$ and such that the germ $(F, P) \subset (\mathbf{C}^3, 0)$ is defined by an equation

$$f(X, Y, Z) = Z^m + g_1(X, Y)Z^{m-1} + \cdots + g_m(X, Y) = 0$$

where the g_i are power series; and "P quasi-ordinary" means that, for some such choice of x, y, z, the *discriminant* $D(X, Y)$ of f (f being considered as a polynomial in Z) is of the form

$$D(X, Y) = X^a Y^b \varepsilon(X, Y), \qquad \varepsilon(0,0) \neq 0.$$

Such singularities arise naturally in the Jungian approach to desingularization, where one begins with a projection of an arbitrary $F \subset \mathbf{C}^3$ into $\mathbf{C}^2$, then applies blowups to the discriminant locus until it has no singularities other than ordinary double points (cf. [$\mathbf{L}_2$, lecture 2] and [$\mathbf{Z}_2$]). In this way, using no more than desingularization of curves, one can modify any surface locally to one with quasi-ordinary singularities. This, first of all, reduces the problem of resolving

1980 *Mathematics Subject Classification.* Primary 14B05, 32C40; Secondary 57Q45.
[1]Partially supported by NSF grant MCS-7903444 at Purdue University.

surface singularities to the quasi-ordinary case; and, secondly, suggests that quasi-ordinary singularities might be useful in analyzing arbitrary surface singularities.

For example, the results on quadratic and monoidal transforms to be given in §3 allow us to describe quite explicitly how to resolve quasi-ordinary singularities by a sequence of transformations in which no singularities other than quasi-ordinary ones appear; and this provides a basis for a proof that a family of germs of surfaces in $\mathbf{C}^3$ which is *equisingular* (in the sense that it has a projection whose discriminant locus is an equisingular family of curves in $\mathbf{C}^2$) admits a *simultaneous embedded resolution* (starting, à la Jung, with a simultaneous resolution of the family of discriminant curves).

Now I will describe the questions to be considered here. (Other questions are not hard to come by: as will soon become apparent, almost any question about plane curve singularities immediately suggests an analogous—and more difficult—one for quasi-ordinary singularities, and not just in the two-dimensional case!) For the most part, proofs, or even complete statements of results, are not given; they can be found in $[\mathbf{L}_1]$.

For simplicity we assume unless otherwise indicated that our singularities are locally irreducible.

A quasi-ordinary singularity P lends itself to detailed analysis, because some neighborhood V of P can be parametrized by a *fractional power series* $\zeta = H(X^{1/n}, Y^{1/n})$; in other words, with suitable local coordinates we have $(x, y, z) \in V \Leftrightarrow z = H(s, t)$ where $s^n = x$ and $t^n = y$. What this amounts to is that if $f = f(Z)$ is as above (and irreducible), with discriminant $D = X^a Y^b \varepsilon$, then the roots of f can be represented in the form

$$(1.1) \qquad \zeta_i = H_i(X^{1/n}, Y^{1/n}) = H\big(\omega_{i1} X^{1/n}, \omega_{i2} Y^{1/n}\big), \qquad 1 \leq i \leq m,$$

with ω_{ij} an n-th root of unity ($j = 1, 2$). This goes back to Jung, the underlying idea being that the local fundamental group of the complement of a plane curve at an ordinary double point is $\mathbf{Z} \times \mathbf{Z}$. (For an algebraic proof cf. $[\mathbf{A}_1$, Theorem 3].) Since

$$X^a Y^b \varepsilon = D = \prod_{i \neq j} \big(H_i(X^{1/n}, Y^{1/n}) - H_j(X^{1/n}, Y^{1/n})\big),$$

unique factorization of (fractional) power series gives

$$(1.2) \quad \zeta_i - \zeta_j = H_i - H_j = X^{u/n} Y^{v/n} \varepsilon_{ij}(X^{1/n}, Y^{1/n}), \qquad \varepsilon_{ij}(0,0) \neq 0.$$

(Here u, v depend on i, j.) The fractional monomials $X^{u/n}Y^{v/n}$ so obtained are the *characteristic monomials* of ζ. The exponents $(u/n, v/n)$ are called the *distinguished pairs* of ζ; they satisfy certain conditions described in the appendix at the end of this section. This parallels exactly the situation for plane curves, where we use fractional (Puiseux) power series and characteristic monomials in one variable X.

In the case of plane curves, two suitably normalized fractional power series in X parametrize *equivalent* singularities if and only if they have the same characteristic monomials [$\mathbf{Z}_4$, Proposition 4.6]. (By "suitably normalized" we mean that the parameter X is chosen to be transversal, so that the denominator n is as small as possible, viz. $n =$ the multiplicity of the singularity.) For a quasi-ordinary singularity P on a surface F, it turns out that quite a lot of the geometry of F near P is determined by the characteristic monomials of a parametrization, for example: the local topology, the tangent cone and the multiplicity at P, the nature of the singular locus, the equivalence class of the singularity of a generic plane curve germ on F at P, or of a generic plane section of F transversal near P to a component of the singular locus. So we may anticipate that the distinguished pairs will provide a good basis for *classifying* quasi-ordinary singularities.

The first problem with this idea is one of uniqueness. Namely, there may be many different possible choices of the projection π, and correspondingly of ζ; and furthermore some projections may have a discriminant with only a normal crossing while the discriminant of the generic projection has a more complicated singularity! (The simplest example of this phenomenon is provided by the surface $Z^3 = XY$.) So we are led to the question which is the central one dealt with in this paper:

(1.3) *For a given quasi-ordinary singularity, are the distinguished pairs of a somehow normalized parametrization ζ as above independent of the possible choices of ζ?*

The answer turns out to be yes. And this is not just a technical matter: approaches to the question which are suggested by the plane curve case lead to algebraic and geometric considerations which add much substance to the idea of classifying quasi-ordinary singularities by distinguished pairs.

For instance, for plane curve singularities one knows, via the classical knot theory of Brauner, Burau, Kähler and Zariski (cf. [**R**]), that *the characteristic monomials of any normalized parametrization are determined by the local topology and vice-versa* (hence the characteristic monomials do not depend on the parametrization). For quasi-ordinary singularities, the distinguished pairs still determine the local topology, but the argument given for this in §2, using saturation, does not give much geometric insight, and does not help much with the converse question. So it would be nice to *find a topological interpretation of the distinguished pairs*. Such an interpretation might well involve an interesting higher-dimensional analogue of the knot theory for curves.

Saturation-theoretic criteria for equivalence of plane curve singularities [$\mathbf{Z}_3$, §2; $\mathbf{Z}_5$, II, §7] motivate one method for getting an affirmative answer to (1.3) for a large class of quasi-ordinary singularities characterized, for example, by the condition $\dim C_4 = 2$, where C_4 is the Whitney cone consisting of "limits of tangents" (cf. beginning of §2). The result for such a singularity P is that *the absolute saturation of the local ring of P determines the distinguished pairs*.

Yet another—and finally successful—approach is suggested by the following well-known result (cf. [$\mathbf{Z}_1$, p. 8] or [$\mathbf{A}_2$, §4]): for an irreducible plane curve singularity Q, the characteristic monomials of any normalized Puiseux parametrization determine and are determined by the sequence of multiplicities of the local rings appearing in the resolution of Q by successive quadratic transformations (where by "resolution..." we mean the sequence of one-dimensional local domains $R_1 < R_2 < \cdots < R_m$ such that R_1 is the local ring of Q, R_{i+1} is obtained from R_i by blowing up the maximal ideal of R_i ($1 \leqslant i < m$), and R_m is regular while R_{m-1} is not). A corresponding result for a quasi-ordinary singularity P should run along the following lines:

There is some natural way of resolving P, say by a sequence of quadratic and "permissible" monoidal transformations (i.e. blowing up points, resp. smooth equimultiple curves), such that certain invariants associated with the local rings appearing in this resolution determine and are determined by the distinguished pairs of any normalized parametrization of P.

In particular, the distinguished pairs are indeed independent of the parametrization.

It *is* possible to establish such a connection between resolutions and distinguished pairs [**L**$_1$, §§5, 6]. The precise statement is too detailed to give here; but let us at least look at some of the problems which arise in carrying through the idea.

First of all, here is what is meant by a "normalized" parametrization of a quasi-ordinary singularity: because of the following lemma, which is a two-dimensional elaboration of the "inversion formula" for plane curve singularities [**Z**$_3$, §3], we can always find a parametrization of the form

$$\zeta = X^{a/n}Y^{b/n}H(X^{1/n}, Y^{1/n}), \qquad H(0,0) \neq 0,$$

where

(i) a and b are not both divisible by n,

(ii) if $a + b < n$, then both $a > 0$ and $b > 0$,

(iii) labelling the distinguished pairs $(\lambda_i, \mu_i)_{1 \leqslant i \leqslant s}$ of ζ so that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_s$ and $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_s$ (cf. Proposition (1.5)) we have $(\lambda_1, \lambda_2, \ldots, \lambda_s) \geqslant (\mu_1, \mu_2, \ldots, \mu_s)$ (lexicographically).

LEMMA (1.4). *Let $\zeta = X^{a/n}H(X^{1/n}, Y^{1/n})$ be a parametrization with $0 < a < n$ and $H(0,0) \neq 0$. Then, by a definite procedure (given in [**L**$_1$, §2]) we can derive from ζ a parametrization of the form $\zeta' = X^{n/a}H'(X^{1/n}, Y^{1/n})$ with $H'(0,0) \neq 0$; and the distinguished pairs of ζ' depend only on those of ζ (cf. Table (3.4)).*

Second, while resolving we want to stay inside the class of quasi-ordinary singularities, so that we can argue by induction on the number of steps in the resolution process. We need then to show that *a quadratic or permissible monoidal transform P' of a quasi-ordinary singularity P is again quasi-ordinary*; and moreover, *from any normalized parametrization ζ for P we can construct one for P' whose distinguished pairs depend only on those of ζ.*

This result is discussed in more detail in §3. For a simple example, blow up the origin on the surface $Z^4 = XY^2$.

Third, which "infinitely near" points shall we consider (where infinitely near points are those which can be obtained from P by a sequence of quadratic and permissible monoidal transformations)? After all, a quadratic transformation produces a whole curve of infinitely near points. And how do we know at any stage of resolution whether to blow up a point or a "permissible" curve (and which curve, if there is more than one)? Roughly speaking, these problems are handled as follows. In the case of quadratic transformations, i.e. blowing up a point Q, it turns out to be sufficient to consider only those finitely many points in the fibre over Q at which the blown-up surface is *not equisingular* along the fibre. As for the choice of what to blow up, what we do is to blow up a curve whenever possible (and a point otherwise), and if more than one curve can be blown up at any stage, we can always choose the one which appeared as the "exceptional curve" (inverse image of the blown-up curve or point) at the previous stage. In this way, we can associate to each quasi-ordinary singularity P a "resolution tree" consisting of finitely many infinitely near points, with branching structure depending only on the distinguished pairs of a normalized parametrization of P. (See below for an illustration of how the singular locus—and in particular the possible curves to be blown up—depends only on the distinguished pairs.)

Finally, which numbers do we attach to the various points in a resolution tree? The numbers we use must be intrinsically associated to the points; and furthermore, given any normalized parametrization for such a point, the numbers in question should be determined by the corresponding distinguished pairs, hence ultimately by the distinguished pairs of some parametrization of the original point P. Conversely, these numbers should give us enough information so that when we look at the entire tree, with its attached numbers, we can reconstruct the original distinguished pairs. Some possibilities which present themselves (as numbers attached to a singularity P) are: the multiplicity, the number of components of the Zariski tangent cone; the number of curves in the singular locus, together with the multiplicities of these curves on the surface, and the multiplicity of P on the curves; the characteristic pairs and intersection numbers of the branches of the plane curve singularities obtained by taking a plane section transversal to the singular locus (away from P), etc. In practice this is more than enough information; and the reconstruction of distinguished pairs from a resolution tree can be carried out, by a very tedious induction based on the transformation formulas of distinguished pairs under quadratic and monoidal transformations (cf. Table (3.4)). Moreover, we do not need the whole tree, but only one specific path through it. Once again, precise statements can be found in [**L**$_1$, §§5, 6]. For a simplified treatment, cf. [**Lu**].

We close this section with some examples to illustrate the relation between distinguished pairs and some geometric invariants (tangent cone, singular locus).

(**A**) *Tangent cone*. If a quasi-ordinary P is represented locally, as before, by an equation

$$f(X, Y, Z) = \prod_{i=1}^{m} (Z - \zeta_i) = 0$$

then the Zariski tangent cone (= Whitney's C_3) is the zero set of the leading form f_0 of f (f_0 is the sum of the monomials in f of lowest total degree). The "algebraic" tangent cone is the Spec of the graded ring $\mathbf{C}[X, Y, Z]/f_0$. One finds that this graded ring is determined up to isomorphism by the smallest distinguished pair of any normalized parametrization. The possibilities are typified by equations of the form $Z^n = X^aY^b$ (with one distinguished pair, viz. $(a/n, b/n)$), for which

$$\begin{aligned} f_0 &= Z^n && \text{if } a+b>n, \\ f_0 &= Z^n - X^aY^b && \text{if } a+b=n, \\ f_0 &= X^aY^b && \text{if } a+b<n. \end{aligned}$$

(We assume that the equation is normalized, so that if $a + b < n$ then both a and b are greater than zero.)

(B) *Singular locus*. (i) Every *normal* quasi-ordinary singularity can be given by an equation of the form $Z^n = XY$.

(ii) There may be one or two smooth curves in the singular locus. For example:

$$\begin{aligned} Z^3 &= X^2Y && (\text{Sing. locus: } [Z=X=0]). \\ Z^5 &= X^2Y^3 && (\text{Sing. locus: } [Z=X=0] \cup [Z=Y=0]). \end{aligned}$$

(iii) The following is typical of the worst that can happen: the singular locus consists of two plane curves, one of which is itself singular. For an example, consider the normalized fractional series

$$\zeta = X^{3/2} + X^{7/4} + X^{9/4}Y^{1/4} + X^{19/8}Y^{11/8}.$$

Multiplying $X^{1/8}, Y^{1/8}$ by various 8th roots of unity, one finds that ζ has 32 distinct conjugates ζ_i; $f(X, Y, Z) = \prod_i(Z - \zeta_i)$ has order 32; and the corresponding singularity has multiplicity 32. The discriminant is

$$\prod_{i \neq j} (\zeta_i - \zeta_j) = X^{1724}Y^{92}\varepsilon(X, Y), \qquad \varepsilon(0,0) \neq 0.$$

Either X or Y vanishes on the singular locus S; and we find that S consists of two curves:

$$\begin{aligned} &C_1: X = Z = 0 \qquad (\text{32-fold line}), \\ &C_2: Y = 0, Z = X^{3/2} + X^{7/4} \end{aligned}$$

(2-fold curve, having itself a 4-fold point at the origin).

Exercise. What happens to the multiplicity of C_2 if we replace the term $X^{9/4}Y^{1/4}$ in ζ by $X^{9/4}Y^{3/4}$? by $X^{9/4}Y^{5/4}$? (Answers (cf. [$\mathbf{L}_1$, p. 39]): 6, 8.)

Appendix. *Characterization of distinguished pairs*. The following description of distinguished pairs turns out to be quite useful. If (λ, μ), (σ, τ) are ordered pairs of rational numbers, we write $(\lambda, \mu) \leqslant (\sigma, \tau)$ to signify $\lambda \leqslant \sigma$ and $\mu \leqslant \tau$. We write $(\lambda, \mu) < (\sigma, \tau)$ if $(\lambda, \mu) \leqslant (\sigma, \tau)$ and $(\lambda, \mu) \neq (\sigma, \tau)$. Also, we let Γ_n be the set of non-negative rational numbers α such that $n\alpha$ is an integer. As usual, $\mathbf{Z}$ denotes the set of all integers.

PROPOSITION (1.5). *Let $\zeta = \sum_{(\alpha,\beta)\in\Gamma_n\times\Gamma_n} c_{\alpha\beta}X^\alpha Y^\beta$ be a fractional power series. Then ζ parametrizes some quasi-ordinary singularity if and only if there exist pairs $(\lambda_i, \mu_i) \in \Gamma_n \times \Gamma_n$ $(1 \le i \le s)$ such that*

(1) $(0,0) < (\lambda_1, \mu_1) < (\lambda_2, \mu_2) < \cdots < (\lambda_s, \mu_s)$,

(2) $c_{\lambda_i\mu_i} \neq 0$ *for* $1 \le i \le s$,

(3) *if* $c_{\lambda\mu} \neq 0$ *then* $(\lambda, \mu) \in \mathbf{Z} \times \mathbf{Z} + \sum_{(\lambda_i,\mu_i)\le(\lambda,\mu)} \mathbf{Z}(\lambda_i, \mu_i)$,

(4) $(\lambda_j, \mu_j) \notin \mathbf{Z} \times \mathbf{Z} + \sum_{(\lambda_i,\mu_i)<(\lambda_j,\mu_j)} \mathbf{Z}(\lambda_i, \mu_i)$ $(1 \le j \le s)$.

If such pairs exist, they are uniquely determined by ζ; in fact they are the distinguished pairs of ζ.

2. Saturation and local topology. In the sequel, $\mathbf{C}\langle \cdots \rangle$ denotes a ring of convergent power series with complex coefficients. We say that

$$\zeta = H(X^{1/n}, Y^{1/n}) \in \mathbf{C}\langle X^{1/n}, Y^{1/n} \rangle$$

is a *quasi-ordinary branch* if ζ parametrizes a quasi-ordinary singularity, i.e. if for any two distinct conjugates $\zeta_i \neq \zeta_j$ of ζ (cf. (1.1)) we have (as in (1.2))

$$\zeta_i - \zeta_j = X^{u/n}Y^{v/n}\varepsilon_{ij}(X^{1/n}, Y^{1/n}), \qquad \varepsilon_{ij}(0,0) \neq 0.$$

The distinguished pairs (λ_i, μ_i) of a quasi-ordinary branch ζ determine the $\mathbf{C}\langle X, Y\rangle$-saturation of $\mathbf{C}\langle X, Y, \zeta\rangle$, and hence [$\mathbf{Z}_3$, *Theorem* 6.1] *determine the local topology of the corresponding analytic germ together with its embedding in* $\mathbf{C}^3$.

Indeed, as is readily checked, the saturation in question is the $\mathbf{C}\langle X, Y\rangle$-algebra generated by all the monomials $X^\lambda Y^\mu$ for which (λ, μ) satisfies

$$(0,0) \le (\lambda, \mu) \in \mathbf{Z} \times \mathbf{Z} + \sum_{(\lambda_i,\mu_i)\le(\lambda,\mu)} \mathbf{Z}(\lambda_i, \mu_i) \tag{2.1}$$

(cf. Proposition (1.5) for notation).

In case $(1,1) \le (\lambda_1, \mu_1)$, the above saturation is the *absolute saturation* $\tilde{A}$ of $A = \mathbf{C}\langle X, Y, \zeta\rangle$ (cf. [$\mathbf{Z}_5$, III, paragraph preceding Proposition 3.6]). Now $\tilde{A}$ depends only on A (a nontrivial fact!); and, I claim, *the distinguished pairs of ζ can be recovered from $\tilde{A}$*, hence do not depend on the choice of ζ.

To justify this claim, note first that the condition $(1,1) \le (\lambda_1, \mu_1)$ is intrinsic to A: it is in fact equivalent to the condition that the singular locus of Spec(A) consists of two nonsingular m-fold curves intersecting transversally, where m is the multiplicity of A (cf. example B(ii) in §1). (It is also equivalent to the Whitney cone C_4 for the corresponding singularity P having dimension 2; or to *every* transversal projection of a neighborhood of P to $(\mathbf{C}^2, 0)$ having a normal-crossing discriminant, where "transversal" means "with local degree equal to the multiplicity of P".) Now since saturation commutes with localization [$\mathbf{Z}_5$, III, Proposition 1.2], and since any saturation of a one-dimensional local domain is a local domain with the same multiplicity [$\mathbf{Z}_5$, I, Proposition 2.5], we see that the singular locus of Spec($\tilde{A}$) consists of two m-fold curves (m is also the multiplicity of $\tilde{A}$, cf. [$\mathbf{Z}_5$, III, Theorem 4.1]). Let ν_1, ν_2 be the discrete valuations belonging to the integral closures of the local rings of these curves. (The integral closures are discrete valuation rings because

$$\mathbf{C}\langle X, Y\rangle \subset \tilde{A} \subset \mathbf{C}\langle X^{1/n}, Y^{1/n}\rangle$$

and because XY vanishes on the singular locus.) Then, with suitable labelling,

$$\nu_1(X) = \nu_2(Y) = m, \quad \nu_2(X) = \nu_1(Y) = 0;$$

and from Proposition (1.5) and (2.1) we find that if

$$S(\tilde{A}) = \{m^{-1}(\nu_1(\alpha), \nu_2(\alpha)) \mid \alpha \in \tilde{A}\}$$

then (λ_1, μ_1) is the smallest pair in $S(\tilde{A}) - (\mathbf{Z} \times \mathbf{Z})$, (λ_2, μ_2) is the smallest pair in $S(\tilde{A}) - (\mathbf{Z} \times \mathbf{Z}) - \mathbf{Z}(\lambda_1, \mu_1)$ and, in general, for $1 \leqslant j \leqslant s$, (λ_j, μ_j) is the smallest pair in $S(\tilde{A}) - (\mathbf{Z} \times \mathbf{Z}) - \Sigma_{i<j}\mathbf{Z}(\lambda_i, \mu_i)$. The claim is thereby proved.

In general, when there is no condition on (λ_1, μ_1), we know for other reasons (indicated in §1) that the distinguished pairs of a normalized ζ depend only on $A = \mathbf{C}\langle X, Y, \zeta\rangle$, so that the $\mathbf{C}\langle X, Y\rangle$-saturation A^* described in (2.1) is still determined up to isomorphism by A (independently of the choice of X, Y, ζ). But A^* may be strictly larger than $\tilde{A}$. It would be interesting to have a more direct intrinsic description of A^*. This could lead again to an affirmative answer to (1.3).

Of course one would like a more explicit description of the local topology, as one has via knots for plane curve singularities (cf. **[R]**).

Conversely, it would be nice to know *whether or not the local topology determines the distinguished pairs*.

(I do not know whether the local topology determines even the multiplicity.)

In attempting to understand the local topology of a surface $F \subset \mathbf{C}^3$ having a quasi-ordinary singularity at the origin 0, one thinks first of the intersection of F with a small 5-sphere S^5 centered at 0. Since the singularity at 0 is usually not isolated (example (B)(i), §1), $F \cap S^5$ may not be a manifold. It might then be better to look at the normalization $\bar{F} \to F$, induced say by a linear projection $\bar{F} \subseteq \mathbf{C}^r \xrightarrow{\lambda} \mathbf{C}^3$, and the resulting map

$$L = \bar{F} \cap \lambda^{-1}(S^5) \cap B \to S^5$$

where B is a suitable neighborhood of 0 in $\mathbf{C}^r$. The link L of $\bar{F}$ is a manifold obtained from a 3-sphere by factoring out a finite cyclic group action, i.e. L is a "lens space". (This corresponds to the fact that the origin on $\bar{F}$ is a quotient singularity: for suitable n the fraction field of $\mathbf{C}\langle X^{1/n}, Y^{1/n}\rangle$ is a cyclic Galois extension of that of $\mathbf{C}\langle X, Y, \zeta\rangle$.)

This raises the question, which seems to me worthwhile pursuing, and perhaps not too difficult for knowledgeable topologists:

Can one relate the distinguished pairs of ζ to some homotopy invariants of the above map $L \to S^5$?

3. Quadratic and monoidal transforms. We say for convenience that a local ring A is quasi-ordinary if $A \cong \mathbf{C}\langle X, Y, \zeta\rangle$ with ζ a quasi-ordinary branch (cf. §2), i.e. A is the analytic local ring of some quasi-ordinary singularity. (We could just as well work with *formal* power series over any algebraically closed field of characteristic zero.)

Let A be a quasi-ordinary local ring, and let $t: T \to \operatorname{Spec}(A)$ be the quadratic transformation, i.e. the map obtained by blowing up the maximal ideal $\mathfrak{m}$ of A. After choosing three generators X, Y, ζ of $\mathfrak{m}$, we have an embedding of the closed fibre $t^{-1}(\mathfrak{m})$ into the projective plane $\mathbf{P}^2_{\mathbf{C}}$. By the *quadratic transform of A in the direction* $(\alpha:\beta:\gamma)$ we mean the analytic local ring of the closed point on T corresponding to $(\alpha, \beta, \gamma) \in t^{-1}(\mathfrak{m}) \subsetneq \mathbf{P}^2_{\mathbf{C}}$. (The reader may prefer to rephrase all this entirely in terms of local analytic geometry.)

A *monoidal transform* of A is a local ring A' obtained by blowing up a *permissible curve*, i.e. a height one prime ideal p in A such that A and A_p have the same multiplicity. If $A = \mathbf{C}\langle X, Y, \zeta\rangle$ with a normalized quasi-ordinary branch ζ having distinguished pairs $(\lambda_1, \mu_1) < (\lambda_2, \mu_2) < \cdots$, we find that if $\lambda_1 \geqslant 1$ then p may be the ideal (X, ζ), if $\mu_1 \geqslant 1$ then p may be the ideal (Y, ζ), and there are no other possibilities for p. We see then that A' is a finite A-module, uniquely determined by p.

Definition (3.1). *Let A be a quasi-ordinary local ring. We say that A' is a* special transform *of A if A' is not a regular local ring and if one of the following conditions holds*:

(1) A has a permissible curve p, and A' is the corresponding monoidal transform of A.

(2) A has no permissible curve, A' is a quadratic transform of A, and there is a curve q in the singular locus of A whose proper transform passes through A' (i.e. some prime ideal in A' contracts to q).

(3) A has no permissible curve, and A' is a quadratic transform of A in a direction corresponding to a singular point of the (reduced) closed fibre of the quadratic transformation of $\operatorname{Spec}(A)$.

When A is represented by a normalized quasi-ordinary branch, then it is easily verified that any special quadratic transform of A must occur in one of the directions (1:0:0), (0:1:0), (0:0:1).

Remark (3.2). It can be shown that a quadratic transform A' of A is nonspecial if and only if the blow-up T is *equisingular* along the closed fibre at the point whose analytic local ring is A'.

Theorem (3.3). *Let A be a quasi-ordinary local ring. Any special transform A' of A is again a quasi-ordinary local ring. If ζ is a normalized branch representing A, i.e. $A \cong \mathbf{C}\langle X, Y, \zeta\rangle$, then, by one of the processes given in* [$\mathbf{L}_1$, §3], *we can find a "standard" quasi-ordinary branch ζ' (not necessarily normalized) which represents A', and whose distinguished pairs depend only on those of ζ and on the process employed, the exact nature of the dependence being as in Table* 3.4.

TABLE 3.4

Transformation	*Distinguished Pairs of Resulting Branch* (omit $i = 1$ if the corresponding pair consists of integers)
LEMMA (1.4)	$(\lambda_i + 1 - \lambda_1)/\lambda_1, \mu_i$
MONOIDAL TRANSFORMATION	
Center (X, ζ)	$\lambda_i - 1, \mu_i$
Center (Y, ζ)	$\lambda_i, \mu_i - 1$
QUADRATIC TRANSFORMATION	
"Transversal Case" $(\lambda_1 + \mu_1 \geq 1)$	
Direction (1:0:0)	$\lambda_i + \mu_i - 1, \mu_i$
Direction (0:1:0)	$\lambda_i, \lambda_i + \mu_i - 1$
"Non-Transversal Case" $(\lambda_1 + \mu_1 < 1)$	
Direction (1:0:0)	$\lambda_i + [(1 + \mu_i)(1 - \lambda_1)/\mu_1] - 2, [(1 + \mu_i)/\mu_1] - 1$
Direction (0:1:0)	$\mu_i + [(1 + \lambda_i)(1 - \mu_1)/\lambda_1] - 2, [(1 + \lambda_i)/\lambda_1] - 1$
Direction (0:0:1)	$(\lambda_i(1 - \mu_1) + \mu_i\lambda_1)/1 - \lambda_1 - \mu_1, (\lambda_i\mu_1 + \mu_i(1 - \lambda_1))/1 - \lambda_1 - \mu_1$

We have given the distinguished pairs only for "special" directions in the case of quadratic transformations. This information can always be used to give us the distinguished pairs for non-special directions, cf. [$\mathbf{L}_1$, §4].

To illustrate what is involved in Theorem (3.3), let us examine more closely the quadratic transform A' in a direction $(1{:}0{:}\gamma)$, in case A is represented by a normalized quasi-ordinary branch

$$\zeta = X^{u/n}Y^{v/n}H(X^{1/n}, Y^{1/n}), \qquad H(0,0) \neq 0,$$

where u, v, n are integers with $u + v < n$ ("non-transversal case"). Let $f(X, Y, Z)$ be the minimum polynomial of ζ over $\mathbf{C}\langle X, Y\rangle$, of degree, say, m. Then $A' = \mathbf{C}\langle X, Y, Z\rangle / f'$ where

$$f'(X, Y, Z) = [f(X, XY, X(Z + \gamma))]/X^{(mu+mv)/n}.$$

It is clear that $f'(0, Y, 0) = Y^{mv/n} \cdot$(unit in $\mathbf{C}\langle Y\rangle$). Thus, by the Weierstrass preparation theorem, there is a *unique* power series $g'(X, Y, Z)$ such that $g' \in \mathbf{C}\langle X, Z\rangle[Y]$ is a monic polynomial of degree mv/n in Y, and such that $g' = f' \cdot$(unit in $\mathbf{C}\langle X, Y, Z\rangle$); g' is called the distinguished polynomial in Y associated with f'. Since $\mathbf{C}\langle X, Y, Z\rangle/(f') = \mathbf{C}\langle X, Y, Z\rangle/(g')$, it will be sufficient for our purpose to study the roots of g'.

Let $G(Z^{1/v})$, $G_i(X^{1/n}, Y^{1/n})$ be such that

$$[G(Z^{1/v})]^v = Z + \gamma, \qquad [G_i(X^{1/n}, Y^{1/n})]^v = H_i(X^{1/n}, Y^{1/n}).$$

Let $\xi = G(Z^{1/v})$, $\xi_i = G_i(X^{1/n}, X^{1/n}Y^{1/n})$. Then

$$f'(X, Y, Z) = \prod_{i=1}^{m} \left\{ \left[X^{(n-u-v)/nv}\xi\right]^v - \left[Y^{1/n}\xi_i\right]^v \right\}$$

$$= \pm \prod_{i=1}^{m} \prod_{j=1}^{v} \left\{ \omega_j X^{(n-u-v)/nv}\xi - Y^{1/n}\xi_i \right\},$$

where ω_j runs through the vth roots of unity.

Let W be an indeterminate and let $E_i(X, Y, W)$ be such that $E_i(0,0,0) \neq 0$, and

$$E_i(X, Y, W)(W - YG_i(X, XY)) = Y - W\overline{G}_i(X, W), \qquad \overline{G}_i(0,0) \neq 0.$$

Since $G_i(0,0) \neq 0$, the existence of E_i is guaranteed by the preparation theorem. Setting

$$\varepsilon_i = E_i\left(X^{1/n}, Y^{1/n}, \omega_j X^{(n-u-v)/nv}\xi\right)$$

and

$$\bar{\xi}_i = \overline{G}_i\left(X^{1/n}, \omega_j X^{(n-u-v)/nv}\xi\right)$$

we have

$$\varepsilon_i\left(\omega_j X^{(n-u-v)/nv}\xi - Y^{1/n}\xi_i\right) = Y^{1/n} - \omega_j X^{(n-u-v)/nv}\xi\bar{\xi}_i.$$

Hence for some unit ε in $\mathbf{C}\langle X^{1/nv}, Y^{1/n}, Z^{1/v}\rangle$ we have

$$f'(X, Y, Z) = \varepsilon \prod_{i=1}^{m} \prod_{j=1}^{v} \left\{Y^{1/n} - \omega_j X^{(n-u-v)/nv}\xi\bar{\xi}_i\right\}.$$

The double product on the right is clearly the distinguished polynomial in $Y^{1/n}$ associated with $f'(X, Y, Z)$ when $f'(X, Y, Z)$ is thought of as an element of $\mathbf{C}\langle X^{1/nv}, Y^{1/n}, Z^{1/v}\rangle$; but so also is $g'(X, Y, Z)$. By uniqueness we must have

$$g'(X, Y, Z) = \prod_{i=1}^{m} \prod_{j=1}^{v} \left\{Y^{1/n} - \omega_j x^{(n-u-v)/nv}\xi\bar{\xi}_i\right\}.$$

It follows that the roots of g' (considered as a polynomial in Y over $\mathbf{C}\langle X, Z\rangle$) are the n-th powers of the fractional power series $\omega_j X^{(n-u-v)/nv}\xi\bar{\xi}_i$. Thus the roots of g' are fractional power series which are non-units.

Now we have to show that the roots of g' are quasi-ordinary branches, with distinguished pairs depending only on those of ζ, and on whether or not $\gamma = 0$. (For uniformity of notation, we can interchange Y and Z.) For this we study the behavior of the semigroup generated by the pairs of exponents of the non-zero terms in a power series under the operations used above (extraction of v-th roots, passage to associated distinguished polynomial...); and then we apply Proposition (1.5). After obtaining Table 3.4, we can calculate the number of conjugates of ζ' (a root of g'), and compare this number with the degree mv/n of g', to conclude that g' is irreducible when $\gamma = 0$ (g' need not be irreducible if $\gamma \neq 0$). Details are in [$\mathbf{L}_1$].

References

[$\mathbf{A}_1$] S. S. Abhyankar, *On the ramification of algebraic functions*, Amer. J. Math. 77 (1955), 575–592.

[$\mathbf{A}_2$] ______, *Singularities of algebraic curves*, Analytic Methods in Mathematical Physics, Conf. Proc., Gordon & Breach, New York, 1970, pp. 3–14.

[$\mathbf{L}_1$] J. Lipman, *Quasi-ordinary singularities of embedded surfaces*, Thesis, Harvard University, 1965.

[$\mathbf{L}_2$] ______, *Introduction to resolution of singularities*, Algebraic Geometry, Proc. Sympos. Pure Math., vol. 29, Amer. Math. Soc., Providence, R.I., 1975, pp. 187–230.

[Lu] I. Luengo, *On the structure of embedded algebroid surfaces*, these Proceedings.

[R] J. E. Reeve, *A summary of results in the topological classification of plane algebroid singularities*, Rend. Sem. Mat. Univ. e Politec: Torino **14** (1954–55), 159–187.

[Z_1] O. Zariski, *Algebraic surfaces*, second ed., Ergebnisse der Math., vol. 61, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

[Z_2] ———, *Exceptional singularities of an algebroid surface and their reduction*, Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Ser. VIII, **43** (1967), 135–146; reprinted in Oscar Zariski, *Collected papers*, vol. 1, MIT Press, Cambridge, pp. 532–543.

[Z_3] ———, *Studies in equisingularity*. III. *Saturation of local rings and equisingularity*, Amer. J. Math. **90** (1968), 961–1023; reprinted in *Collected papers*, vol. 4, pp. 96–158.

[Z_4] ———, *Contributions to the problem of equisingularity*, Centro Internazionale Matematico Estivo (C.I.M.E.), Questions on Algebraic Varieties. III ciclo, Varenna, 7–17 Settembre 1969, Edizioni Cremonese, Roma, 1970, pp. 261–343; reprinted in *Collected papers*, vol. 4, pp. 159–237.

[Z_5] ———, *General theory of saturation and of saturated local rings*. I, Amer. J. Math. **93** (1971), 573–648; II, Amer. J. Math. **93** (1971), 872–964; III, Amer. J. Math. **97** (1975), 415–502; all reprinted in *Collected papers*, vol. 4.

PURDUE UNIVERSITY