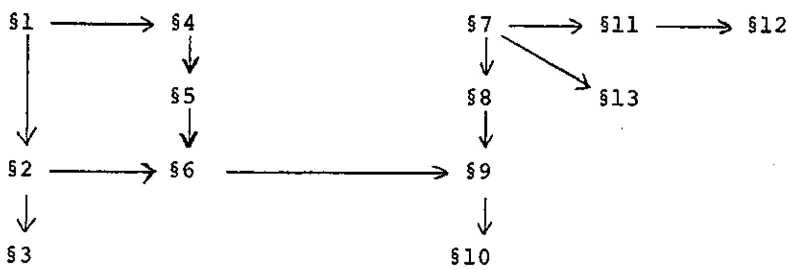
CONTENTS

Leitfaden

§0



INTRODUCTION

This is a semi-expository account about the role of differential forms and residues in the duality theory of algebraic varieties over a perfect field. The main results are summarized in the Residue Theorem (0.6) stated near the end of §0, and generalized in §10.

In some sense there is little here which cannot be dug out from other sources: the basic ideas involved were announced by Grothendieck in [G2]; the foundations of duality theory were then worked out from a very general point of view (derived categories dualizing complexes, etc.) by Hartshorne [RD], Deligne [RD, Appendix] and Verdier [V]; and the "fundamental class", a canonical map from differential forms to dualizing complexes, was studied by El Zein and Angéniol [E]. But, fundamental, powerful, and beautiful as the resulting theory is, the formalism in which it is ensconced and some lack of detail in the literature[(1)] have prevented it from becoming as well-known and understood as it should be. For me at any rate, reaching even the level of understanding represented by these notes has been a long and arduous process. And the reactions of audiences to lectures which I have given over the past twelve years on this subject have suggested that an exposition in the spirit of [S, pp. 76-81] (case of curves, after Rosenlicht) and [K2] (case of projective Cohen-Macaulay varieties)- i.e. accessible in principle to someone familiar with, say, Chapter III of [H]- may not be superfluous.

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Various forms of the main results to be presented have appeared in the literature. In this Introduction, and in §0, we gather some variants together and point out their interconnections. In the process indications about the contents of the paper will emerge.

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(1) Of course in the writing of any exposition (this one included) the choices about which details to include and which to leave to the reader are a matter of taste, judgement, mood,...

Throughout V will be a d-dimensional variety (reduced and irreducible) over a fixed perfect field k (V and d may vary). We set

$$\Omega_V = \Lambda^d(\Omega^1_{V/k})$$

the d (= dim V)-th exterior power of the sheaf of Kähler differential one-forms.

For V a non-singular projective curve ($d = 1$) the classical duality theorem states that for any invertible (or, more generally, coherent) $\mathcal{O}_V$-module $\mathcal{G}$, the k-vector spaces $H^1(V,\mathcal{G})$ and $\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G},\Omega_V)$ are naturally dual. In fact there is a natural isomorphism

$$\int_V : H^1(V,\Omega_V) \xrightarrow{\sim} k^{(1)}$$

such that for all $\mathcal{G}$, the composition

$$(*)_1 \qquad \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G},\Omega_V) \xrightarrow[\text{natural}]{} \mathrm{Hom}_k(H^1(V,\mathcal{G}),H^1(V,\Omega_V))$$

$$\xrightarrow[\text{via } \int_V]{} \mathrm{Hom}_k(H^1(V,\mathcal{G}),k)$$

is an isomorphism. In other words, the pair $(\Omega_V, \int_V)$ represents the functor

$$H'(\mathcal{G}) = \mathrm{Hom}_k(H^1(V,\mathcal{G}),k) .$$

This theorem is sometimes proved in two steps, as follows. First, by means of certain injective complexes (Weil's "repartitions") the functor $H'(\mathcal{G})$ is shown to be representable by some pair (ω,θ), which is necessarily unique up to isomorphism. We say then that the pair (ω,θ) is dualizing; and in particular that ω is a dualizing sheaf. Then, by the theory of residues of differentials, the canonical pair $(\Omega_V, \int_V)$ is shown to be dualizing.

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(1) realizable if $k = \mathbb{C}$, via a $\bar{\partial}$-Dolbeault resolution of Ω_V, by integrating (1,1)-forms.

More specifically, the residue res'_v at a closed point $v \in V$ is the unique k-linear map from the meromorphic one-forms $\Omega_{k(V)} = \Omega^1_{k(V)/k}$ to k ($k(V)$ = field of rational functions on V) such that, for any local coordinate t at v with differential δt we have:

$$\mathrm{res}'_v(t^a \delta t) = 0 \qquad\qquad a \in \mathbb{Z},\ a \neq -1$$

$$\mathrm{res}'_v(t^{-1}\delta t) = 1 .$$

In particular res'_v factors as

$$\Omega_{k(V)} \xrightarrow[\text{natural}]{} \Omega_{k(V)}/\Omega_{V,v} = H^1_v(\Omega_V) \xrightarrow[\mathrm{res}_v]{} k$$

where H^1_v denotes local cohomology supported at v. The <u>local duality</u> theorem says that the pair $[(\Omega_{V,v})^\wedge, \mathrm{res}_v]$ (where $^\wedge$ denotes completion with respect to the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{O}_{V,v}$) represents the functor

$$\mathrm{Hom}_k(H^1_{\hat{\mathfrak{m}}_v}(G), k)$$

of finitely generated $(\mathcal{O}_{V,v})^\wedge$-modules G.

Now if $\bar{\Omega}_{k(V)}$ is the constant sheaf with sections $\Omega_{k(V)}$, and Ω^*_V is the sheaf whose sections over an open $U \subset V$ are given by

$$\Omega^*_V(U) = \bigoplus_{v\in U} \Omega_{k(V)}/\Omega_{V,v} = \bigoplus_{v\in U} H^1_v(\Omega_V)$$

(sheaf of "differential repartitions") then

$$0 \to \Omega_V \to \bar{\Omega}_{k(V)} \xrightarrow{\text{natural}} \Omega^*_V \to 0$$

is an injective resolution of Ω_V; and taking global sections we get the exact row in the following diagram:

$$\begin{array}{ccccccc}
\Omega_{k(V)} & \longrightarrow & \bigoplus_{v\in V} H^1_v(\Omega_V) & \longrightarrow & H^1(V,\Omega_V) & \longrightarrow & 0 \\
 & & \quad\searrow{\scriptstyle \oplus\mathrm{res}_v} & & \swarrow{\scriptstyle \int_V} & & \\
 & & & k & & &
\end{array}$$

The key residue theorem says that "the sum of the residues of a meromorphic differential is zero"; which means that $\oplus\, \mathrm{res}_v$ annihilates the image of $\Omega_{k(V)}$, i.e. there is a unique map $H^1(V,\Omega_V) \to k$ making the diagram commute. This map is $\int_V$. The proof that $(\Omega_V, \int_V)$ is dualizing can be found e.g. in [S,p.26]. What we want to bring out here is that for non-singular projective curves, differentials and residues give us a canonical realization of and compatibility between local and global duality.

Our principal Theorem (0.6) establishes a similar canonical compatibility for arbitrary proper k-varieties.

Here are some historical highlights in the development of such a generalization. In his 1952 thesis, Rosenlicht constructed (in essence) an isomorphism like $(*)_1$ for V any (possibly singular) projective curve and $\mathcal{G}$ invertible, with Ω_V replaced by a certain sheaf of "regular" meromorphic differential forms, definable e.g. through residues (cf. [S,pp.76-81]). Shortly afterwards, Serre established for any d-dimensional normal projective V an isomorphism of functors of coherent $\mathcal{O}_V$-modules $\mathcal{G}$:

$$(*)_d \qquad \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G},\Omega_V^{\vee\vee}) \xrightarrow{\sim} \mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$$

where $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{F},\mathcal{O}_V)$ for any $\mathcal{O}_V$-module $\mathcal{F}$ (cf. [Z,§8]). In other words: $\Omega_V^{\vee\vee}$ is a dualizing sheaf on V. Then Grothendieck showed for arbitrary projective V that the functor $\mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$ is representable ([Gl], [AK], [H,§7]).[(1)] Grothendieck's method is to deduce from the fact that $\Omega_{\mathbb{P}}$ is dualizing on $\mathbb{P} = \mathbb{P}_k^n$ = projective n-space over k (a fact proved, following Serre, by explicit calculation), that for a closed embedding $\iota: V \to \mathbb{P}$, the sheaf $\omega_\iota = \iota^*\mathcal{E}xt^{n-d}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_V,\Omega_{\mathbb{P}})$ is dualizing on V. Of course ω_ι is not canonical on V; but if V is non-singular, then using suitable

(1) Though we concentrate here on the dual of $H^d(V,\mathcal{G})$, that is only one aspect of duality theory. For example the isomorphism $(*)_d$ extends uniquely to a homomorphism of δ-functors $\mathrm{Ext}^i(\mathcal{G},\Omega_V^{\vee\vee}) \to \mathrm{Hom}_k(H^{d-i}(V,\mathcal{G}),k)$ $(i \geq 0)$, which is an isomorphism for all i when V is Cohen-Macaulay (cf. [H,§7]; and also (13.8.7) below for the local case). Moreover the general theory requires the consideration of dualizing complexes, one of whose homologies is a dualizing sheaf. All this lies beyond the scope of these notes.

Koszul complexes Grothendieck shows that ω_ι is isomorphic to Ω_V, so that the canonical sheaf Ω_V is dualizing. This approach and its elaborations involve a considerable amount of homological algebra (cf. §13 below, where a closed embedding $\iota: V \to X$ of a singular d-dimensional V into a smooth n-dimensional X is considered, with the object of constructing an "adjunction" isomorphism of $\iota^* \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V, \Omega_X)$ onto a canonical dualizing sheaf of V).

There is however a more elementary approach (indicated e.g. in [H,p.249,Ex.7.2]), which we will follow everywhere except in §13. Namely, instead of embedding V in $\mathbb{P}^n_k$, choose a finite surjective map $\pi: V \to \mathbb{P}' = \mathbb{P}^d_k$. Then there exists a coherent $\mathcal{O}_V$-module ω_π together with an isomorphism

$$\pi_* \omega_\pi \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}'}}(\pi_* \mathcal{O}_V, \Omega_{\mathbb{P}'}).$$

As above, $\Omega_{\mathbb{P}'}$ is dualizing on $\mathbb{P}'$; and it follows easily that ω_π is dualizing on V. Moreover, using the trace map for differential forms, Kunz constructs, in [K1], [K3], a concrete realization of such an ω_π as a sheaf of meromorphic d-forms, which turns out to depend only on V (not on the choice of π)! Thus we have a <u>canonical dualizing sheaf</u> $\tilde{\omega}_V$, the sheaf of "regular differentials", which (as Kunz shows) coincides with Ω_V at the smooth points of V, and which is identical with Rosenlicht's sheaf of regular differentials when V is a curve.

Now since $\tilde{\omega}_V$ is dualizing, there is a map

$$\theta_V : H^d(V, \tilde{\omega}_V) \to k$$

corresponding to the identity map of $\tilde{\omega}_V$. This θ_V is determined, a priori, only up to multiplication by non-zero elements of k (which give automorphisms of $\tilde{\omega}_V$). For non-singular V, where $\tilde{\omega}_V = \Omega_V$, Grothendieck describes a <u>canonical</u> θ_V, via the "fundamental class of a point" ([G1,p.149-13]). For general V, the existence of a canonical θ_V is closely related to a theory of residues, as in the above discussion of $\int_V$ when $d = 1$; when $d > 1$ this idea is worked out explicitly in [K2], at least for Cohen-Macaulay characteristic zero V.

Actually a higher-dimensional theory of residues was announced by Grothendieck in 1958 [G2], in connection with his proof of a duality theorem for arbitrary proper k-varieties. (Details appear in [RD].) Most of what we discuss in these notes is implicit in Grothendieck's theory; but it needs to be brought out into the open.

Our global existence and uniqueness statements (for instance (0.6)(d) and (0.6)(e)) make use of the following results of Grothendieck: (i) the representability of the functor $\mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$ of coherent $\mathcal{O}_V$-modules for any proper d-dimensional V; and (ii) surjectivity of the natural maps $H^d_v(\mathcal{G}) \to H^d(V,\mathcal{G})$ $(v \in V)$ from local to global cohomology. Fortunately, relatively simple proofs of these results have been provided by Kleiman in [Km 2] resp. [Km1].

We will not repeat Kleiman's proofs, but rather concentrate on defining a <u>canonical dualizing pair</u> for any proper k-variety V. We will show, first of all, that Kunz's sheaf $\tilde{\omega}_V$ - which can be defined locally via a finite surjective map to affine space, then globally by patching - is still dualizing. Note that for arbitrary proper V, finite maps onto smooth varieties exist only locally; so that for example if ω_V is a dualizing sheaf (for whose existence we refer to [Km2]), then Kunz's methods give us isomorphisms

$$\tilde{\omega}_W \xrightarrow{\sim} \omega_V|W$$

only for affine (or quasi-projective) open $W \subset V$. Why should these isomorphisms patch together to give a single isomorphism over all of V?[1] This problem typifies the essential difficulty in the non-projective case.

Another one of our main concerns is to describe higher-dimensional <u>residue maps</u>

$$\mathrm{res}^{\sim}_v : H^d_v(\tilde{\omega}_V) \to k \qquad (v \in V).$$

With these maps we have a <u>local duality</u> theorem (cf. (0.6)(c)), which

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[1] Following Verdier [V], Kleiman shows in [Km 2, Prop. (22)] that if V is <u>Cohen-Macaulay</u>, then the restriction of a dualizing sheaf to any smooth open $V_0 \subset V$ is isomorphic to Ω_{V_0}. It is not clear a priori that Kleiman's isomorphism coincides with Kunz's when V_0 is affine.

says in particular that $\tilde{\omega}_{V,v}$ is a canonical $\mathcal{O}_{V,v}$-module in the sense of [HK]. Furthermore, residues provide the means for overcoming the patching problems just mentioned; they may be used to give a local description of $\tilde{\omega}_V$ (cf. (11.4)); and for proper V they glue together (as in the above-described residue theorem for curves) to a k-linear map

$$\tilde{\theta}_V : H^d(V, \tilde{\omega}_V) \to k,$$

giving us the desired canonical dualizing pair $(\tilde{\omega}_V, \tilde{\theta}_V)$ as well as a natural compatibility between local and global duality (cf. the main Theorem (0.6)).

As we are limiting ourselves more or less to the topics already indicated, we do not go very far into the theory of residues. There are numerous other treatments in the literature (cf. remarks following the proof of (7.2) below), each of which illuminates some interesting facets of the theory, though none seems to be definitive.

* * *

In §0 we give a more complete discussion of the main results. As this discussion will be rather long, a few orienting remarks are in order.

First of all, as already indicated, the <u>main theorem</u> is (0.6), which gives a local characterization of residues, and describes via residues and the canonical sheaf $\tilde{\omega}_V$ the compatibility of local and global duality. A generalization to the "relative case" is given in §10.

The statement of (0.6) should be understood, even if nothing else is. The reader may wish to begin with this statement, referring back to (0.2A) and (0.4) as needed. The proof of (0.6) occupies most of §§1-9, and some of §0, proceeding roughly as follows:

$$\left.\begin{array}{r} (4.2) = (0.3A) \\ \S 9 \xRightarrow[(9.3)]{} \S 6 \longrightarrow (0.3B) \end{array}\right\} \xRightarrow[(0.3.2)]{} (0.2) \xRightarrow[(0.6.1)]{} (0.6)$$

Theorem (0.1) and the remarks following it are not really needed in the sequel. They are included to set the mood, and to indicate one of many possible ways to think about matters related to (0.6).

Finally, a word about the style in which results are stated. We are concerned here with <u>concrete</u> realizations of certain aspects of duality theory. These realizations may not be constructed directly, but, for example, by non-obvious patching procedures (which, incidentally, are greatly facilitated by the language of $\mathcal{O}$-modules reviewed in §1). So it is important to enunciate a minimal number of characteristic properties which make explicit the <u>canonicity</u> of the objects in question. When it comes to details, the subject is not a simple one; and if some statements of theorems seem lengthy, it is because they compress a lot of information which seems to me essential for a proper understanding. The reader is therefore encouraged to take the time to absorb these statements.

§0. Discussion of results

As before, V is a d-dimensional variety over the perfect field k, and $\Omega_V = \Lambda^d(\Omega^1_{V/k})$ is the sheaf of holomorphic Kähler d-forms.

Among our principal results, the easiest to state is the following portion of [E, p.34, Théorème 3.1]:

THEOREM (0.1). There exists a unique family of k-linear maps

$$\int_V : H^d(V,\Omega_V) \to k$$

indexed by proper d-dimensional k-varieties V, and satisfying the following conditions (a), (b), (c):

(a) If V is the projective space $\mathbb{P}^d_k$, then $\int_V$ is the well-known canonical isomorphism (defined, for example, in (8.4) below).

(b) For any finite surjective map $f: V \to W$ with W a proper d-dimensional normal k-variety, if τ is the map

$$(\text{trace} \otimes 1): f_*\mathcal{O}_V \otimes_{\mathcal{O}_W} \Omega_W \longrightarrow \mathcal{O}_W \otimes_{\mathcal{O}_W} \Omega_W = \Omega_W$$

then the following diagram commutes (trivially if f is not separable, i.e. if the function field k(V) is not separable over k(W)):

$$\begin{array}{ccc}
H^d(W, f_*\mathcal{O}_V \otimes \Omega_W) & \xrightarrow{\text{natural}} & H^d(W, f_*\Omega_V) = H^d(V,\Omega_V) \\
\Big\downarrow \text{via } \tau & & \Big\downarrow \int_V \\
H^d(W,\Omega_W) & \xrightarrow[\int_W]{} & k
\end{array}$$

(c) *If* $g : V \to W$ *is a birational map of proper* d-*dimensional* k-*varieties, then the following diagram commutes:*

$$\begin{array}{ccccc} H^d(W,\Omega_W) & \xrightarrow{\text{natural}} & H^d(W,g_*\Omega_V) & \xrightarrow{\text{natural}} & H^d(V,\Omega_V) \\ & {\scriptstyle \int_W} \searrow & & \swarrow {\scriptstyle \int_V} & \\ & & k & & \end{array}$$

Furthermore:

(d) *If* V *is a* smooth d-*dimensional proper* k-*variety, then the pair* $(\Omega_V, \int_V)$ *is dualizing,* i.e. *represents the functor* $\mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$ *of coherent* $\mathcal{O}_V$-*modules* $\mathcal{G}$.

Remarks. (i) To introduce some of the ideas which play an important role in these notes, we sketch a partial proof of (0.1) (even though this proof is somewhat different than the one we will use, cf. Remark (i) following Theorem (0.2B) below).

One can reduce (a), (b) and (c) of (0.1) to the projective case via Chow's Lemma (for (b) this is messy!). For projective V, the *uniqueness* of $\int_V$ follows from (a), (b), and Noether normalization (Appendix A), which gives us a finite *separable* (=generically étale) $f : V \to \mathbb{P}^d_k = W$. (Note that then $f_*\mathcal{O}_V \otimes \Omega_W \to f_*\Omega_V$ is a generic isomorphism, so that the map in (b) labelled "natural" is surjective.) As for *existence*, given a finite separable $f: V \to \mathbb{P}^d_k = W$ it is well-known that there is a unique map $\tau' : f_*\Omega_V \to \Omega_W$ whose composition with the natural map $f_*\mathcal{O}_V \otimes \Omega_W \to f_*\Omega_V$ is τ (cf. [Kl, p. 15, Satz 5.5], which uses the "equality of Kähler and Dedekind differents"); so we can use τ' and the canonical isomorphism $\int_W$ to define $\int_V$, which we denote temporarily by $\int_V^f$. The crux of the problem is to show that $\int_V^f$ *does not depend on the choice of* f; and further, (a) and (b) being then straightforward, to prove (c).

This can be done, roughly, as follows.

For any closed point $v \in V$ we consider the composition

$$(0.1.1) \qquad \mathrm{res}_v^f : H_v^d(\Omega_V) \xrightarrow{\ \varphi_v\ } H^d(V,\Omega_V) \xrightarrow{\ \int_V^f\ } k$$

where H_v^d denotes cohomology supported at v and φ_v is the natural map. It follows from the results in §8 that *if* f *is étale at* v (so that V is smooth at v) *then* res_v^f *is the classical residue map* (reviewed in §7). So for any two finite separable maps $f_1, f_2 : V \to \mathbb{P}_k^d$, if $v \in V$ is a closed point where both are étale, then

$$\int_V^{f_1} \circ\, \varphi_v = \int_V^{f_2} \circ\, \varphi_v \ ;$$

and since φ_v is *surjective* (cf. (9.6)), we conclude that $\int_V^{f_1} = \int_V^{f_2}$, i.e. $\int_V^f$ does not depend on f. Thus we can set

$$\int_V^f = \int_V \ , \ \mathrm{res}_v^f = \mathrm{res}_v \ .$$

Furthermore if v is *any* closed point where V is smooth, then there exists an f which is étale at v (cf. Appendix A), and so res_v is still the classical residue map. Hence (c) can be proved by picking a v around which V is smooth and g is an isomorphism, and considering the resulting diagram

$$(0.1.2) \qquad \begin{array}{ccccc} H_{g(v)}^d(\Omega_W) & \xrightarrow{\ \varphi_{g(v)}\ } & H^d(W,\Omega_W) & & \\ \| & & \downarrow & \searrow & \\ & & & & k \\ & & & \nearrow & \\ H_v^d(\Omega_V) & \xrightarrow[\ \varphi_v\ ]{} & H^d(V,\Omega_V) & & \end{array}$$

This kind of local-global interplay is one of our basic themes.

Now (d) of (0.1) follows, in the projective case, from the well-known duality theorem for $\mathbb{P}^d_k$ and the fact that when V is smooth the above map τ' corresponds to an $f_*\mathcal{O}_V$-isomorphism

$$f_*\Omega_V \xrightarrow{\sim} Hom_{\mathcal{O}_W}(f_*\mathcal{O}_V, \Omega_W) \tag{0.1.3}$$

(cf. [K1, Korollar 5.2 and Satz 2.2]).[1] Ultimately we will prove (d) (and everything else) for arbitrary proper smooth V by reduction to the projective case; but there doesn't seem to be any relatively simple reduction as there was for (a), (b), (c).

(ii) Theorem (0.1) enables us to define, for every closed point $v \in V$ (smooth or not), a canonical residue map res_v (cf. (0.1.1), ignoring "f"). By means of diagrams like (0.1.2), it is not hard to deduce from (0.1)(c) that res_v depends only on the local ring $\mathcal{O}_{V,v}$ (and not on V), cf. §11. In fact it then follows from (a) and (b) and the results in §8 that res_v depends only on the completion $(\mathcal{O}_{V,v})^\wedge$.

Such a definition of (local) residues proceeding from the smooth case via a global theorem to the general case, is not very appealinn. A direct and much more general homological definition can be found in [HL], [Ho]. It can be shown that this definition is equivalent to the preceeding one, where applicable, but there is as yet no published proof.

* * *

We will deduce Theorem (0.1) from a stronger result, which we now describe.

In [K3, p. 68], Kunz defines the sheaf $\tilde{\omega}_V$ of regular differential forms on V, as follows. Consider first an integral domain C which is a finitely generated k-algebra. Then by Noether normalization there exists a polynomial ring

$$B = k[X_1,\cdots,X_d] \subset C$$

[1] In turn, the existence of τ' and the fact that (0.1.3) is an isomorphism can be deduced directly from (0.1).

such that C is a finite B-module and the corresponding extension of fraction fields $k(B) \subset k(C)$ is separable (cf. [N1, p. 152, (39.11)]; or Appendix A). A trace map τ for degree d Kähler differential forms is then given by

$$\tau\colon \Omega^d_{k(C)/k} = k(C) \otimes_{k(B)} \Omega^d_{k(B)/k} \xrightarrow{\text{trace}\otimes 1} k(B) \otimes_{k(B)} \Omega^d_{k(B)/k} = \Omega^d_{k(B)/k} .$$

The "generalized Dedekind complementary module" $\tilde{\omega}_{C/B}$ is defined as

$$\tilde{\omega}_{C/B} = \{\nu \in \Omega^d_{k(C)/k} \mid \tau(C\nu) \subset \Omega^d_{B/k}\} .$$

For any d-dimensional k-variety V, Kunz shows (and this is the main result in [K1]):

THEOREM (0.2A). <u>There exists a unique $\mathcal{O}_V$-submodule $\tilde{\omega}_V$ of the constant sheaf $\Omega^d_{k(V)/k}$ of meromorphic d-forms on V such that for any affine open subset</u>

$$U = \mathrm{Spec}(C) \subset V$$

<u>and any $B \subset C$ as above, we have</u>

$$\Gamma(U,\tilde{\omega}_V) = \tilde{\omega}_{C/B}$$

The $\mathcal{O}_V$-module $\tilde{\omega}_V$ is clearly coherent. In [K1, §5] the following statements are proved: (i) The image of the natural map $\Omega_V \to \Omega^d_{k(V)/k}$ of holomorphic into meromorphic forms lies in $\tilde{\omega}_V$, with equality at smooth points. (ii) Hence when V is normal, $\tilde{\omega}_V$, being reflexive, consists of those meromorphic forms which are holomorphic in codimension one, i.e. $\tilde{\omega} = (\Omega^d)^{\vee\vee}$, where $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{F},\mathcal{O}_V)$ for any $\mathcal{O}_V$-module $\mathcal{F}$. (iii) Also, when V is a curve $\tilde{\omega}_V$ is the sheaf of regular differentials in the sense of Rosenlicht (cf. [S, p. 76]). (Cf. (11.4) in §11 below for a higher-dimensional generalization of (iii).)

In case V is projective, it is not hard to see that $\tilde{\omega}_V$ is a dualizing sheaf, i.e. there exists an isomorphism of functors of coherent $\mathcal{O}_V$-modules $\mathcal{G}$

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G},\tilde{\omega}_V) \xrightarrow{\sim} \mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$$

(cf. [Kl, Satz 2.2]). One of our main results is that *for any* V *proper over* k, $\tilde{\omega}_V$ *is in a natural way dualizing*, i.e. there is a natural k-linear map

$$\tilde{\theta}_V : H^d(V, \tilde{\omega}_V) \to k$$

such that the pair $(\tilde{\omega}_V, \tilde{\theta}_V)$ represents the functor $\mathrm{Hom}_k(H^d(V, \mathcal{G}), k)$.

Thus we have a *dualizing pair* $(\tilde{\omega}_V, \tilde{\theta}_V)$ which is *canonical* (not just unique up to isomorphism).

Here, precisely, is what is meant by the "naturality" of $\tilde{\theta}_V$. A *dualizing structure* on $\tilde{\omega}$ is a family of maps $\{\tilde{\theta}_V\}$ as above (i.e. $(\tilde{\omega}_V, \tilde{\theta}_V)$ represents $\mathrm{Hom}_k(H^d(V, \mathcal{G}), k)$ for each proper d-dimensional V) such that, for each birational map $f : V \to W$ of proper d-dimensional k-varieties, the following diagram commutes:

$$\text{(0.2.1)} \qquad \begin{array}{ccc} H^d(W, f_*\tilde{\omega}_V) & \xrightarrow{\ \alpha\ } & H^d(W, \tilde{\omega}_W) \\ \Big\downarrow {\scriptstyle \text{canonical}} & & \Big\downarrow {\scriptstyle \tilde{\theta}_W} \\ H^d(V, \tilde{\omega}_V) & \xrightarrow[\ \tilde{\theta}_V\ ]{} & k \end{array}$$

where α is induced by the inclusion map $f_*\tilde{\omega}_V \hookrightarrow \tilde{\omega}_W$ (cf. Lemma (3.2)). (This description of "dualizing structure" is equivalent to the one given in Definition (4.1)). The dualizing structure is *normalized* if for projective space $\mathbb{P} = \mathbb{P}^d_k$ $(d \geq 0)$, $\tilde{\theta}_{\mathbb{P}}$ is the well-known canonical isomorphism

$$H^d(\mathbb{P}, \tilde{\omega}_{\mathbb{P}}) = H^d(\mathbb{P}, \Omega_{\mathbb{P}}) \xrightarrow{\ \sim\ } k$$

(cf. e.g. Proposition (8.4)). Then our assertion is:

THEOREM (0.2B). *There exists a unique normalized dualizing structure* $\{\tilde{\theta}_V\}$ *on* $\tilde{\omega}$ *such that for any finite separable* (= generically étale) *map* $f: V \to W$ *of proper* d-*dimensional* k-*varieties, the following diagram commutes:*

$$\begin{array}{ccc} H^d(W, f_*\tilde{\omega}_V) & \xrightarrow{H^d(\tilde{\tau}_f)} & H^d(W, \tilde{\omega}_W) \\ \| & & \downarrow \tilde{\theta}_W \\ H^d(V, \tilde{\omega}_V) & \xrightarrow[\tilde{\theta}_V]{} & k \end{array}$$

where $\tilde{\tau}_f : f_*\tilde{\omega}_V \to \tilde{\omega}_W$ is induced by the trace map τ for meromorphic forms (cf. example (2.1.2)).

Remarks. (i) Theorem (0.1) can be derived from (0.2B) as follows: define $\int_V$ to be the composition

$$H^d(V, \Omega_V) \xrightarrow{\text{natural}} H^d(V, \tilde{\omega}_V) \xrightarrow{\tilde{\theta}_V} k$$

and check ... .

(ii) Just as for (0.1), uniqueness in (0.2B) follows from Noether normalization and Chow's lemma. (Note that in (0.2.1), α is surjective since $f_*\tilde{\omega}_V \hookrightarrow \tilde{\omega}_W$ is a generic isomorphism.)

As for existence, giving $\tilde{\theta}_V$ is equivalent to giving an $\mathcal{O}_V$-isomorphism $\tilde{c}_V : \tilde{\omega}_V \to \omega_V$ where ω_V is some dualizing sheaf on V (whose existence is guaranteed by Grothendieck's duality theory, or also by the simplified duality theory of Kleiman [Km 2]). Kunz's methods provide a $\tilde{c}$ over any quasi-projective open subset U of V, but this $\tilde{c}$ is defined in terms of a choice of a Noether normalization of a chosen projective compactification of U; the basic difficulty is to show that these local $\tilde{c}$'s actually do not depend on the choices involved, hence patch together to give a global $\tilde{c}_V$. In addition, of course, we must show that the resulting family $\{\tilde{\theta}_V\}$ does form a dualizing structure as in (0.2B).

* * *

We will prove (0.2A) and (0.2B) in an equivalent —somewhat technical— form, given below as (0.3A) and (0.3B). First we need some definitions (whose length will be justified by their convenience).

In §1, we review the notion of <u>coherent $\mathcal{O}$-module</u>, which means (cf. (1.3)) a family $\{\mathcal{F}_V\}$ indexed by k-varieties V, with $\mathcal{F}_V$ a coherent $\mathcal{O}_V$-module for each V, together with a family $\{\beta_i\}$ indexed by open immersions $i: U \to U'$, such that for each i, β_i is an $\mathcal{O}_U$-<u>isomorphism</u>

$$\beta_i : i^*\mathcal{F}_{U'} \xrightarrow{\sim} \mathcal{F}_U ,$$

and such that for each pair of open immersions $U \xrightarrow{i} U' \xrightarrow{j} U''$ we have the transitivity relation

$$\beta_{ji} = \beta_i \circ i^*\beta_j \text{ (1)}$$

A <u>canonical structure</u> on a coherent $\mathcal{O}$-module $\{\omega_V\}$ consists of the data (a), (b) below, subject to conditions (1), (2), (3) (cf. §2 for an equivalent - and more complete - treatment):

(a) For each smooth d-dimensional variety V an $\mathcal{O}_V$-isomorphism

$$\gamma_V : \Omega_V \xrightarrow{\sim} \omega_V .$$

(b) For each finite separable map $f: V \to W$, an $f_*\mathcal{O}_V$-isomorphism

$$T_f : f_*\omega_V \xrightarrow{\sim} Hom_{\mathcal{O}_W}(f_*\mathcal{O}_V, \omega_W).$$

(1) γ_V is compatible (in an obvious sense, via β) with open immersions into V.

(2) T_f is compatible (via β) with open immersions into W (precise formulation left to the reader).

(3) If V and W are smooth, and $f: V \to W$ is finite and étale, then T_f corresponds to the trace map (cf. (0.1)(b))

$$\tau : f_*\Omega_V = f_*\mathcal{O}_V \otimes \Omega_W \longrightarrow \Omega_W$$

---

(1) We could also work with the étale topology, i.e. substitute étale maps for open immersions.

in the sense that the following diagram commutes:

$$\begin{array}{ccc}
f_*\Omega_V & \xrightarrow[f_*\gamma_V]{\sim} & f_*\omega_V \\
 & & \downarrow{\scriptstyle T_f} \\
\tau \Big\downarrow & & \mathcal{H}om(f_*\mathcal{O}_V,\omega_W) \\
 & & \downarrow{\scriptstyle \text{evaluate at } 1} \\
\Omega_W & \xrightarrow[\gamma_W]{\sim} & \omega_W
\end{array}$$

The $\mathcal{O}$-module $\tilde{\omega} = \{\tilde{\omega}_V\}$ (where the sheaves $\tilde{\omega}_V$ of regular differentials are as before and the β_i are the obvious maps) has a natural canonical structure (cf. example (2.1.2)). Moreover (2.4) says that *for any coherent* $\mathcal{O}$-*module* $\omega = \{\omega_V\}$, *the canonical structures on* ω *are in one-one correspondence with* $\mathcal{O}$-*module isomorphisms* $\lambda : \omega \xrightarrow{\sim} \tilde{\omega}$, the structure on ω corresponding to λ being obtained by pullback from the natural one on $\tilde{\omega}$. Thus the notion of canonical structure is simply a characterization of $\tilde{\omega}$, up to isomorphism. And the proof of (2.2) shows that statement (0.2A) above is equivalent to:

THEOREM (0.2A'). *There exists a canonical* $\mathcal{O}$-*module* (*i.e. a coherent* $\mathcal{O}$-*module together with a canonical structure*).

But what is the point of all this elaboration? It's that we have isolated the properties needed to show how $\tilde{\omega}$ relates to existing duality theory, and to prove (0.2), as we shall now see.

We define a *dualizing structure* $\{\theta_V\}$ (V proper) on a coherent $\mathcal{O}$-module ω in a manner similar to that used above for $\tilde{\omega}$ - cf. Definition (4.1) for a precise statement. Given (0.2B), Remark (4.8) says that *the dualizing structures on* ω *are in one-one correspondence with* $\mathcal{O}$-*module isomorphisms* $\lambda : \omega \xrightarrow{\sim} \tilde{\omega}$.

Thus we have a *one-one correspondence* between dualizing structures and canonical structures. We can also describe this correspondence by the following statements (0.3A) and (0.3B), which are together equivalent to (0.2A') and (0.2B).

THEOREM (0.3A)(cf.(4.2)). There exists a dualizing $\mathcal{O}$-module (i.e. a coherent $\mathcal{O}$-module together with a dualizing structure).

THEOREM (0.3B)(cf. §6). Every dualizing $\mathcal{O}$-module $(\{\omega_V\}, \{\theta_V\})$ has a unique canonical structure $(\{\gamma_V\},\{T_f\})$ such that

(a) for projective space $\mathbb{P} = \mathbb{P}^d_k$ $(d \geq o)$, the composition

$$H^d(\mathbb{P},\Omega_{\mathbb{P}}) \xrightarrow{H^d(\gamma_{\mathbb{P}})} H^d(\mathbb{P},\omega_{\mathbb{P}}) \xrightarrow{\theta_{\mathbb{P}}} k$$

is the canonical isomorphism ; and

(b) for any finite separable map $f : V \to W$ of proper d-dimensional k-varieties, if $t_f : f_*\omega_V \to \omega_W$ is T_f followed by "evaluation at 1", then the following diagram commutes:

$$\begin{array}{ccc} H^d(W,f_*\omega_V) & \xrightarrow{H^d(t_f)} & H^d(W,\omega_W) \\ \Vert & & \downarrow{\scriptstyle \theta_W} \\ H^d(V,\omega_V) & \xrightarrow[\theta_V]{} & k \end{array}$$

Remarks.(0.3.1) The passage from dualizing to canonical structures described in (0.3B) respects $\mathcal{O}$-isomorphisms, in the following sense. If ω, ω' are two dualizing modules, and (γ,T), (γ',T') are the corresponding canonical structures given by (0.3B), then the unique $\mathcal{O}$-isomorphism $\lambda : \omega \to \omega'$ given by (4.7) coincides with the isomorphism of (2.3), because by uniqueness in (0.3B) we see that (γ',T') is the canonical structure obtained from (γ,T) by "push forward" via λ (cf.(2.4)). It follows that the two foregoing correspondences between dualizing and canonical structures (one via(0.2B) and isomorphisms $\lambda : \omega \to \tilde{\omega}$, the other as in (0.3B)) are the same.

(0.3.2) The equivalence of (0.2) and (0.3) can be seen as follows. Trivially (0.3A) and (0.3B) imply (0.2A'), and (0.2A) and (0.2B) imply (0.3A). Given (0.3A) and (0.3B), hence (0.2A), Corollary (2.3) provides an isomorphism of $\mathcal{O}$-modules $\lambda : \omega \xrightarrow{\sim} \tilde{\omega}$ which is compatible with the respective canonical structures; and the existence part of (0.2B) follows easily (use λ to push the dualizing structure on ω forward to $\tilde{\omega}$). Similarly, using (4.7) we see that existence in (0.2B) gives existence in (0.3B). The corresponding implications for uniqueness can be proved in a like manner with the help of (2.4) and (4.8).

(0.3.3) We have already noted that (0.2) $\Rightarrow$ (0.1). On the other hand, (0.2) can be deduced from (0.3A) and (0.1). For if ω is a dualizing $\mathcal{O}$-module, then for each proper V, ω_V is torsion-free of rank one (cf. (4.4)), and so $\int_V$ (which one checks to be non-zero) corresponds to an $\mathcal{O}_V$-homomorphism $\Omega_V \to \omega_V$ which becomes an isomorphism when tensored with the constant sheaf $k(V)$ of rational functions. The inverse of this isomorphism takes ω_V isomorphically onto an $\mathcal{O}_V$-submodule $\approx\!\omega_V$ of the constant sheaf $\Omega^d_{k(V)/k}$ $(d = \dim V)$, and $\approx\!\omega_V$ contains the image of the natural map

$\Omega_V \to \Omega^d_{k(V)/k}$. For an arbitrary variety V, choose a compactification $V \subset \bar{V}$ (cf. §4), and set

$$\approx\!\omega_V = \approx\!\omega_{\bar{V}}\,|V.$$

Using (0.1) and the defining properties of a dualizing structure, one can show in a straightforward -if somewhat tedious- way that this $\approx\!\omega_V$ does not depend on the choice of $\bar{V}$, and satisfies (0.2A) and (0.2B).[1]

In summary, we have indicated the following implications:

[(0.3A) + (0.1)] $\Leftrightarrow$ [(0.2A) + (0.2B)] $\Leftrightarrow$ [(0.3A) + (0.3B)]

---

[1] For verifying (0.2A), note that any finite map

$$f : U = \mathrm{Spec}(C) \to \mathrm{Spec}(B) = W$$

has a compactification $f_1 : U_1 \to W_1 = \mathbb{P}^d_k$ (cf. (5.4)), which may be assumed to be finite (replace U_1 by $\mathrm{Spec}(f_{1*}\mathcal{O}_{U_1})$).

(0.3.4) Let ω be a dualizing module. In [E, p.34, Théorème] ElZein defines a canonical map

$$c_V : \Omega^d_{V/k} \to \omega_V ,$$

the fundamental class of V. (The existence of such a map was asserted by Grothendieck in [G2, p. 114]). This map expresses the basic relation of differential forms to duality theory. It appears quite naturally in our setup, because as we have seen ω is canonically isomorphic to $\tilde{\omega}$ and, as we have also seen, there is a natural map

$$\Omega^d_{V/k} \to \tilde{\omega}_V \subset \Omega^d_k(V)/k \ .$$

Of course if V is proper, then c_V corresponds to $\int_V$. For any V, if V_o is the smooth part of V, then c_V is the unique extension to V of γ_{V_o}, where γ comes from the canonical structure on ω given by (0.3B). A more complete discussion (including an explanation of the terminology "fundamental class") is given in §3.

* * *

Statements (0.3A) and (0.3B) are consequences (more or less) of [RD, p. 383, Corollary 3.4]. However one of our main purposes in this paper is to provide a proof of (0.3) for which loc. cit. is not a prerequisite. (We use instead the simpler, though less flexible, duality theory given by Kleiman in [Km 2].) The other main purpose is to describe the connection between local and global duality, via residues (cf. [RD, p. 386, Proposition 3.5]). In fact, what was referred to in Remark (ii) following (0.2B) as the "basic difficulty" (which becomes for (0.3B) the problem of defining γ_V for non-proper smooth V)[1] will be resolved by means of this connection (cf. §9).

---

[1] We might like to define γ_V via [Km 2, p. 55, Prop. 3.3]; but as far as we know, a smooth V may not have a Cohen-Macaulay compactification, at least if k has positive characteristic.

More specifically, two principal ingredients of our proof are Theorem (9.1), which asserts roughly that *local duality is induced by global duality*, and the following *primitive residue theorem*, proved in §§7-8 by means of explicit constructions.

Let $\mathcal{R}_d^o$ be the collection of all d-dimensional *regular* local k-algebras R whose residue field R/m_R is finite over k (m_R = maximal ideal of R), and such that the universal finite differential module $\Omega^1_{R/k}$ exists (i.e. there is a k-derivation $R \to \Omega^1_{R/k}$ which is universal for k-derivations of R into *finitely generated* R-modules cf. e.g. [SS1, §1]). For $R \in \mathcal{R}_d^o$ set

$$\Omega_R = \Lambda^d_R(\Omega^1_{R/k}).$$

THEOREM (0.4) ("Primitive Residue Theorem"). *With preceding notation, there is a unique family of k-linear maps*

$$\mathrm{res}_R : H^d_{m_R}(\Omega_R) \to k \qquad (R \in \mathcal{R}_d^o)$$

(*where* $H^d_{m_R}$ *denotes local cohomology*) *such that:*

(a) *If* $\hat{R}$ *is the completion of* R—*so that* $\Omega_{\hat{R}}$ *is the completion of* Ω_R [SS1, p. 141, Korollar 1.6] *and*

$H^d_{m_{\hat{R}}}(\Omega_{\hat{R}}) = H^d_{m_R}(\Omega_R)$ — *then* $\mathrm{res}_R = \mathrm{res}_{\hat{R}}$.

(b) *If* $R, R' \in \mathcal{R}_d^o$, *and* $R \to R'$ *is a* k-*homomorphism via which* R' *becomes a finite étale* R-*algebra, whence* $\Omega_{R'} = \Omega_R \otimes_R R'$ *and*

$$H^d_{m_{R'}}(\Omega_{R'}) = H^d_{m_R}(\Omega_R)\otimes_R R' \ ,$$

then

$$\mathrm{res}_{R'} = \mathrm{res}_R \circ (1\otimes \mathrm{trace}).$$

(c) *If* x *is a closed point of the projective space* $\mathbb{P} = \mathbb{P}^d_k$ *and* $R = \mathcal{O}_{\mathbb{P},x}$, *then the following diagram (with* $\int_{\mathbb{P}}$ *the canonical* *isomorphism) commutes:*

$$\begin{array}{ccc} H^d_{\mathfrak{m}_R}(\Omega_R) = H^d_x(\Omega_{\mathbb{P}}) & \xrightarrow{\text{canonical}} & H^d(\mathbb{P},\Omega_{\mathbb{P}}) \\ & {\scriptstyle \mathrm{res}_R}\searrow \qquad \swarrow{\scriptstyle \int_{\mathbb{P}}} & \\ & k & \end{array}$$

Remarks. It should be noted that while this result as stated contains a (global) definition of the (local) maps res_R (if $\int_{\mathbb{P}}$ is assumed known, since every complete $R \in \mathcal{R}^o_d$ is étale over some $(\mathcal{O}_{\mathbb{P},x})^{\wedge}$), the proof itself begins (§7) with the standard *purely local* description of res_R, which is then used in (8.4) to define $\int_{\mathbb{P}}$.
The above mentioned Theorem (9.1) is proved by reduction to the case of projective space, where, in view of (a) and (c) of (0.4), it amounts to the following explicit version of *local duality* (cf. Theorem (7.4)):

If $R \in \mathcal{R}^o_d$ *is complete, then the pair* $(\Omega_R, \mathrm{res}_R)$ *represents the functor* $\mathrm{Hom}_k(H^d_{\mathfrak{m}_R}(G),k)$ *of finitely generated* R-*modules* G.

* * *

The preceding facts are summarized in the following stronger *Residue Theorem*, which is the central result of these notes.

(In §10 we will give a more general "relative" residue theorem, and in §12 some consequences of the form

$$\sum_{v\in V} \mathrm{residue}_v(\text{something}) = 0. \qquad)$$

Let $\mathcal{R}_d$ be the collection of all d-dimensional local domains R which are localizations of finitely generated k-algebras and whose residue field $R/\mathfrak{m}_R$ ($\mathfrak{m}_R$ = maximal ideal of R) is finite over k.

As before (cf. (0.2A)), we denote by $\tilde{\omega}$ the $\mathcal{O}$-module of regular differential forms. For any $R \in \mathcal{R}_d$, $R = C_{\mathfrak{p}}$ where C is an integral domain finitely generated over k and $\mathfrak{p}$ is a prime ideal in C. We define $\tilde{\omega}_R$ to be the localization $(\tilde{\omega}_{C/B})_{\mathfrak{p}}$, where $\tilde{\omega}_{C/B}$ is as in the remarks preceding (0.2A); then (0.2A) implies that the R-module $\tilde{\omega}_R$ depends only on R. If $S \supset R$ is a domain which is a finite R-module, with fraction field separable over that of R, then we define

$$\tilde{\omega}_{S/R} = \{\nu \in \Omega^d_{k(S)/k} \mid \tau(S\nu) \subset \tilde{\omega}_R\}$$

(cf. definition of $\tilde{\omega}_{C/B}$), and check that if $S_i (1 \le i \le n)$ are the localizations of S at its various maximal ideals (so that $S_i \in \mathcal{R}_d$) then $\tilde{\omega}_{S_i}$ is the localization $(\tilde{\omega}_{S/R}) \otimes_S S_i$. Note that if $\mathfrak{m}_i = \mathfrak{m}_{S_i}$, then the trace map τ induces a map

$$(0.5) \qquad \bigoplus_{i=1}^{n} H^d_{\mathfrak{m}_i}(\tilde{\omega}_{S_i}) = H^d_{\mathfrak{m}_R}(\tilde{\omega}_{S/R}) \xrightarrow{\text{via } \tau} H^d_{\mathfrak{m}_R}(\tilde{\omega}_R)$$

We can now state the

RESIDUE THEOREM (0.6). <u>There exists a unique family of k-linear maps</u>

$$\widetilde{\mathrm{res}}_R : H^d_{\mathfrak{m}_R}(\tilde{\omega}_R) \to k \qquad (R \in \mathcal{R}_d)$$

<u>satisfying the following conditions</u> (a) <u>and</u> (b) :

(a) (Normalization). <u>If</u> $R \in \mathcal{R}_d$ <u>is regular</u>, <u>then</u> $\widetilde{\mathrm{res}}_R$ <u>coincides with the map</u> res_R <u>of Theorem (0.4)</u> (see also the remarks following that Theorem).

(b) (Trace property). <u>For any</u> R, S_i <u>as above</u>, <u>the following</u>

diagram commutes:

$$\begin{array}{ccc} H^d_{\mathfrak{m}_i}(\tilde{\omega}_{S_i}) & \xrightarrow{\text{cf. (0.5)}} & H^d_{\mathfrak{m}_R}(\tilde{\omega}_R) \\ & {}_{\operatorname{res}_{S_i}^{\sim}}\searrow \quad \swarrow_{\operatorname{res}_R^{\sim}} & \\ & k & \end{array}$$

Furthermore:

(c) (Local duality). *If* ^ *denotes* $\mathfrak{m}_R$-*adic completion, so that*

$$H^d_{\mathfrak{m}_R}(\tilde{\omega}_R) = H^d_{\hat{\mathfrak{m}}_R}(\hat{\tilde{\omega}}_R)$$

then the pair $(\hat{\tilde{\omega}}_R, \operatorname{res}_R^{\sim})$ *represents the functor* $\operatorname{Hom}_k(H^d_{\hat{\mathfrak{m}}_R}(G), k)$ *of finitely generated* $\hat{R}$-*modules* G.

(d) (Globalization). *There exists for each proper* d-*dimensional* k-*variety* V *a unique* k-*linear map*

$$\tilde{\theta}_V : H^d(V, \tilde{\omega}_V) \to k$$

such that for each closed point $v \in V$, *the following diagram commutes:*

$$\begin{array}{ccc} H^d_v(\tilde{\omega}_V) & \xrightarrow{\text{canonical}} & H^d(V, \tilde{\omega}_V) \\ & {}_{\operatorname{res}_{\mathcal{O}_{V,v}}^{\sim}}\searrow \quad \swarrow_{\tilde{\theta}_V} & \\ & k & \end{array}$$

(e) (Global duality). For each V as in (d), the pair $(\tilde{\omega}_V, \tilde{\theta}_V)$ is dualizing, i.e. represents the functor $\mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$ of coherent $\mathcal{O}_V$-modules $\mathcal{G}$.

Remarks. (0.6.1) One can readily derive (0.6) — except for (c) — from (0.2B), using (0.6)(d) as a definition of $\mathrm{res}_R^{\sim}$, which makes sense by straightforward considerations following from the commutativity of (0.2.1), cf. the proof of (9.1)(a). (For (0.6)(b) note that finite maps have finite compactifications, cf. footnote in Remark (0.3.3); for (0.6)(a), use (0.6)(b) to reduce to the case where R is the local ring of a closed point of $\mathbb{P}_k^d$ ... .) Then (0.6)(c) is a restatement of (9.1)(b).

Conversely, in view of (9.6), (a), (b), (d) and (e) of (0.6) easily imply (0.2B).

(0.6.2) The existence of the family $\mathrm{res}_R^{\sim}$ of (0.6) will be proved here in a roundabout and indirect manner, via global considerations. A more satisfying local approach is given, under restrictive hypotheses, by Kunz [K2]. He defines $\tilde{\omega}_S$ and the map

$$\mathrm{res}_S^{\sim} : H^d_{\mathfrak{m}_S}(\tilde{\omega}_S) \to k$$

for any complete local Cohen-Macaulay k-algebra S with residue field finite over k, k being assumed to have characteristic zero. He has informed me that it is possible to eliminate the Cohen-Macaulay hypothesis by the use of techniques such as are found in [K1, §4].

Of course even when (a), (b), (c) of (0.6) are worked out in a purely local way, proving (d) and (e) is still difficult.

* * *

In §§11-13, we give various complements to the Residue Theorem, as described in their respective introductory remarks. Suffice it here to mention that §13 gives an alternative approach to the construction of $\tilde{\omega}$, via embeddings and the "fundamental local homomorphism" (Theorem (13.5)); and the essential local property of residues for this purpose is given in Theorem (13.12). (In the simpler approach described in this Introduction, the corresponding principle ingredients were Noether normalization, "evaluation at 1", and the "trace formula" (0.4)(b) for residues.)

It should be noted that all the main results can be extended to varieties over non-perfect fields, or indeed over regular local rings. The main technical requirement for such an extension is an adequate notion of the trace of a differential, with respect, say, to inseparable extensions. Such a notion is available (cf. [K3], [A], [L]); and is treated in great detail in an unpublished manuscript of Kunz.

Other topics which could have been dealt with are the behavior of $\tilde{\omega}$ with respect to smooth morphisms, and the corresponding local property of residues (cf. (R4) on p. 198 of [RD]); and an explicit local description of the relative residue map ρ of Theorem (10.2) (cf. [Ke]). I hope to return to these questions - in a more general context - at a later time; but for now, enough is enough.

I. CANONICAL MODULES

§1. Zariski sheaves ($\mathcal{O}$-modules)

(1.1) Let k be a perfect field, and let $\mathcal{V}$ be the category of k-varieties, i.e. non-empty reduced irreducible separated k-schemes of finite type. For fixed $V \in \mathcal{V}$, we will often consider data defined in terms of various open immersions $V \to W$, and so it will be quite convenient to use the notion of Zariski sheaf on $\mathcal{V}$, which we now recall.

Let $\mathcal{V}_{Zar}$ be the subcategory of $\mathcal{V}$ having the same objects as $\mathcal{V}$ and having the open immersions in $\mathcal{V}$ as its morphisms. Let $\mathcal{F}:\mathcal{V}_{Zar} \to$ (abelian groups) be a contravariant functor, and suppose that for each $V \in \mathcal{V}_{Zar}$, $\mathcal{F}(V)$ has an $\mathcal{O}(V) = \Gamma(V,\mathcal{O}_V)$-module structure, and further that for each open immersion $i:U \to V$ the corresponding map $\mathcal{F}(V) \to \mathcal{F}(U)$ is a homomorphism of $\mathcal{O}(V)$-modules ($\mathcal{F}(U)$ being an $\mathcal{O}(V)$-module via the ring homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$ corresponding to i). For each V we denote by $\mathcal{F}_V$ the restriction of $\mathcal{F}$ to the subcategory of $\mathcal{V}_{Zar}$ whose objects are the open subvarieties of V and whose morphisms are inclusion maps. Then $\mathcal{F}_V$ is a presheaf of $\mathcal{O}_V$-modules; and we say that $\mathcal{F}$ is a Zariski sheaf of $\mathcal{O}_V$-modules - or, for brevity, that $\mathcal{F}$ is an $\mathcal{O}$-module - if $\mathcal{F}_V$ is a sheaf in the usual sense for every V.

We say that $\mathcal{F}$ is a quasi-coherent (resp. coherent) $\mathcal{O}$-module if $\mathcal{F}_V$ is a quasi-coherent (resp. coherent) $\mathcal{O}_V$-module for every V.

The notion of homomorphism of $\mathcal{O}$-modules is defined in the obvious way.

Examples. (1.1.1) The functor $\mathcal{O}$ such that, as above, $\mathcal{O}(V) = \Gamma(V,\mathcal{O}_V)$ and $\mathcal{O}(V) \to \mathcal{O}(U)$ is the natural map for open immersions $U \to V$, is a coherent $\mathcal{O}$-module. For each V, the restriction $\mathcal{O}_V$ is identical with the usual structure sheaf.

(1.1.2) There is a quasi-coherent $\mathcal{O}$-module $\mathcal{K}$ with

$$\mathcal{K}(V) = \text{field of rational functions on } V$$

(a field which we also denote by $k(V)$).

(1.1.3) The tensor product $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ of two $\mathcal{O}$-modules is defined in the obvious way.

(1.1.4) For each V let $\Omega^1_{V/k}$ be the usual sheaf of relative Kähler differentials; and for $e \geq 0$ let $\Omega^e_{V/k} = \Lambda^e_{\mathcal{O}_V}(\Omega^1_{V/k})$, the e-th exterior power. Set

$$\Omega(V) = \Gamma(V, \Omega^{\dim V}_{V/k}) \qquad (\text{"dim"} = \text{"dimension"}).$$

Then Ω is, in an obvious way, a coherent $\mathcal{O}$-module. For each V, we have

$$\Omega_V = \Omega^{\dim V}_{V/k} \quad .$$

(1.1.5) For each V, let $\tilde{\omega}_V$ be the sheaf of <u>regular differential forms</u> on V, as described in the Introduction (cf. (0.2A)). Then there is a coherent $\mathcal{O}$-module $\tilde{\omega}$ such that, for all V,

$$\tilde{\omega}(V) = \Gamma(V, \tilde{\omega}_V) \quad .$$

(The definition of $\tilde{\omega}(V) \to \tilde{\omega}(U)$ for open immersions $i: U \to V$ is left to the reader.) In this example, the two possible meanings of the symbol "$\tilde{\omega}_V$" coincide.

(1.2) The main reason for introducing $\mathcal{O}$-modules into this exposition will not emerge until §4, where we deal with dualizing $\mathcal{O}$-modules (cf. Definition (4.1)). As mentioned in the Introduction, one of our principal results will be that the $\mathcal{O}$-module $\tilde{\omega}$ of example (1.1.5) is dualizing.

(1.3) The category of $\mathcal{O}$-modules is <u>equivalent</u> to the category whose objects are: families of (sheaves of) $\mathcal{O}_V$-modules $(\mathcal{F}_V)_{V \in \mathcal{V}}$ together with families of isomorphisms $\beta_i : i^*\mathcal{F}_V \to \mathcal{F}_U$ ($i: U \to V$ an open immersion) satisfying the transitivity condition

$$\beta_{ji} = \beta_i \circ i^* \beta_j$$

vis-à-vis couples of open immersions $U \xrightarrow{i} V \xrightarrow{j} W$.

More precisely (but with some details left to the reader):

If $\mathcal{F}'$ is an $\mathcal{O}$-module, and $i: U \to V$ is an open immersion, then there is an $\mathcal{O}_U$-<u>isomorphism</u>

$$\beta'_i : i^*\mathcal{F}'_V \xrightarrow{\sim} \mathcal{F}'_U$$

corresponding by adjointness to the obvious map $\mathcal{F}'_V \to i_*\mathcal{F}'_U$; and for any pair of open immersions $U \xrightarrow{i} V \xrightarrow{j} W$ the following diagram commutes:

$$\begin{array}{ccc} i^*j^*\mathcal{F}'_W & = & (ji)^*\mathcal{F}'_W \\ {\scriptstyle i^*\beta'_j}\downarrow & & \downarrow{\scriptstyle \beta'_{ji}} \\ i^*\mathcal{F}'_V & \xrightarrow[\beta'_i]{} & \mathcal{F}'_U \end{array}$$

And conversely:

Suppose given for each $V \in \mathcal{V}$ an $\mathcal{O}_V$-module $\mathcal{F}_V$, and for each open immersion $i:U \to V$ an $\mathcal{O}_U$-isomorphism

$$\beta_i : i^*\mathcal{F}_V \xrightarrow{\sim} \mathcal{F}_U$$

such that for any pair of open immersions $U \xrightarrow{i} V \xrightarrow{j} W$ the preceding diagram, with $\mathcal{F}, \beta$ in place of $\mathcal{F}', \beta'$, commutes. Then there is a functor $\mathcal{F}' : \mathcal{V}_{Zar} \to$ (abelian groups) defined by

$$\mathcal{F}'(V) = \Gamma(V, \mathcal{F}_V)$$

and (for open immersions $i:U \to V$) by

$$\begin{array}{ccccc} \mathcal{F}'(i):\Gamma(V,\mathcal{F}_V) & \xrightarrow[\text{canonical}]{} & \Gamma(U, i^*\mathcal{F}_V) & \xrightarrow[\text{via } \beta_i]{} & \Gamma(U,\mathcal{F}_U) \\ \| & & & & \| \\ \mathcal{F}'(V) & & & & \mathcal{F}'(U) \end{array}$$

and this $\mathcal{F}'$ has an obvious $\mathcal{O}$-module structure. In fact for each V there is an $\mathcal{O}_V$-isomorphism

$$\alpha_V : \mathcal{F}_V \xrightarrow{\sim} \mathcal{F}'_V$$

such that for any open subset $U \subset V$, with inclusion map $i:U \to V$, $\alpha_V(U)$ is given by

$$\alpha_V(U) : \Gamma(U,\mathcal{F}_V) = \Gamma(U, i^*\mathcal{F}_V) \xrightarrow[\text{via } \beta_i]{} \Gamma(U,\mathcal{F}_U) = \mathcal{F}'(U) = \Gamma(U,\mathcal{F}'_V);$$

and moreover for any open immersion $j:V \to W$, the following diagram commutes:

$$\begin{array}{ccc} j^*\mathcal{F}_W & \xrightarrow{\beta_j} & \mathcal{F}_V \\ {\scriptstyle j^*\alpha_W}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\ j^*\mathcal{F}'_W & \xrightarrow[\beta'_j]{} & \mathcal{F}'_V \end{array}$$

(1.4) Let $\mathcal{F}$ be an $\mathcal{O}$-module, and let R be a (commutative) local domain with maximal ideal $\mathfrak{m}$, such that R is a localization of a finitely generated k-algebra. We can then define an R-module $\mathcal{F}_R$, <u>the stalk of</u> $\mathcal{F}$ <u>at</u> R, as follows:

There exists a k-morphism

$$\varphi : \mathrm{Spec}(R) \to V$$

where $V \in \mathcal{V}$, such that the corresponding map

$$\mathcal{O}_{V,v} \to R \qquad\qquad (v = \varphi(\mathfrak{m}))$$

is an <u>isomorphism</u>. We order the collection of all such φ by setting $\varphi_1 \geq \varphi_2$ if there exists an open immersion $i : V_1 \to V_2$ making the following diagram commute:

$$\begin{array}{ccc} & \overset{\varphi_1}{\nearrow} & V_1 \\ \mathrm{Spec}(R) & & \big\downarrow i \\ & \underset{\varphi_2}{\searrow} & V_2 \end{array}$$

Such an i, if it exists, is uniquely determined by φ_1 and φ_2 [EGA 0I, p.311,(6.6.1)(i)]. Furthermore [<u>ibid</u>, p.312,(6.6.2),(6.6.4)] shows that for any φ_1, φ_2 there exists a φ_3 with $\varphi_3 \geq \varphi_1$, $\varphi_3 \geq \varphi_2$. Now if $\varphi_1 \geq \varphi_2$ then corresponding to $i : V_1 \to V_2$ we have a map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$; and thus we have a filtered inductive system. We can then set

$$\mathcal{F}_R = \varinjlim_{\varphi} \mathcal{F}(V)$$

The following assertions are easily checked.

(i) Let K be the fraction field of R, and d the transcendence degree of K over k. Then there are natural isomorphisms

$$R \xrightarrow{\sim} \mathcal{O}_R$$

$$K \xrightarrow{\sim} \mathcal{K}_R$$

$$\Omega^d_{R/k} \xrightarrow{\sim} \Omega_R$$

(ii) $\mathcal{F}_R$ is in a natural way an $\mathcal{O}_R$-module, hence (by (i)) an R-module.

(iii) For any φ as above, with $v = \varphi(\mathfrak{m})$, there is a natural isomorphism of R-modules

$$R \otimes_{\mathcal{O}_{V,v}} \mathcal{F}_{V,v} \xrightarrow{\sim} \mathcal{F}_R$$

§2. Canonical $\mathcal{O}$-modules

DEFINITION (2.1). A canonical $\mathcal{O}$-module is a quasi-coherent $\mathcal{O}$-module ω together with the following data - which we call a "canonical structure" on ω:

(a) an isomorphism of functors

$$\gamma : \Omega|\mathcal{V}_0 \to \omega|\mathcal{V}_0$$

where $\mathcal{V}_0$ is the full subcategory of $\mathcal{V}_{Zar}$ whose objects are all the smooth k-varieties, and Ω is as in (1.1.4);

(b) for each finite surjective map $f:V \to W$ (in $\mathcal{V}$) which is separable (i.e. the corresponding function field extension $k(W) \subset k(V)$ is separable), an $f_*\mathcal{O}_V$-isomorphism

$$T_f : f_*\omega_V \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V, \omega_W)$$

whose composition with "evaluation at 1" we denote by $t_f : f_*\omega_V \to \omega_W$;

these data being subject to the following condition (which states, roughly speaking, that t_f is generically identical-via γ-with the trace map τ for differential forms):

(2.1.1) Let $f:V \to W$ be as in (b). Let v be the generic point of V, set $k(V) = \mathcal{O}_{V,v}$ (the function field of V), and $\Omega_{k(V)} = \Omega_{V,v}$. Let $V_0 \subset V$ be the open subvariety consisting of all the smooth points of V, and set

$$\gamma_v = \gamma_{V_0,v} : \Omega_{k(V)} \xrightarrow{\sim} \omega_{V,v} .$$

Similarly define (with w the generic point of W)

$$\gamma_w : \Omega_{k(W)} \xrightarrow{\sim} \omega_{W,w} .$$

Then the following diagram commutes:

$$\begin{array}{ccccc}
k(V) \otimes_{k(W)} \Omega_{k(W)} & = \Omega_{k(V)} & \xrightarrow[\gamma_V]{} & \omega_{V,v} = (f_*\omega_V)_w \\
\Big\downarrow {\scriptstyle \tau = \text{trace} \otimes 1} & & & \Big\downarrow {\scriptstyle t_{f,w}} \\
k(W) \otimes_{k(W)} \Omega_{k(W)} & = \Omega_{k(W)} & \xrightarrow[\gamma_W]{} & \omega_{W,w}
\end{array}$$

Remark. Of course the $f_*\mathcal{O}_V$-isomorphism T_f and the $\mathcal{O}_W$-homomorphism t_f determine each other; we can specify a canonical $\mathcal{O}$-module by (ω,γ,T) or by (ω,γ,t).

Example (2.1.2). The $\mathcal{O}$-module $\tilde{\omega}$ of regular differential forms (cf. (1.1.5)) has a canonical structure. For V smooth we have $\tilde{\omega}_V = \Omega_V$ (cf. [K1, Korollar 5.2], whose proof holds in the present context) so that γ may be taken to be the identity map. As for t_f, we have, by [K3, p.69, Korollar 3.7], with the sheaf of meromorphic forms $\bar{\Omega} = \Omega \otimes_{\mathcal{O}} \mathcal{K}$ (cf. (1.1.2), (1.1.3), (1.1.4)), that the image of $f_*\tilde{\omega}_V \subset f_*\bar{\Omega}_V$ under the trace map $\tau: f_*\bar{\Omega}_V \to \bar{\Omega}_W$ is contained in $\tilde{\omega}_W$, so that we can take $t_f = \tau | f_*(\tilde{\omega}_V)$. To see that the corresponding map $T_f: f_*\tilde{\omega}_V \to \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V, \tilde{\omega}_W)$ is an isomorphism is a local problem, easily settled by choosing (locally) a Noether normalization of W and applying [K3, p.56 Satz 2.2 and p.61, Satz 2.12].

Remark (2.1.3). If ω is a canonical $\mathcal{O}$-module then ω is coherent, and for any $V \in \mathcal{V}$, ω_V satisfies the Serre condition (S_2). (In particular, ω_V is a torsion-free $\mathcal{O}_V$-module.)

Proof. The question is local, so we may assume that there exists a finite surjective separable map $f: V \to W = \mathrm{Spec}(B)$, where $B = k[X_1, \ldots, X_d]$ is a polynomial ring; then (a) and (b) in (2.1) give an $f_*\mathcal{O}_V$-isomorphism

$$f_*\omega_V \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V, \Omega_W)$$

and we see easily that ω_V is coherent and that $f_*\omega_V$ satisfies (S_2), whence [EGA IV, (5.7.11)] ω_V satisfies (S_2). Q.E.D.

We shall now see that any two canonical $\mathcal{O}$-modules are canonically isomorphic.

Let ω be a canonical $\mathcal{O}$-module, and set

$$\bar{\omega} = \omega \otimes_{\mathcal{O}} \mathcal{K}$$

(cf. (1.1.2),(1.1.3)). Similarly set

$$\bar{\Omega} = \Omega \otimes_{\mathcal{O}} \mathcal{K}$$

(sheaf of highest order meromorphic forms).

Clearly γ induces an isomorphism

$$\bar{\Omega}|\mathcal{V}_0 \xrightarrow{\sim} \bar{\omega}|\mathcal{V}_0$$

which extends to an isomorphism

$$\bar{\gamma}:\bar{\Omega} \xrightarrow{\sim} \bar{\omega} .$$

We have canonical maps $\Omega \to \bar{\Omega}$, $\omega \to \bar{\omega}$. Moreover for any V, $\omega_V \to \bar{\omega}_V$ is injective because ω_V is torsion-free (2.1.3). We consider then the composed map

$$\lambda:\omega \to \bar{\omega} \xrightarrow{\bar{\gamma}^{-1}} \bar{\Omega}$$

which gives an <u>isomorphism of</u> ω <u>onto an</u> $\mathcal{O}$-<u>submodule of</u> $\bar{\Omega}$.

LEMMA (2.2). <u>Let</u> $V = \mathrm{Spec}(C)$ <u>be an affine</u> k-<u>variety of dimension</u> d, <u>and</u> $B = k[X_1,\dots,X_d]$ <u>a polynomial</u> k-<u>subalgebra of</u> C <u>such that</u> C <u>is a finite</u> B-<u>module and the corresponding extension of fraction fields</u> $k(B) \subset k(C)$ <u>is separable</u>. <u>Then</u>

(2.2.1) $$\lambda(\omega)(V) = \{\nu \in \Omega^d_{k(C)/k} \mid \tau(C\nu) \subseteq \Omega^d_{B/k}\}$$

<u>where</u> τ . <u>is defined by</u>

(2.2.2) $$\tau:\Omega^d_{k(C)/k} = k(C)\otimes_{k(B)}\Omega^d_{k(B)/k} \xrightarrow{\text{trace}\otimes 1} k(B)\otimes_{k(B)}\Omega^d_{k(B)/k} = \Omega^d_{k(B)/k}.$$

<u>Remark</u>. What this says is that $\lambda(\omega) = \tilde{\omega}$ (cf. (2.1.2)). However we avoid using $\tilde{\omega}$ because we want to make clear that we do not need here the main result of [Kl] to the effect that the right side of equation (2.2.1) depends only on C. In fact the Lemma shows that any proof of the existence of a canonical $\mathcal{O}$-module <u>implies</u> that result.

<u>Proof</u>. The proof is essentially a matter of unravelling definitions. Let $f:V \to W = \mathrm{Spec}(B)$ correspond to the inclusion $B \subseteq C$. The right hand side of (2.2.1) is the C-module of global

sections of the image of

$$Hom_{\mathcal{O}_W}(f_*\mathcal{O}_V, \Omega_W) \subset Hom_{\mathcal{K}_W}(f_*\mathcal{K}_V, \bar{\Omega}_W)$$

under the isomorphism

$$Hom_{\mathcal{K}_W}(f_*\mathcal{K}_V, \bar{\Omega}_W) \xrightarrow{\sim} f_*\bar{\Omega}_V$$

corresponding to τ. So it suffices to show that the following diagram (in which unlabelled maps are canonical) commutes:

$$\begin{array}{ccccc}
f_*\omega_V & \xrightarrow{\tau_f} & Hom_{\mathcal{O}_W}(f_*\mathcal{O}_V, \omega_W) & & \\
\downarrow & & \downarrow & \searrow{\scriptstyle \text{via } \gamma_W} & \\
f_*\bar{\omega}_V & \xrightarrow[\text{via } \tau_f]{} & Hom_{\mathcal{K}_W}(f_*\mathcal{K}_V, \bar{\omega}_W) & & Hom_{\mathcal{O}_W}(f_*\mathcal{O}_V, \Omega_W) \\
{\scriptstyle f_*\bar{\gamma}_V^{-1}}\downarrow & ? & \downarrow{\scriptstyle \text{via } \bar{\gamma}_W^{-1}} & \swarrow & \\
f_*\bar{\Omega}_W & \xrightarrow[\text{via } \tau]{} & Hom_{\mathcal{K}_W}(f_*\mathcal{K}_V, \bar{\Omega}_W) & &
\end{array}$$

The only problem is with the subdiagram labelled ?; but this is easily disposed of by means of condition (2.1.1).

COROLLARY (2.3). *If* (ω,γ,t), (ω',γ',t') *are two canonical* $\mathcal{O}$-*modules, then there is a unique* $\mathcal{O}$-*isomorphism* $\lambda:\omega \xrightarrow{\sim} \omega'$ *compatible with* γ *and* γ', *i.e. if* V *is smooth, then* $\lambda_V = \gamma_V'\circ\gamma_V^{-1}$. *Moreover this* λ *is also compatible with* t, t', *i.e. for each finite surjective separable* $f:V \to W$ *the following diagram commutes:*

$$(2.3.1) \qquad \begin{array}{ccc}
f_*\omega_V & \xrightarrow{t_f} & \omega_W \\
{\scriptstyle f_*\lambda_V}\downarrow & & \downarrow{\scriptstyle \lambda_W} \\
f_*\omega_V' & \xrightarrow[t_f']{} & \omega_W'
\end{array}$$

Proof. The first assertion follows from Lemma (2.2) and the remarks preceding it. The commutativity of (2.3.1) can be checked at the generic point of W (since ω_W' is torsion-free), where it follows at once from (2.1.1).

Remark (2.4). Let ω' be a canonical $\mathcal{O}$-module, and let ω be any $\mathcal{O}$-module. It is easily checked that for any $\mathcal{O}$-isomorphism $\lambda:\omega \xrightarrow{\sim} \omega'$ there is a unique canonical structure on ω such that λ is an isomorphism of canonical modules (as in (2.3)). Hence, and by (2.3), *the canonical structures on* ω *correspond one-one to the* $\mathcal{O}$-*isomorphisms* $\lambda:\omega \xrightarrow{\sim} \omega'$.

§3. *The fundamental class*

Notation remains as in §2.

PROPOSITION (3.1) (cf.[E,p.34]). *If* ω *is a canonical* $\mathcal{O}$-*module, then there exists a unique* $\mathcal{O}$-*homomorphism*

$$c = c(\omega):\Omega \to \omega$$

whose restriction to $\mathcal{V}_0$ *is* γ. *Moreover* c *satisfies the following* trace property: *if* $f:V \to W$ *is finite surjective and separable, with* W *normal, then the following diagram commutes:*

$$\begin{array}{ccccc} & f_*\Omega_V & \xrightarrow{f_*c_V} & f_*\omega_V & \\ \text{canonical} \nearrow & & & & \searrow t_f \\ f_*(\mathcal{O}_V)\otimes\Omega_W & \xrightarrow[\text{trace}\,\otimes 1]{} & \mathcal{O}_W\otimes\Omega_W = \Omega_W & \xrightarrow[c_W]{} & \omega_W \end{array}$$

Remark (3.1.1). If ω' is another canonical $\mathcal{O}$-module, $c':\Omega \to \omega'$ is the map given by (3.1), and $\lambda:\omega \xrightarrow{\sim} \omega'$ is the canonical isomorphism of (2.3), then

$$c' = \lambda\circ c.$$

(Since ω' is torsion free, this need only be checked on $\mathcal{V}_0$, where it is clear.)

Proof of (3.1). This is a straightforward consequence of Lemma (2.2) and [Kl,p.15,Satz 5.5]. For completeness we give a proof which is basically that of *loc. cit.*, dressed up in the terminology and notation of this paper.

We note first that since ω_V satisfies (S_2), cf.(2.1.3), we have by [EGA IV, (5.10.2) and (5.10.5)]:

LEMMA (3.1.2). <u>Let</u> U <u>be a non-empty open subset of</u> $V \in \mathcal{V}$, <u>with inclusion map</u> $i:U \to V$. <u>Then the canonical map</u> $\varphi:\omega_V \to i_*i^*\omega_V$ <u>is injective; and if</u> $V - U$ <u>has codimension</u> ≥ 2 <u>in</u> V <u>then</u> φ <u>is bijective</u>.

Now the Lemma gives an <u>injection</u>

$$\eta:\mathrm{Hom}_{\mathcal{O}_V}(\Omega_V,\omega_V) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\Omega_V,i_*i^*\omega_V)$$

$$= \mathrm{Hom}_{\mathcal{O}_U}(i^*\Omega_V,i^*\omega_V) = \mathrm{Hom}_{\mathcal{O}_U}(\Omega_U,\omega_U)$$

from which we see (taking U to be any non-empty smooth open subvariety of V) that there is at most one c as in Proposition (3.1).

To show that such a c exists, suppose first that V is <u>normal</u> and $U \subset V$ is the open subset consisting of all the smooth points, so that $V - U$ has codimension ≥ 2 in V. Then the above η is <u>bijective</u>, so there exists a unique $\mathcal{O}_V$-homomorphism

$$c_V:\Omega_V \to \omega_V$$

whose restriction to U is γ_U. If W is any open subset of V then $c_V|W$ and c_W both restrict to $\gamma_{U\cap W}$ on $U \cap W$ whence, as above,

$$c_W = c_V|W .$$

For arbitrary V, let $\pi:\bar{V} \to V$ be the normalization, and let c_V be the composed map

$$\text{(3.1.3)} \qquad c_V:\Omega_V \xrightarrow{\text{canonical}} \pi_*\Omega_{\bar{V}} \xrightarrow{\pi_*c_{\bar{V}}} \pi_*\omega_{\bar{V}} \xrightarrow{t_\pi} \omega_V .$$

Let us show that <u>for any smooth open</u> $U \subset V$, c_V <u>restricts on</u> U <u>to</u> γ_U. As above, it will follow that for any open immersion $i:W \to V$, we have a natural identification $i^*c_V = c_W$; and then we can define an $\mathcal{O}$-homomorphism $c:\Omega \to \omega$ restricting to γ on $\mathcal{V}_0$ by setting, for each $V \in \mathcal{V}$,

$$\begin{array}{ccc} c(V) = \Gamma(c_V):\Gamma(V,\Omega_V) & \to & \Gamma(V,\omega_V) \\ \| & & \| \\ \Omega(V) & & \omega(V) \end{array}$$

Let $i:U \to V$ be the inclusion. Apply i^* to (3.1.3) to obtain a factorization

$$\begin{array}{l} i^*c_V : i^*\Omega_V = i^*\pi_*\Omega_{\bar V} \xrightarrow[i^*\pi_* c_{\bar V}]{} i^*\pi_*\omega_{\bar V} \xrightarrow[i^*t_\pi]{} \omega_U \\ \qquad\quad \| \\ \qquad\quad \Omega_U \end{array}$$

So it must be shown that $i^*t_\pi \circ i^*\pi_* c_{\bar V} = \gamma_U$. Since ω_U is torsion-free (2.1.3), this need only be verified at the generic point of U, where it follows easily from (2.1.1) because, $\bar V$ being normal, we have

$$c_{\bar V} | \pi^{-1}(U) = \gamma_{\pi^{-1}(U)} .$$

It remains to prove the trace property. Again, since ω_W is torsion free, this need only be checked at the generic point of W, where it is nothing but (2.1.1). Q.E.D.

LEMMA (3.2). *Let* (ω,γ,t) *be a canonical* $\mathcal{O}$*-module, and let* $f:V \to W$ *be a proper surjective map such that the corresponding extension of function fields* $k(W) \subset k(V)$ *is finite and separable* (*i.e.* f *is generically étale*). *Then there is a unique map* $t_f^{\#}: f_*\omega_V \to \omega_W$ *which localizes* (*modulo* γ) *to* trace $\otimes 1$ *at the generic point of* W (*and hence is injective if* f *is birational*).

Proof. There is a cartesian diagram

$$\begin{array}{ccc} V' & \xrightarrow{\ j\ } & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow[\ i\ ]{} & W \end{array}$$

with f' *finite*, i and j open immersions, and $W - i(W')$ of codimension ≥ 2. By (2.1.1) and (3.1.2), the map $t_{f'}$ extends to the desired $t_f^{\#}$. Uniqueness and birational injectivity of $t_f^{\#}$ hold because ω is torsion free. Q.E.D.

Remark (3.2.1). If f is finite then $t_f^{\#} = t_f$.

The following variant of (3.1) is more general in appearance.

COROLLARY (3.3). *With* $f:V \to W$, (ω,γ,t) *as in* (3.2), *there is a unique map*

$$c_f = c_f(\omega): f_*\Omega_V \to \omega_W$$

generically equal (modulo γ) to trace $\otimes 1$, namely $c_f = t_f^{\#} \circ f_* c_V$. In particular, if W is smooth we have a unique map $f_*\Omega_V \to \Omega_W$ generically equal to trace $\otimes 1$ (hence independent of ω, if ω exists), namely $\gamma_W^{-1} \circ c_f$.

Remarks (i). Of course the trace map $\tau: f_*\bar{\Omega}_V \to \bar{\Omega}_W$ of §2 gives rise to a map $\tau': f_*\Omega_V \to \bar{\Omega}_W$; and (3.3) tells us in particular that the image of τ' lies in Ω_W if W is smooth. This is well-known (e.g. [K3, p.69, Korollar 3.7] and [K1, Korollar 5.2]), but not trivial. (It depends classically on the "equality of Dedekind and Kähler differents".) Here we have exhibited it as a consequence of the existence of a canonical $\mathcal{O}$-module.

(ii). (Not used elsewhere). The map c_V is called the fundamental class on V for the following reason. One of our main results will be that there exists an $\mathcal{O}$-module ω which is both canonical and dualizing (cf. Introduction). Suppose then that V is a closed subvariety of a smooth proper (over k) variety X. With $d = \dim V$, $N = \dim X$, it is well known that (ω being dualizing) there is an isomorphism

$$\omega_V \xrightarrow{\sim} \mathcal{E}xt_{\mathcal{O}_X}^{N-d}(\mathcal{O}_V, \Omega_X)$$

(cf. [H, p. 242]). Hence c gives rise to a canonical element via

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_V}(\Omega_V, \mathcal{E}xt_{\mathcal{O}_X}^{N-d}(\mathcal{O}_V, \Omega_X)) \\
&= \mathrm{Ext}_{\mathcal{O}_X}^{N-d}(\Omega_V, \Omega_X) \\
&\to \mathrm{Ext}_{\mathcal{O}_X}^{N-d}(\Omega_{X/k}^d \otimes \mathcal{O}_V, \Omega_{X/k}^N) \\
&= \mathrm{Ext}_{\mathcal{O}_X}^{N-d}(\mathcal{O}_V, \Omega_{X/k}^{N-d}) \\
&\to H_{|V|}^{N-d}(X, \Omega_{X/k}^{N-d})
\end{aligned}$$

where the arrows are canonical maps, and $H_{|V|}^{N-d}$ denotes cohomology with supports in V. This way of associating a cohomology class to $V \subset X$, was introduced by Grothendieck in [G1]. It will play no role in this paper; but it is important e.g. in Angeniol's theory of Chow schemes [A].

§4. Existence and uniqueness of dualizing $\mathcal{O}$-modules

DEFINITION (4.1). Let ω be a quasi-coherent $\mathcal{O}$-module, and for each open immersion $i: U \to V$ let $\beta_i : i^*\omega_V \xrightarrow{\sim} \omega_U$ be the natural isomorphism. A dualizing structure on ω is a family of k-linear maps

$$\theta_V : H^d(V, \omega_V) \to k \qquad (d = \dim V)$$

one for each proper k-variety V, such that:

(i) for each such V the pair (ω_V, θ_V) represents the functor $\mathrm{Hom}_k(H^d(V, \mathcal{G}), k)$ of quasi-coherent $\mathcal{O}_V$-modules $\mathcal{G}$; and

(ii) for each commutative diagram

$$\begin{array}{ccc} & & V \\ & \overset{j}{\nearrow} & \big\downarrow f \\ U & \underset{i}{\searrow} & W \end{array}$$

with i, j open immersions and V, W proper over k, the following diagram commutes:

$$(4.1.1) \qquad \begin{array}{ccc} i^*f_*\omega_V & \xrightarrow{\;i^*(\theta_f)\;} & i^*\omega_W \\ {\scriptstyle \text{canonical}}\big\downarrow & & \big\downarrow{\scriptstyle \beta_i} \\ j^*\omega_V & \xrightarrow[\;\beta_j\;]{} & \omega_U \end{array}$$

where $\theta_f : f_*\omega_V \to \omega_W$ is the map (whose existence and uniqueness is guaranteed by (i)) such that, with $d = \dim U = \dim V = \dim W$,

$$(4.1.2) \qquad \begin{array}{ccc} H^d(W, f_*\omega_V) & \xrightarrow{\;H^d(\theta_f)\;} & H^d(W, \omega_W) \\ {\scriptstyle \text{canonical}}\big\downarrow & & \big\downarrow{\scriptstyle \theta_W} \\ H^d(V, \omega_V) & \xrightarrow[\;\theta_V\;]{} & k \end{array}$$

commutes.

A dualizing $\mathcal{O}$-module is a quasi-coherent $\mathcal{O}$-module together with a dualizing structure.

We will see below (Remark (4.4)) that a dualizing $\mathcal{O}$-module is necessarily coherent.

THEOREM (4.2). There exists a dualizing $\mathcal{O}$-module (essentially unique, cf. (4.7)).

Proof. We define a quasi-coherent $\mathcal{O}_V$-module ω_V for each $V \in \mathcal{V}$ as follows:

- if V is proper over k, of dimension d, choose an ω_V which represents the above functor $\mathrm{Hom}_k(H^d(V,\mathcal{G}),k)$ [Km 2, p.43, Theorem 4]; in particular ω_V comes equipped with a k-linear map

$$\theta_V(\omega): H^d(V,\omega_V) \to k;$$

- for arbitrary V, choose a compactification, i.e. an open immersion $e_V: V \to \bar{V}$ with $\bar{V}$ proper over k, cf. [N2] (choose e_V = identity if V is already proper), and set

$$\omega_V = e_V^* \omega_{\bar{V}} \text{.}^{(1)}$$

In view of (1.3), it clearly suffices for proving (4.2) to find isomorphisms

$$\beta_i : i^*\omega_V \xrightarrow{\sim} \omega_U \qquad (i:U \to V \text{ an open immersion})$$

such that

(4.2.1): for any couple of open immersions $U \xrightarrow{i} V \xrightarrow{j} W$ we have

$$\beta_{ji} = \beta_i \circ i^*\beta_j \ ;$$

and such that moreover

(4.2.2): condition (4.1)(ii) is satisfied.

This will take up most of the rest of this section.

We need the following preliminary version of relative duality which will be used in this section only in case f' = identity, but which will also form the basis for §5.

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(1) It is possible to bypass Nagata's compactification theorem by first defining ω_V only for quasi-projective V (via a projective compactification), then for arbitrary V choosing an affine covering $\{V_\alpha\}$, and with the following results pasting the ω_{V_α} together to obtain ω_V.. Then one must verify...

PROPOSITION (4.3) (Deligne). *Given a commutative diagram*

$$\begin{array}{ccc} V' & \overset{j}{\hookrightarrow} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \underset{i}{\hookrightarrow} & W \end{array}$$

where V *and* W *are proper* k-*varieties*, i *and* j *are open immersions*, *and* f' *is proper, surjective and with all its fibers of the same dimension*, *say* d, *set* r = dim W (*so that* dim V = r + d), *let*

$$\theta_f : R^d f_* \omega_V \to \omega_W$$

be the $\mathcal{O}_W$-*homomorphism corresponding by the defining property of* ω_W *to the composed* k-*linear map*

$$H^r(W, R^d f_* \omega_V) \xrightarrow[\text{canonical}]{} H^{r+d}(V, \omega_V) \xrightarrow[\theta_V]{} k$$

(ω, θ as above, and cf. Remark (4.3.1) below), *and for any quasi-coherent* $\mathcal{O}_V$-*module* $\mathcal{F}$ *let*

$$\Theta = \Theta_{f,\mathcal{F}} : f_* Hom_{\mathcal{O}_V}(\mathcal{F}, \omega_V) \xrightarrow[\text{natural}]{} Hom_{\mathcal{O}_W}(R^d f_* \mathcal{F}, R^d f_* \omega_V)$$

$$\xrightarrow[\text{via } \theta_f]{} Hom_{\mathcal{O}_W}(R^d f_* \mathcal{F}, \omega_W)$$

be the induced $f_*\mathcal{O}_V$-*homomorphism*. *Then*, *with* $\mathcal{F}' = j^*\mathcal{F}$, *we have that*

$$\boxed{i^*\Theta : f'_* Hom_{\mathcal{O}_{V'}}(\mathcal{F}', j^*\omega_V) \to Hom_{\mathcal{O}_{W'}}(R^d f'_* \mathcal{F}', i^*\omega_W)}$$

$$\begin{array}{ccc} \| & & \| \\ i^* f_* Hom_{\mathcal{O}_V}(\mathcal{F}, \omega_V) & & i^* Hom_{\mathcal{O}_W}(R^d f_* \mathcal{F}, \omega_W) \end{array}$$

is an isomorphism.[1]

Remark (4.3.1). Since dim W = r, we have, for all q ≥ 0 and any quasi-coherent $\mathcal{O}_V$-module $\mathcal{G}$,

---

(1) i.e., with the notation of [Km 2,p.42], $j^*\omega_V = f'^{!} i^*\omega_W$.

$$H^p(W,R^q f_*\mathcal{G}) = 0 \qquad (p > r)$$

and hence the Leray spectral sequence gives rise to a canonical map $H^r(W,R^q f_*\mathcal{G}) \to H^{r+q}(V,\mathcal{G})$. Moreover $R^q f_*\mathcal{G}$ is supported in the subvariety of W over which the fibers of f have dimension $\geq q$ (use [EGA III, (4.2.2)] and the fact that $\mathcal{G}$ is a direct limit of coherent sheaves [EGA 01, p.320, (6.9.12)]), a subvariety which, if $q > d$, has dimension at most $r + d - q - 1$, so that

$$H^p(W,R^q f_*\mathcal{G}) = 0 \qquad (\text{if } q > d \text{ and } p + q \geq r + d);$$

and the spectral sequence shows then that <u>the canonical map</u> $H^r(W,R^d f_*\mathcal{G}) \to H^{r+d}(V,\mathcal{G})$ <u>is surjective</u>.

<u>Proof of (4.3)</u>. Note that, f' being proper, $j(V')$ is open and closed in - hence equal to - $f^{-1}(i(W'))$. In particular $(R^d f'_*)\circ j^* = i^*\circ(R^d f_*)$.

Now the question is clearly local on W'; so it suffices to show that $i^*\Theta$ induces an isomorphism on global sections over W', i.e. that $\Gamma(i^*\Theta)$ <u>is an isomorphism</u>. Also, we may assume, since $\mathcal{F}$ is a direct limit of coherent $\mathcal{O}_V$-modules, that $\mathcal{F}$ itself is coherent.

Let I be a coherent $\mathcal{O}_W$-ideal defining the reduced closed subscheme $W - i(W')$ of W, so that $I\mathcal{O}_V$ defines a (not necessarily reduced) subscheme of V whose support is $V - f^{-1}i(W') = V - j(V')$. Then for any $n \geq 0$ we have $j^*(I^n\mathcal{F}) = \mathcal{F}'$, and there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_V(I^n\mathcal{F},\omega_V) & \xrightarrow{\Gamma(\Theta_n)} & \mathrm{Hom}_W(R^d f_* I^n\mathcal{F},\omega_W) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{V'}(\mathcal{F}',j^*\omega_V) & \xrightarrow[\Gamma(i^*\Theta)]{} & \mathrm{Hom}_{W'}(R^d f'_*\mathcal{F}',i^*\omega_W) \end{array}$$

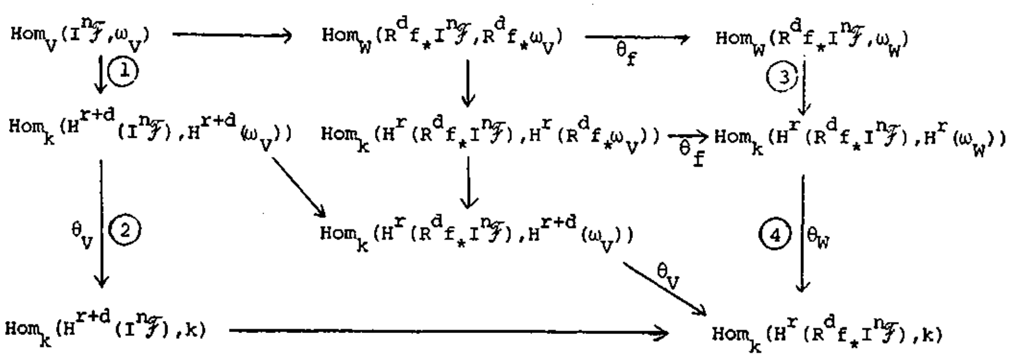
where $\Theta_n = \Theta_{f,I^n\mathcal{F}}$. For some n_0 and all $n \geq n_0$, the image of the natural map $\varphi_n : R^d f_* I^n\mathcal{F} \to R^d f_* I^{n_0}\mathcal{F}$ is $I^{n-n_0} R^d f_* I^{n_0}\mathcal{F}$ (cf. [RD, p.412]); and the kernel K_n of φ_n is supported on $W - W'$, whence

$$\mathrm{Hom}_W(K_n,\omega_W) \cong \mathrm{Hom}_k(H^d(W,K_n),k) = 0;$$

it follows that φ_n induces a <u>bijection</u>

$$\mathrm{Hom}_W(I^{n-n_0}R^d f_* I^{n_0}\mathcal{F},\omega_W) \xrightarrow{\sim} \mathrm{Hom}_W(R^d f_* I^n\mathcal{F},\omega_W).$$

In view of this, and of [EGA 01, p.323, (6.9.17)], we see that applying $\varinjlim$ (over n) to the above diagram makes the vertical arrows into isomorphisms. Furthermore the following natural diagram commutes:



Here the top row is $\Gamma(\Theta_n)$; ②∘① is an isomorphism by the definition of ω_V; and similarly ④∘③ is an isomorphism. We conclude: it suffices to show that the canonical map

$$(4.3.2) \qquad \varinjlim_n \mathrm{Hom}_k(H^{r+d}(I^n\mathcal{F}),k) \to \varinjlim_n \mathrm{Hom}_k(H^r(R^d f_* I^n\mathcal{F}),k)$$

is an isomorphism.

We remarked in (4.3.1) that the canonical map

$$\lambda_n: H^r(R^d f_* I^n\mathcal{F}) \to H^{r+d}(I^n\mathcal{F})$$

is surjective; hence (4.3.2) is injective. Surjectivity of (4.3.2) amounts to the following: if K_n is the kernel of λ_n, then for some $N > 0$ the natural map $K_{n+N} \to K_n$ is the zero map. Again using the Leray spectral sequence, we see that it suffices to show that

$$R^q f_* I^{n+N}\mathcal{F} \to R^q f_* I^n\mathcal{F}$$

is the zero map for $q > d$ and N large. As above, if n is sufficiently large the image of this map is $I^N R^q f_* I^n\mathcal{F}$ for any N; and since $R^q f'_* I^n\mathcal{F} = 0$ (the fibers of f' being of dimension d, cf. (4.3.1)), we have $I^N R^q f_* I^n\mathcal{F} = 0$ for large N. Q.E.D.

Remark (4.3.3). At the end of the preceding proof it would have been enough to show that

$$H^p(W, R^q f_* I^{n+N}\mathcal{F}) \to H^p(W, R^q f_* I^n \mathcal{F})$$

is the zero map for $q > d$, $p + q \geq r + d - 1$, and N large; and so it would suffice for such p, q, N that

$$H^p(W, I^N R^q f_* I^n \mathcal{F}) = 0,$$

which is certainly so if for $p < r - 1$ the generic fiber of f over any p-dimensional subvariety of W' has dimension $< r + d - 1 - p$ (so that for $q > d$ the support of $R^q f_* I^n \mathcal{F}$ has dimension $< r + d - 1 - q$). Hence we can weaken the assumption in (4.3) on the fibers of f' to the following assumption:

if $E \subset V'$ is a closed subvariety of codimension one, then $f'(E) \subset W'$ also has codimension one (in other words, the subvariety

$$\{v \in V' \mid \dim_v(f'^{-1}f'v) > d\}$$

has codimension ≥ 2 in V').

Remark (4.4). We can see that any dualizing module ω is coherent as follows: the question being local, we need only show that ω_V is coherent when V is projective, so that there exists a finite map $f: V \to W = \mathbb{P}^d_k$ ($d = \dim V$); by the duality theorem on $\mathbb{P}^d_k$ ([H,p.240],[Km2,p.55]), we know that there is an isomorphism $\omega_W \xrightarrow{\sim} \Omega_W$; and then by the simple case [$d = 0$, $f = f'$ finite, i = identity, j = identity, $\mathcal{F} = \mathcal{O}_V$] of (4.3), we have an isomorphism

$$f_*\omega_V \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V, \Omega_W),$$

whence ω_V is indeed coherent.

* * *

Next, to define β_i and prove (4.2.1) and (4.2.2) we need a few remarks on compactifications. Given two compactifications $i: V \to X$, $j: V \to Y$ we say that $j \geq i$ if there exists a map $f: Y \to X$ such that $fj = i$. Note that such an f, if it exists, is uniquely determined by i and j, since $j(V)$ is dense in Y and $\mathcal{O}_Y \to j_*\mathcal{O}_V$ is injective. Finally any two compactifications i, j have a least upper bound, namely the map $e_{ij}: V \to Z$ where Z is the closure of the image of the composed immersion

$$V \xrightarrow{\text{diagonal}} V \times_k V \xrightarrow{i \times j} X \times_k Y$$

and e_{ij} is the obvious map.

Now given two compactifications $i:V \to X$, $j:V \to Y$, we define an isomorphism

$$\mu_{ij}: j^*\omega_Y \xrightarrow{\sim} i^*\omega_X$$

as follows:

<u>first</u>, if $j \geq i$, then, applying Proposition (4.3), to the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{j} & Y \\ \| & & \downarrow f \\ V & \xrightarrow[i]{} & X \end{array}$$

we set

(4.5.1) $$\mu_{ij} = i^*\Theta_{f,\mathcal{O}_V} = i^*\theta_f$$

(notation as in (4.3));

<u>second</u>, for arbitrary i, j, let $e = e_{ij}$ (see above) and set

(4.5.2) $$\mu_{ij} = \mu_{ie} \circ (\mu_{je})^{-1} .$$

The definitions (4.5.1), (4.5.2) agree when $j \geq i$; in fact in (4.5.2) we can take e to be <u>any</u> compactification such that $e \geq i$ and $e \geq j$, and this does not affect μ_{ij}. Indeed:

LEMMA 4.6. (i) <u>For any three compactifications</u> $i:V \to X$, $j:V \to Y$, $h:V \to Z$, <u>we have</u>

$$\mu_{ij} \circ \mu_{jh} = \mu_{ih}$$

(ii) <u>Given compactifications</u> $i:V \to X$, $j:V \to Y$ <u>and an open immersion</u> $\ell:U \to V$, <u>we have</u>

$$\mu_{i\ell,j\ell} = \ell^*\mu_{ij} .$$

The (slightly tedious) <u>proof</u> is left to the reader. (The basic point is that for a composition $V \xrightarrow{f} W \xrightarrow{g} X$, with V, W, X all proper and of dimension d, we have $\theta_{gf} = \theta_g \circ g_*\theta_f$.)

Finally we can define β_i for an open immersion $i:U \to V$. Let $e_U:U \to \bar{U}$, $e_V:V \to \bar{V}$ be the compactifications chosen as at the beginning of this section, so that

$$\omega_U = e_U^*\omega_{\bar{U}} \qquad \omega_V = e_V^*\omega_{\bar{V}} .$$

Then $e_V i$ is a compactification of U, and we set

$$\begin{array}{ccc} \beta_i = \mu_{e_U, e_V i} : (e_V i)^*\omega_{\bar{V}} & \longrightarrow & e_U^*\omega_{\bar{U}} \\ \| & & \| \\ i^*\omega_V & & \omega_U \end{array} .$$

(4.2.1) is now a direct consequence of Lemma (4.6):

$$\beta_{ji} = \mu_{e_U, e_W ji} = \mu_{e_U, e_V i} \circ \mu_{e_V i, e_W ji}$$

$$= \beta_i \circ i^*\beta_j \quad .$$

As for (4.2.2), it simply amounts to $\mu_{ij} = \beta_i^{-1} \circ \beta_j$, which again follows from (4.6) because V and W are now proper, so that e_V and e_W are identity maps:

$$\beta_i^{-1} \circ \beta_j = \mu_{i, e_U} \circ \mu_{e_U, j} = \mu_{ij} \quad .$$

This completes the proof of the existence of a dualizing $\mathcal{O}$-module.

Finally, for uniqueness, we have the following analogs of (2.3) and (2.4).

PROPOSITION (4.7). Let $(\omega, \{\theta_V\})$, $(\omega', \{\theta'_V\})$ be two dualizing $\mathcal{O}$-modules. Then there is a unique $\mathcal{O}$-isomorphism $\lambda : \omega \xrightarrow{\sim} \omega'$ compatible with θ and θ', i.e. such that for each proper V, the following diagram commutes;

(4.7.1)
$$\begin{array}{ccc} H^d(V, \omega_V) & \xrightarrow{H^d(\lambda_V)} & H^d(V, \omega'_V) \\ & {\scriptstyle \theta_V} \searrow \quad \swarrow {\scriptstyle \theta'_V} & \\ & k & \end{array}$$

Proof. For each proper V let $\lambda_V : \omega_V \xrightarrow{\sim} \omega'_V$ be the unique $\mathcal{O}_V$-isomorphism making (4.7.1) commute (λ_V exists because ω and ω' represent the same functor). For arbitrary V, choose a compactification $i : V \to \bar{V}$ and set $\lambda_V = i^*\lambda_{\bar{V}}$; that this λ_V does not depend on the choice of i follows in a straightforward way from the definition of dualizing structure and the fact that any two compactifications have a least upper bound. It is then simple to verify that the family $\{\lambda_V\}$ gives an $\mathcal{O}$-isomorphism, as asserted.

Remark (4.8). Same as (2.4), with "canonical" replaced by "dualizing", and (2.3) by (4.7).

Exercise (4.9). Generalize the results of this section by using Kleiman's notion of dualizing pair [Km 2,pp.41-44]. (Caution: the proof of [ibid, p.58, example (viii)] is globally, but not locally, correct.)

§5 Relative duality

As a first step toward the proof of Theorem (0.3B) of the Introduction, we want to define, for a given dualizing module ω and a finite surjective map $f:V \to W$, a natural $\mathcal{O}_W$-homomorphism

$$t_f : f_*\omega_V \to \omega_W$$

such that the corresponding $f_*\mathcal{O}_V$-homomorphism

$$T_f : f_*\omega_V \to \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V, \omega_W)$$

is an isomorphism.

With a little extra effort, we can deal with arbitrary proper surjective maps, and prove the following relative duality theorem.

THEOREM (5.1). Let $(\omega, \{\theta_V\})$ be a dualizing $\mathcal{O}$-module; and for each open immersion $i:U \to V$ let $\beta_i : i^*\omega_V \xrightarrow{\sim} \omega_U$ be the natural isomorphism. It is then possible, in just one way, to assign to each proper surjective map $f:V \to W$ of k-varieties an $\mathcal{O}_W$-homomorphism

$$\theta_f : R^d f_*\omega_V \to \omega_W \qquad (d = \dim V - \dim W)$$

so that the following conditions (i) and (ii) hold:

(i) If W (hence V) is proper over k, then θ_f is the unique map making the following diagram commute ($r = \dim W$):

$$\begin{array}{ccc} H^r(W, R^d f_*\omega_V) & \xrightarrow{H^r(\theta_f)} & H^r(W, \omega_W) \\ \text{cf.(4.3.1)} \downarrow & & \downarrow \theta_W \\ H^{r+d}(V, \omega_V) & \xrightarrow{\theta_V} & k \end{array}$$

(ii) For any commutative diagram of maps

$$\begin{array}{ccc} V & \xhookrightarrow{j} & V_1 \\ f \downarrow & & \downarrow f_1 \\ W & \xhookrightarrow{i} & W_1 \end{array}$$

with f, f_1 both proper and surjective, and i, j both open immersions (so that $j(V)$ is open and closed in - hence equal to - $f_1^{-1}i(W)$, i.e. the diagram is cartesian), the following diagram commutes $(d = \dim V - \dim W)$:

$$\begin{array}{ccc}
i^*R^d f_{1*}\omega_{V_1} & \xrightarrow{\quad i^*\theta_{f_1}\quad} & i^*\omega_{W_1} \\
\Big\downarrow{\scriptstyle \text{canonical}} & & \\
R^d f_* j^*\omega_{V_1} & & \Big\downarrow{\scriptstyle \beta_i} \\
\Big\downarrow{\scriptstyle R^d f_*\beta_j} & & \\
R^d f_*\omega_V & \xrightarrow{\quad \theta_f \quad} & \omega_W
\end{array}$$

<u>Furthermore, if all the fibers of</u> f <u>have the same dimension</u> d,[1] <u>then for any quasi-coherent</u> $\mathcal{O}_V$<u>-module</u> $\mathcal{F}$, <u>the</u> $f_*\mathcal{O}_V$<u>-homomorphism</u>

$$\Theta_f : f_*\mathcal{H}om_{\mathcal{O}_V}(\mathcal{F},\omega_V) \to \mathcal{H}om_{\mathcal{O}_W}(R^d f_*\mathcal{F},\omega_W)$$

<u>induced by</u> θ_f <u>is an isomorphism</u>.

<u>Remark</u> (5.2). In case $f: V \to W$ is finite and surjective, we set $t_f = \theta_f$, and then taking $\mathcal{F} = \mathcal{O}_V$ in the last assertion of the Theorem, we have that the $f_*\mathcal{O}_V$-homomorphism

$$f_*\omega_V \to \mathcal{H}om_{\mathcal{O}_W}(f_*\mathcal{O}_V,\omega_W)$$

corresponding to t_f is indeed an isomorphism. So we have, for ω, condition (b) in Definition (2.1) (without any separability assumption).

Later on, in the proof of (9.1), we will use the following trivial case of (5.1):

<u>Exercise</u> (5.3). Let $f: V \to W$ be an isomorphism, so that we have the canonical identification of functors $f_* = (f^{-1})^*$. Then

---

(1) cf. (4.3.3) for a weaker assumption.

$$\theta_f : f_*\omega_V \to \omega_W$$

$$= \beta_{f^{-1}} : (f^{-1})^*\omega_V \to \omega_W \quad .$$

Proof of (5.1). The underlying idea is quite simple - we will observe that any proper surjective $f:V \to W$ can be compactified, i.e. embedded in a diagram as in (5.1)(ii) with V_1 and W_1 proper over k. Then θ_{f_1} is uniquely specified by (5.1)(i), and hence θ_f is determined by (5.1)(ii); and the last assertion of (5.1) is given by (4.3). The problem then is to show that:

(5.1.1): θ_f, as just described, does not depend on the chosen compactification of f.

So let us begin with some further remarks on compactifications. A compactification of a map $f:V \to W$ is a commutative diagram

(5.4)
$$\begin{array}{ccc} V & \xrightarrow{j_1} & V_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ W & \xrightarrow[i_1]{} & W_1 \end{array}$$

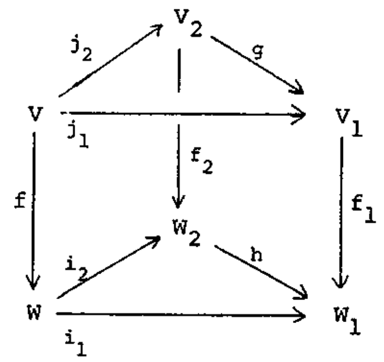
with V_1, W_1 proper over k and i_1, j_1 open immersions. Such compactifications always exist: for example we can first choose compactifications $i_1:W \to W_1$, $j:V \to V'$ of the varieties W, V; and then we can take V_1 to be the closure in $V' \times_k W_1$ of the graph of the map $i_1 \circ f$ (j_1 and f_1 being then the obvious maps).

We say that a compactification

(5.5)
$$\begin{array}{ccc} V & \xrightarrow{j_2} & V_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_2} \\ W & \xrightarrow[i_2]{} & W_2 \end{array}$$

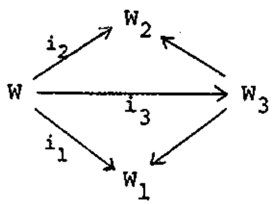
dominates (5.4) if there is a commutative diagram

(5.6)



Note that for commutativity in (5.6), it suffices that $gj_2 = j_1$ and $hi_2 = i_1$: for then $f_1 g = hf_2$ because $j_2(V)$ is dense in V_2, $\mathcal{O}_{V_2} \to j_{2*}\mathcal{O}_V$ is injective, and $f_1 g j_2 = hf_2 j_2$.

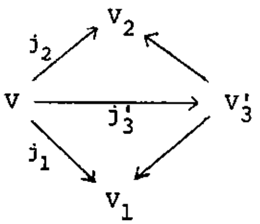
<u>Any two compactifications</u> (5.4), (5.5) <u>of</u> f <u>are dominated by a third one</u>: we first choose a compactification $i_3 : W \to W_3$ such that there is a commutative diagram



(for example, as in §4 we can take W_3 to be the closure of the image of the composed immersion

$$W \xrightarrow{\text{diagonal}} W \times_k W \xrightarrow{i_1 \times i_2} W_1 \times_k W_2);$$

similarly, choose a commutative diagram



and finally take V_3 to be the closure in $V_3' \times_k W_3$ of the graph of the map $i_3 \circ f$ (so that we have obvious maps $V_3 \to W_3$, $V_3 \to V_2$, $V_3 \to V_1 \ldots$).

We return now to (5.1.1).

We need to consider two compactifications (5.4) and (5.5) of f; and since these are both dominated by a third compactification, we may assume that (5.5) dominates (5.4). The problem becomes then to show, with reference to (5.6), that the following diagram commutes:

$$
\begin{array}{ccccccccc}
 & & R^d f_* j_2^* \omega_{V_2} & \longrightarrow & i_2^* R^d f_{2*} \omega_{V_2} & \xrightarrow{i_2^*\theta_{f_2}} & i_2^* \omega_{W_2} & & \\
 & \nearrow{\scriptstyle \beta_{j_2}^{-1}} & & & & & & \searrow{\scriptstyle \beta_{i_2}} & \\
R^d f_* \omega_V & \text{(a)} & \downarrow{\scriptstyle j_1^*\theta_g} & & \text{(b)} & & \downarrow{\scriptstyle i_1^*\theta_h} \quad \text{(c)} & & \omega_W \\
 & \searrow{\scriptstyle \beta_{j_1}^{-1}} & & & & & & \nearrow{\scriptstyle \beta_{i_1}} & \\
 & & R^d f_* j_1^* \omega_{V_1} & \longrightarrow & i_1^* R^d f_{1*} \omega_{V_1} & \xrightarrow{i_1^*\theta_{f_1}} & i_1^* \omega_{W_1} & &
\end{array}
$$

The commutativity of subdiagrams (a) and (c) is given by (4.1)(ii). This leaves us with subdiagram (b), which can be modified and expanded to:

$$
\begin{array}{ccccccc}
i_2^* R^d f_{2*} \omega_{V_2} & \longrightarrow & i_1^* h_* R^d f_{2*} \omega_{V_2} & & \xrightarrow{\theta_{f_2}} & & i_1^* h_* \omega_{W_2} \\
\uparrow & \text{(A)} & \uparrow{\scriptstyle \text{canonical}} & & \text{(C)} & & \downarrow{\scriptstyle \theta_h} \\
R^d f_* j_2^* \omega_{V_2} & \longrightarrow & i_1^* R^d (hf_2)_* \omega_{V_2} & = & i_1^* R^d (f_1 g)_* \omega_{V_2} & \xrightarrow{\theta_{hf_2}} & i_1^* \omega_{W_1} \\
\downarrow & \text{(B)} & \uparrow{\scriptstyle \text{canonical}} & & \text{(D)} & & \uparrow{\scriptstyle \theta_{f_1}} \\
R^d f_* j_1^* g_* \omega_{V_2} & \longrightarrow & i_1^* R^d f_{1*} g_* \omega_{V_2} & & \xrightarrow{\theta_g} & & i_1^* R^d f_{1*} \omega_{V_1}
\end{array}
$$

The commutativity of (A) and (B) is left to the reader (ω_{V_2} can be replaced by any $\mathcal{O}_{V_2}$-module). As for (C), after dropping the initial i_1^*'s, applying the functor $H^r(W_1, \cdot)$ ($r = \dim W_1$) and chasing around the commutative diagrams which define θ_h, θ_{f_2}, and θ_{hf_2}, we end up having to show the commutativity of the following natural diagram (where, again, ω_{V_2} can be replaced by any $\mathcal{O}_{V_2}$-module $\mathcal{F}$):

$$\begin{array}{ccc} H^r(W_1, h_*R^d f_{2*}\omega_{V_2}) & \xrightarrow{\quad (4) \quad} & H^r(W_2, R^d f_{2*}\omega_{V_2}) \\ (1)\uparrow & & \downarrow(3) \\ H^r(W_1, R^d(hf_2)_*\omega_{V_2}) & \xrightarrow[\quad (2) \quad]{} & H^{r+d}(V_2, \omega_{V_2}) \end{array}$$

This is a technical exercise,[1] best carried out with the language of derived categories (cf. remark (5.8) below). Having forsworn such a luxury, we outline the argument as follows:

For any complex of sheaves

$$C^\bullet: \cdots \to C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \to \cdots$$

and any integer e, let $\sigma_e(C^\bullet)$ be the complex

$$\cdots \to 0 \to 0 \to \operatorname{coker}(\delta^{e-1}) \to C^{e+1} \to C^{e+2} \to \cdots$$

For simplicity, write f for f_2. Given an $\mathcal{O}_{V_2}$-module $\mathcal{F}$, let $I^\bullet$ be an injective resolution. There is an obvious map of complexes $\sigma_d h_* f_* I^\bullet \to h_* \sigma_d f_* I^\bullet$, and taking homology H^d, we get the canonical map $R^d(hf)_*\mathcal{F} \to h_* R^d f_* \mathcal{F}$. Hence there is a commutative diagram of complexes

$$\begin{array}{ccc} (h_*R^d f_*\mathcal{F})[-d] & \longrightarrow & h_*\sigma_d f_*(I^\bullet) \\ \uparrow & & \uparrow \\ (R^d(hf)_*\mathcal{F})[-d] & \longrightarrow & \sigma_d h_* f_*(I^\bullet) \end{array} \tag{5.7}$$

(where, for an object G, $G[-d]$ is the complex which is G in degree d and 0 elsewhere). Replacing each complex in (5.7) by an injective complex with the same homology, we obtain a homotopy-

[1] only the simple case $d = 0$ is used in subsequent sections.

commutative diagram, from which, applying sections over W_1 and then homology H^{r+d}, we derive the commutative diagram

$$\begin{array}{ccc}
H^r(W_1, h_*R^df_*\mathcal{F}) & \xrightarrow{\textcircled{5}} & H^{r+d}(V_2,\mathcal{F}) \\
\textcircled{1}\uparrow & & \| \\
H^r(W_1, R^d(hf)_*\mathcal{F}) & \xrightarrow[\textcircled{2}]{} & H^{r+d}(V_2,\mathcal{F})
\end{array} .$$

It remains to show that $\textcircled{5} = \textcircled{3}\circ\textcircled{4}$. For this, consider a homotopy-commutative diagram

$$\begin{array}{ccc}
(H^d\sigma_d f_*I^\bullet)[-d] = (R^df_*\mathcal{F})[-d] & \xrightarrow{\alpha} & \sigma_d f_*I^\bullet \\
\beta\downarrow & & \downarrow\gamma \\
J^\bullet & \longrightarrow & K^\bullet
\end{array}$$

where α is the obvious map, $J^\bullet$ and $K^\bullet$ are injective complexes, and β and γ are <u>quasi-isomorphisms</u> (i.e. they induce isomorphisms on homology). From this we derive a homotopy-commutative diagram

$$\begin{array}{ccccccc}
(h_*R^df_*\mathcal{F})[-d] & & & \longrightarrow & & & h_*\sigma_d f_*I^\bullet \\
\downarrow & \overset{\kappa}{\searrow} & & & & \overset{\mu}{\swarrow} & \downarrow h_*\gamma \\
 & & L^\bullet & \longrightarrow & N^\bullet & & \\
 & & \downarrow & & \| & & \\
 & & M^\bullet & \longrightarrow & N^\bullet & & \\
 & \overset{\lambda}{\nearrow} & & & & \overset{\nu}{\nwarrow} & \\
h_*J^\bullet & & & \longrightarrow & & & h_*K^\bullet
\end{array}$$

where $L^\bullet$, $M^\bullet$, $N^\bullet$ are injective complexes, and κ, λ, μ, ν are quasi-isomorphisms. (Note that, $\sigma_d f_*(I^\bullet)$ being flasque, $h_*\gamma$ is a quasi-isomorphism.) Finally, apply sections over W_1 to the inner square and then take homology H^{r+d} to obtain $\textcircled{5} = \textcircled{3}\circ\textcircled{4}$.

This establishes the commutativity of the above subdiagram $\textcircled{C}$. Subdiagram $\textcircled{D}$ is treated similarly. Q.E.D.

Remark (5.8). In the language of derived categories, the basic point in the preceding proof of $\textcircled{2} = \textcircled{3}\circ\textcircled{4}\circ\textcircled{1}$ is the commutativity of the following natural diagram:

$$\begin{array}{ccccc}
(h_*R^d f_*\mathcal{F})[-d] & & \longrightarrow & & \mathbb{R}h_*(R^d f_*\mathcal{F})[-d] \\
\uparrow & & & & \downarrow \\
(R^d(hf)_*\mathcal{F})[-d] & \longrightarrow & \sigma_d\mathbb{R}h_*\mathbb{R}f_*\mathcal{F} & \longrightarrow & \mathbb{R}h_*\sigma_d\,\mathbb{R}f_*\mathcal{F}
\end{array}$$

(to which one applies $H^{r+d}\mathbb{R}\Gamma(W_1,\cdot)\ldots$).

§6. The canonical structure on a dualizing module

We proceed with the proof of Theorem (0.3B) of the Introduction.

So, given a dualizing module $(\omega,\{\theta_V\})$, we want to construct a canonical structure $(\{\gamma_V\},\{T_f\})$ satisfying conditions (a) and (b) in (0.3B). The dualizing property of ω gives the uniqueness of t_f - or, equivalently, of T_f - satisfying (b): it must be the t_f of Remark (5.2). What is needed then is an isomorphism

$$\gamma:\Omega|\mathcal{V}_0 \xrightarrow{\sim} \omega|\mathcal{V}_0$$

satisfying (2.1.1), and such that $\gamma_{\mathbb{P}}$ is the canonical isomorphism when $\mathbb{P}=\mathbb{P}_k^d$ $(d \geq 0)$. (This canonical isomorphism over $\mathbb{P}$ is well-known, from numerous points of view, e.g. [G1,p.149-13, Théorème 2], [K2,pp.186-187], [RD,p.204, Corollary 10.2], [Km2,p.55, Proposition 22]; we will realize it via the residue map at the vertex of the projecting cone over $\mathbb{P}$, cf. (8.4) below.)

As mentioned before, γ can be derived from the general duality theory of Grothendieck, Hartshorne, Deligne and Verdier. But as we want to avoid using this theory, and anyway wish to bring out relations between the foregoing material and residues and local duality, we will rely ultimately on local considerations to be developed in Chapter III.

* * *

We will now give a local description of γ which is forced by the above requirements (so that γ is unique, if it exists globally); and then in the remainder of this section reduce the existence problem to a question of patching (Proposition (6.3)).

Let, then, V be a d-dimensional smooth variety, so that V is covered by open sets V_α each of which admits an étale map $h = h_\alpha$ into $\mathbb{P} = \mathbb{P}^d_k$ [EGA IV, (17.11.4)]. By Zariski's main theorem there is a commutative diagram

(6.1)
$$\begin{array}{ccc} V_\alpha & \xrightarrow{\ i\ } & \bar{V} = \bar{V}_\alpha \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle \bar{h}=\bar{h}_\alpha} & \\ & \mathbb{P} & \end{array}$$

where i is an open immersion and $\bar{h}$ is finite.[1] We have an $\mathcal{O}_{\mathbb{P}}$-homomorphism

$$\text{trace}\otimes 1 : \bar{h}_*\bar{h}^*\Omega_{\mathbb{P}} = \bar{h}_*\mathcal{O}_{\bar{V}}\otimes\Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}\otimes\Omega_{\mathbb{P}} = \Omega_{\mathbb{P}}$$

whence a composed $\bar{h}_*\mathcal{O}_{\bar{V}}$-homomorphism

$$\begin{aligned} \bar{h}_*\bar{h}^*\Omega_{\mathbb{P}} &\longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\bar{h}_*\mathcal{O}_{\bar{V}}, \Omega_{\mathbb{P}}) \\ &\xrightarrow[\gamma_{\mathbb{P}}]{\sim} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\bar{h}_*\mathcal{O}_{\bar{V}}, \omega_{\mathbb{P}}) \\ &\xrightarrow[(T_{\bar{h}})^{-1}]{} \bar{h}_*\omega_{\bar{V}} \end{aligned}$$

(with $\gamma_{\mathbb{P}}$ the canonical isomorphism). Correspondingly we have an $\mathcal{O}_{\bar{V}}$-homomorphism

(6.2) $$\bar{h}^*\Omega_{\mathbb{P}} \to \omega_{\bar{V}}$$

and finally, restricting to V_α:

$$\gamma_{V_\alpha} : \Omega_{V_\alpha} = h^*\Omega_{\mathbb{P}} = i^*\bar{h}^*\Omega_{\mathbb{P}} \to i^*\omega_{\bar{V}} \xrightarrow[\beta_i]{\sim} \omega_{V_\alpha}$$

(β_i being the natural isomorphism). This γ_{V_α} is actually an isomorphism, as can be seen for example by completing, i.e. making the base change

[1] Such diagrams can also be obtained via Noether normalization, cf. Appendix A.

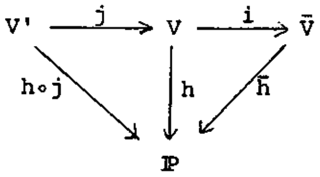
$$\mathrm{Spec}(\hat{\mathcal{O}}_{\mathbb{P},h(v)}) \to \mathbb{P}, \qquad (v \in V_\alpha)$$

and using the fact that since $S = \hat{\mathcal{O}}_{V,v}$ is a finite étale algebra over $R = \hat{\mathcal{O}}_{\mathbb{P},h(v)}$, therefore the map $S \to \mathrm{Hom}_R(S,R)$ corresponding to the trace map $S \to R$ is an isomorphism.

In §9 (following Corollary (9.2)) we will prove:

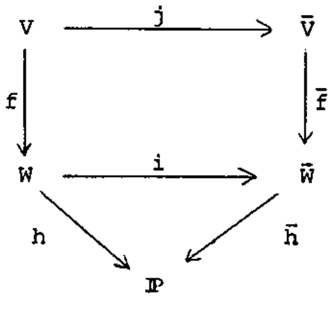
PROPOSITION (6.3). _If_ V_α, V_β, $\bar{h}_\alpha : V_\alpha \to \mathbb{P}^d$, $\bar{h}_\beta : \bar{V}_\beta \to \mathbb{P}^d$ _are as above, then the corresponding maps_ γ_{V_α} _and_ γ_{V_β} _agree on_ $V_\alpha \cap V_\beta$.

This enables us to complete the proof of (0.3B), as follows. Given a smooth $V = \cup V_\alpha$ as above, we can define $\gamma_V : \Omega_V \to \omega_V$ by patching together the V_α. If $j : V' \to V$ is an open immersion, then $\gamma_{V'} = j^*\gamma_V$: to check this we may assume $V = V_\alpha$ and use the diagram



(cf. (6.1)) to define $\gamma_{V'}$. It follows that there exists a unique $\mathcal{O}$-isomorphism $\gamma : \Omega|\mathcal{V}_0 \xrightarrow{\sim} \omega|\mathcal{V}_0$ whose restriction to each smooth V is the above γ_V (which coincides with the canonical isomorphism when $V = \mathbb{P}$).

It remains then to verify (2.1.1) for a finite surjective separable map $f : V \to W$. Using (5.1)(ii) (with $d = 0$) we may replace W by any open subvariety, so we may assume that there exists an etale map $h : W \to \mathbb{P} = \mathbb{P}^d_k$ (so that W is smooth, of dimension d), and that furthermore f is étale. Starting as with (6.1), we get a commutative diagram

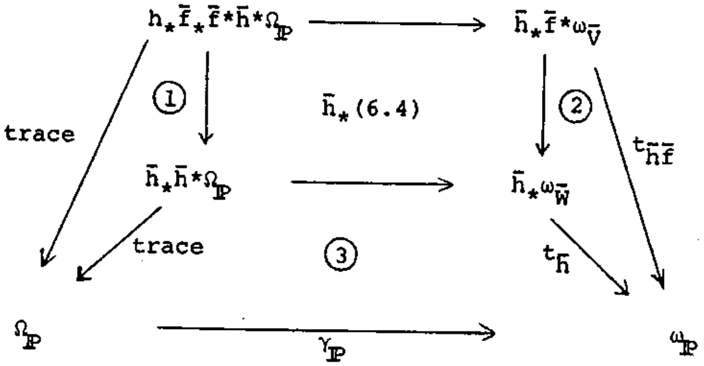


where i is an open immersion, $\bar{h}$ is finite, $\bar{W}$ can be taken to be <u>normal</u>, $\bar{f}$ is the normalization of $\bar{W}$ in the function field of V, and j is an open immersion; and we can use this diagram to calculate the maps γ_V, γ_W.

A little reflection shows then that (2.1.1) (for f) is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \bar{f}_*\bar{f}^*\bar{h}^*\Omega_{\mathbb{P}} & \xrightarrow{(6.2)} & \bar{f}_*\omega_{\bar{V}} \\ \Big\downarrow{\scriptstyle \text{trace}_{\bar{f}}\otimes 1} & & \Big\downarrow{\scriptstyle t_{\bar{f}}} \\ \bar{h}^*\Omega_{\mathbb{P}} & \xrightarrow[(6.2)]{} & \omega_{\bar{W}} \end{array} \tag{6.4}$$

which follows from the (readily proved) commutativity of subdiagrams ①, ②, and ③ in



§7. Residues and local duality for power series rings

In this chapter III we develop some local theory - and its relation to global duality - and use it to prove Proposition (6.3), which is all that remains for the proof of (0.1), (0.2) and (0.3) of the Introduction (cf. (0.3.3)).

The "primitive residue theorem" (0.4) is proved in §§7-8. The connection between local and global duality is then given in (9.1). As pointed out in (0.6.1), the Residue Theorem (0.6) follows from (0.2) and (9.1). Finally, in §10, we give a "relative" generalization of (9.1), and hence of the Residue Theorem.

To begin with we need some notation for specifying elements of local cohomology modules. Let R be any d-dimensional noetherian local ring with maximal ideal m, and let $U = \mathrm{Spec}(R) - m$. We assume $d \geq 1$, leaving the trivial case $d = 0$ to the reader. For any R-module M, let $\tilde{M}$ be the corresponding quasi-coherent sheaf on Spec(R). We have a canonical surjective map (bijective if $d > 1$)

(7.1) $$H^{d-1}(U,\tilde{M}) \to H^d_m(M).$$

If $\underset{\sim}{t} = (t_1,\ldots,t_d)$ is a system of parameters in R, and $U_i \subset U$ is the open set where t_i does not vanish, then $\{U_i\}_{1\leq i\leq d}$ is an affine open covering of U, giving a Čech complex which can be used to compute $H^{d-1}(U,\tilde{M})$. We denote by

$$m/(t_1,\ldots,t_d) \quad \text{or} \quad m/\underset{\sim}{t} \qquad (m \in M)$$

the image under the map (7.1) of the cohomology class of the Čech (d-1)-cocycle

$$m/t_1t_2\ldots t_d \in H^0(U_1 \cap U_2 \cap\ldots\cap U_d,\tilde{M}) = M_{t_1t_2\ldots t_d} .$$

Thus any element $\xi \in H^d_m(M)$ can be represented as a "generalized fraction", in which the denominator is a system of parameters (for example the system $\underset{\sim}{t}^a = (t_1^a,\ldots,t_d^a)$ for some $a > 0$ depending on ξ). The map $m \mapsto m/\underset{\sim}{t}$ is clearly R-linear, i.e. fractions with a given denominator can be added and multiplied by elements of R in the obvious way. To say more we need rules for determining when two generalized fractions represent the same element of $H^d_m(M)$.

LEMMA (7.2) *Let* R, $\underset{\sim}{t}$, M *be as above. Then:*

(a) $m/\underset{\sim}{t} = m'/\underset{\sim}{t}$ $(m, m' \in M)$ *if and only if for some* $n \geq 0$ ($n = 0$ *if* M *has depth* d) *we have*

$$(t_1 t_2 \cdots t_d)^n (m - m') \in (t_1^{n+1}, t_2^{n+1}, \ldots, t_d^{n+1})M = \underset{\sim}{t}^{n+1}M;$$

(b) *if* $\underset{\sim}{t}' = (t_1', t_2', \ldots, t_d')$ *is a system of parameters in* R, *with*

$$t_i' = \sum_{j=1}^{d} r_{ij} t_j \qquad (1 \leq i \leq d,\ r_{ij} \in R)$$

then, denoting determinant by "det", we have

$$m/\underset{\sim}{t} = \det(r_{ij})\, m/\underset{\sim}{t}' .$$

Remark. To see whether $p/\underset{\sim}{u} = q/\underset{\sim}{v}$, where now p, q are any elements of M and $\underset{\sim}{u}$, $\underset{\sim}{v}$ are any two systems of parameters, choose a system of parameters $\underset{\sim}{t}$ such that $\underset{\sim}{t}R \subseteq \underset{\sim}{u}R \cap \underset{\sim}{v}R$, use (b) to write $p/\underset{\sim}{u} = m/\underset{\sim}{t}$, $q/\underset{\sim}{v} = m'/\underset{\sim}{t}$, and then use (a). Similarly (b) allows us to find the sum $p/\underset{\sim}{u} + q/\underset{\sim}{v}$, viz. $(m + m')/\underset{\sim}{t}$.

Proof of (7.2). (Cf. [SZ] for an alternate treatment.) We first recall the Koszul complex interpretation of local cohomology. Let $K_{\bullet}(\underset{\sim}{t})$ be the Koszul complex determined over R by the sequence $\underset{\sim}{t}$ (cf. e.g. [EGA III, §1.1]). As a graded R-module, $K_{\bullet}(\underset{\sim}{t})$ is the exterior algebra $\Lambda(R^d)$; and if $e_1, \ldots, e_d$ is the standard basis of R^d, then the differential

$$\delta : \Lambda^p(R^d) \to \Lambda^{p-1}(R^d) \qquad (0 < p \leq d)$$

is given by

$$\delta(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} t_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_p} .$$

We let $K^{\bullet}(\underset{\sim}{t}, M)$ be the complex

$$K^{\bullet}(\underset{\sim}{t}, M) = \mathrm{Hom}_R(K_{\bullet}(\underset{\sim}{t}), M)$$

and denote the cohomology of $K^{\bullet}(\underset{\sim}{t}, M)$ by

$$H^{\bullet}(\underset{\sim}{t}, M) = H^{\bullet}(K^{\bullet}(\underset{\sim}{t}, M)).$$

The map $\varphi: R^d \to R^d$ given by

$$\varphi(e_i) = \sum_{j=1}^{d} r_{ij} e_j \qquad (1 \le i \le d)$$

extends to a map $\Lambda(\varphi)$ of exterior algebras

$$K_\bullet(\underset{\sim}{t}') = \Lambda(R^d) \xrightarrow{\Lambda(\varphi)} \Lambda(R^d) = K_\bullet(\underset{\sim}{t}),$$

which is moreover a map of complexes. Hence we obtain a map of complexes $K^\bullet(\underset{\sim}{t},M) \to K^\bullet(\underset{\sim}{t}',M)$, and the corresponding cohomology map

$$M/\underset{\sim}{t}M \cong H^d(\underset{\sim}{t},M) \to H^d(\underset{\sim}{t}',M) \cong M/\underset{\sim}{t}'M$$

is induced by multiplication by $\det(r_{ij})$ in M. In particular we see that the R-modules

$$H^d(\underset{\sim}{t}^n,M) = M/\underset{\sim}{t}^n M \qquad (n > 0)$$

form an inductive system, with maps

$$\psi_{ab}: M/\underset{\sim}{t}^b M \to M/\underset{\sim}{t}^a M \qquad (a \ge b)$$

given by multiplication by $(t_1 t_2 \dots t_d)^{a-b}$. As in [G4,p.20, Proposition 5] we have then a canonical <u>isomorphism</u>

$$(7.2.1) \qquad \varinjlim_{n} H^d(\underset{\sim}{t}^n,M) \xrightarrow{\sim} H^d_m(M)$$

under which $m/\underset{\sim}{t}$ is the image of the cohomology class in $H^d(\underset{\sim}{t},M)$ of the map in $K^d(\underset{\sim}{t},M) = \mathrm{Hom}_R(K_d(\underset{\sim}{t}),M)$ which takes the generator $e_1 \wedge e_2 \wedge \dots \wedge e_d$ of $K_d(\underset{\sim}{t}) = \Lambda^d(R^d)$ to m. In other words, after naturally identifying $H^d(\underset{\sim}{t},M)$ with $M/\underset{\sim}{t}M$, we have that $m/\underset{\sim}{t}$ is the canonical image of $(m + \underset{\sim}{t}M) \in M/\underset{\sim}{t}M$.

The assertion (a) of (7.2) should now be clear. (It is well-known - and not hard to show - that ψ_{ab} is <u>injective</u> if the sequence $\underset{\sim}{t}$ is M-regular, i.e. if M has depth d.)

Now we prove (7.2)(b). In case $d = 1$, set $t_1 = t$, $t_1' = t' = rt$. Then $H^0(U,\tilde{M})$ is the module of fractions $M_t = M_{t'}$, $H^1_m(M)$ is the cokernel of the natural map $j: M \to M_t$, and

$$m/\underset{\sim}{t} = m/t + j(M) \in M_t/j(M) = H^1_m(M).$$

So (7.2)(b) follows from the equation (in M_t)

$$m/t = rm/rt = rm/t'.$$

Suppose then that $d > 1$. We have to show that <u>the diagram</u>

$$
(7.2.2)\qquad
\begin{array}{ccccc}
M/\underset{\sim}{t}M & \xrightarrow{\ \sim\ } & H^{d}(\underset{\sim}{t},M) & & \\
 & & & \searrow^{\alpha} & \\
\det(r_{ij})\Big\downarrow & & & & H^{d}_{m}(M) \cong H^{d-1}(U,\widetilde{M}) \\
 & & & \nearrow_{\alpha'} & \\
M/\underset{\sim}{t}'M & \xrightarrow{\ \sim\ } & H^{d}(\underset{\sim}{t}',M) & &
\end{array}
$$

(with maps as described above) <u>commutes</u>. For this, we look more closely at the map α. Let $\mathscr{C}^{\bullet}$ be the Čech resolution of $\widetilde{M}|U$ associated with the covering $\{U_i\}$. (cf. e.g. [H,p.220, Lemma 4.2]). If $\mathscr{I}^{\bullet}$ is an injective resolution of $\widetilde{M}|U$, then there is a homotopy-unique map of complexes $\mathscr{C}^{\bullet} \to \mathscr{I}^{\bullet}$, whence the canonical isomorphism

$$H^{\bullet}(\{U_i\},\widetilde{M}) = H^{\bullet}(\Gamma(U,\mathscr{C}^{\bullet})) \xrightarrow{\sim} H^{\bullet}(\Gamma(U,\mathscr{I}^{\bullet})) = H^{\bullet}(U,\widetilde{M}).$$

Next let $\mathscr{K}^{\bullet}$ be the complex of quasi-coherent sheaves (on Spec(R)):

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K^{1}(\underset{\sim}{t},M)^{\sim} & \longrightarrow & K^{2}(\underset{\sim}{t},M)^{\sim} & \longrightarrow & \cdots \\
 & & \| & & \| & & \\
 & & \mathscr{K}^{0} & & \mathscr{K}^{1} & &
\end{array}
$$

($K^{i}(\underset{\sim}{t},M)$ as above). There is a canonical commutative diagram (cf. EGA III,p.86])

$$
\begin{array}{ccccccc}
 & \nearrow & \mathscr{K}^{0}|U & \longrightarrow & \mathscr{K}^{1}|U & \longrightarrow & \cdots \\
\widetilde{M}|U & & \Big\downarrow & & \Big\downarrow & & \\
 & \searrow & \mathscr{C}^{0} & \longrightarrow & \mathscr{C}^{1} & \longrightarrow & \cdots
\end{array}
$$

and α is the resulting cohomology map

$$
\begin{array}{ccccccc}
H^{d-1}(\Gamma(X,\mathscr{K}^{\bullet})) & \to & H^{d-1}(\Gamma(U,\mathscr{K}^{\bullet})) & \to & H^{d-1}(\Gamma(U,\mathscr{C}^{\bullet})) & \to & H^{d-1}(\Gamma(U,\mathscr{I}^{\bullet})) \\
\| & & & & & & \| \\
H^{d}(\underset{\sim}{t},M) & & & & & & H^{d-1}(U,\widetilde{M})\ .
\end{array}
$$

Replacing $\underset{\sim}{t}$, $\mathscr{C}^{\bullet}$, $\mathscr{K}^{\bullet}$ by $\underset{\sim}{t}'$, $\mathscr{C}'^{\bullet}$, $\mathscr{K}'^{\bullet}$, we get a similar description of α'.

As above, we have a map of complexes $\mathscr{K}^{\bullet} \to \mathscr{K}'^{\bullet}$ which is multiplication by $\det(r_{ij})$ in degree $d-1$. Since $\underset{\sim}{t}\mathcal{O}_U = \mathcal{O}_U$, therefore $\mathcal{K}^{\bullet}|U$ is a <u>resolution</u> of $\widetilde{M}|U$, and so the diagram

$$\begin{array}{ccc} \mathcal{K}^{\bullet}|U & \xrightarrow{\text{via } \mathcal{E}^{\bullet}} & \\ \downarrow & & \mathcal{J}^{\bullet} \\ \mathcal{K}'^{\bullet}|U & \xrightarrow{\text{via } \mathcal{E}'^{\bullet}} & \end{array}$$

is <u>homotopy-commutative</u>. The commutativity of (7.2.2) now follows.

* * *

Now we can move toward the definition of the k-linear <u>residue map</u>

$$\mathrm{res}_R : H^d_{\mathfrak{m}}(\Omega_R) \to k$$

where R is a complete <u>regular</u> d-dimensional local k-algebra with maximal ideal $\mathfrak{m}$ such that the residue field $R/\mathfrak{m}$ is finite over k, $H^d_{\mathfrak{m}}$ denotes cohomology with supports in the closed point $\mathfrak{m}$ of Spec(R), and $\Omega_R = \Omega^d_{R/k}$ is the d-th exterior power of the universal <u>finite</u> differential module $\Omega^1_{R/k}$ (i.e. there is a k-derivation $\delta: R \to \Omega^1_{R/k}$ which is universal for k-derivations of R into <u>finitely generated</u> R-modules).

This residue map was mentioned by Grothendieck in [G4,pp.59-60], though it appears to have been around in analytic garb for a long time [GH,Chapter 5], and might have been known in some algebraic form to Macaulay (always with R regular). There exist various algebraic treatments in the literature, for example [RD,pp.195-199], [V,p.400] and, more explicitly, [B,§4] (R regular), [SS2] (R a complete intersection), [K2,§2] (R Cohen-Macaulay), [HL] and [Ho] (R arbitrary); and, from an intriguingly different viewpoint, [L].

* * *

Since the residue field $R/\mathfrak{m}$ is assumed finite over k, and k is perfect, therefore if K is the integral closure of k in R then K is a finite separable field extension of k, and if $\underset{\sim}{t} = (t_1,\ldots,t_d)$ is a regular parameter system in R (i.e. $\mathfrak{m} = \underset{\sim}{t}R$) then R is the power series ring $K[[t_1,\ldots,t_d]]$. Moreover the universal finite differential module $\Omega^1_{R/k}$ is free over R, with basis $\delta t_1,\ldots,\delta t_d$. We define a k-linear map

$$\operatorname{res}_{\underset{\sim}{t}}\colon H^d_{\mathfrak{m}}(\Omega_R) \to k$$

as follows: as above any $x \in H^d_{\mathfrak{m}}(\Omega_R)$ can be written as

$$x = \nu/\underset{\sim}{t}^a = (\sum_I \alpha_I t^I \delta\underset{\sim}{t})/\underset{\sim}{t}^a$$

for some $\nu \in \Omega_R$ and some integer $a > 0$, where $I = (i_1,\ldots,i_d)$ runs through d-tuples of non-negative integers, $\alpha_I \in K$, $t^I = t_1^{i_1} t_2^{i_2} \ldots t_d^{i_d}$, and $\delta\underset{\sim}{t} = \delta t_1 \delta t_2 \ldots \delta t_d$; and we set

$$\operatorname{res}_{\underset{\sim}{t}}[x] = \operatorname{trace}_{K/k}(\alpha_{a-1,a-1,\ldots,a-1}).$$

By (7.2) this definition does not depend on the choice of the representation $x = \nu/\underset{\sim}{t}^a$ (note that $\Omega_R \cong R$ has depth d). Moreover we have, with $\nu = \sum \alpha_I t^I \delta\underset{\sim}{t}$ as above,

$$\operatorname{res}_{\underset{\sim}{t}}[\nu/(t_1^{a_1},\ldots,t_d^{a_d})] = \operatorname{trace}_{K/k}(\alpha_{a_1-1,\ldots,a_d-1}) \qquad (a_i > 0)$$

LEMMA (7.3). <u>If</u> $\underset{\sim}{t} = (t_1,\ldots,t_d)$, $\underset{\sim}{u} = (u_1,\ldots,u_d)$ <u>are regular parameter systems in</u> R, <u>then</u> $\operatorname{res}_{\underset{\sim}{t}} = \operatorname{res}_{\underset{\sim}{u}}$.

<u>Proof</u>. Since $\operatorname{res}_{\underset{\sim}{t}}$ is k-linear, and since (7.2)(a) implies that

$$\nu/\underset{\sim}{t}^a = \sum_{I}{}_{(a)}\ \alpha_I t^I \delta\underset{\sim}{t}/\underset{\sim}{t}^a$$

where the sum $\sum_{(a)}$ is taken over those $I = (i_1,\ldots,i_d)$ such that $0 \le i_\lambda < a$ for $\lambda = 1,2,\ldots,d$, therefore it is enough to show, for any $\alpha \in K$, that

(7.3.1) $$\operatorname{res}_{\underset{\sim}{u}}[\alpha\delta\underset{\sim}{t}/\underset{\sim}{t}] = \operatorname{trace}_{K/k}(\alpha)$$

and that for any d-tuple of positive integers $(a_1,\ldots,a_d)$, at least one of which is ≥ 2, we have

$$\text{(7.3.2)} \qquad \operatorname{res}_{\underset{\sim}{u}}[\alpha\delta\underset{\sim}{t}/(t_1^{a_1},\ldots,t_d^{a_d})] = 0.$$

To prove (7.3.1) it is clearly enough to show that

$$\text{(7.3.1)}' \qquad \delta\underset{\sim}{t}/\underset{\sim}{t} = \delta\underset{\sim}{u}/\underset{\sim}{u}\ .$$

We can write

$$u_i = \sum_{j=1}^{d} r_{ij}t_j$$

with elements $r_{ij} \in R$ such that $\det(r_{ij})$ is a unit in R. Then

$$\delta u_i = \sum_{j=1}^{d}(\delta r_{ij})t_j + \sum_{j=1}^{d} r_{ij}\delta t_j$$

so that

$$\delta\underset{\sim}{u} - \det(r_{ij})\delta\underset{\sim}{t} \in \underset{\sim}{t}\Omega_R$$

and hence, by (7.2),

$$\det(r_{ij})\delta\underset{\sim}{t}/\underset{\sim}{t} = \delta\underset{\sim}{u}/\underset{\sim}{t} = \det(r_{ij})\delta\underset{\sim}{u}/\underset{\sim}{u}.$$

Since $\det(r_{ij})$ is a unit, (7.3.1)' follows.

For (7.3.2) we use the map

$$\partial_{\mathfrak{m}} : H^d_{\mathfrak{m}}(\Omega^{d-1}_{R/k}) \to H^d_{\mathfrak{m}}(\Omega^d_{R/k}) = H^d_{\mathfrak{m}}(\Omega_R)$$

induced by exterior differentiation considered as a map of sheaves of abelian groups over $\operatorname{Spec}(R)$. Working over $U = \operatorname{Spec}(R) - \mathfrak{m}$, we find that, for $\beta \in K$,

$$\text{(7.3.3)} \qquad \partial_{\mathfrak{m}}[\beta\delta u_1\ldots\widehat{\delta u_i}\ldots\delta u_d/(u_1^{b_1},\ldots,u_d^{b_d})]$$
$$= (-1)^i b_i\beta\delta u_1\ldots\delta u_d/(u_1^{b_1},\ldots,u_i^{b_i+1},\ldots,u_d^{b_d})$$

and it follows easily that

$$\text{(7.3.4)} \qquad \operatorname{res}_{\underset{\sim}{u}}[\partial_{\mathfrak{m}} y] = 0 \quad \text{for all } y \in H^d_{\mathfrak{m}}(\Omega^{d-1}_{R/k}) .$$

Replacing u by t in (7.3.3), we see with $0 \leq p$ = characteristic of K that if $a_i \not\equiv 1$ (mod. p) for some i then for any $\alpha \in K$, $\alpha\delta t_1 \ldots \delta t_d/(t_1^{a_1},\ldots,t_d^{a_d})$ is $\partial_{\mathfrak{m}} y$ for some y, so that (7.3.2) holds. In particular this completes the proof when k has characteristic zero.

Moreover, we have an induced map

$$\overline{\operatorname{res}}_{\underset{\sim}{t}} : H^d_{\mathfrak{m}}(\Omega_R)/\partial_{\mathfrak{m}} H^d_{\mathfrak{m}}(\Omega^{d-1}_{R/k}) = \bar{H}^d_{\mathfrak{m}}(\Omega_R) \to k.$$

To treat the case $a_i \equiv 1$ (mod. p) for all i (where now we may assume $p > 0$) we consider the exterior algebra $\Omega^{\bullet} = \oplus_{n\geq 0}\Omega^n_{R/k}$ as a complex via the exterior differentiation ∂, so that the homology

$$H^{\bullet} = \bigoplus_{n\geq 0} H^n(\Omega^{\bullet})$$

is a graded anticommutative $\mathbb{Z}$-algebra, with $\xi^2 = 0$ for all $\xi \in H^1$. Since exterior differentiation is R^p-linear, we may consider $H^{\bullet}$ to be an R-algebra via the Frobenius map $F : R \to R^p$ ($F(r) = r^p$ for all $r \in R$). Then there is an R-derivation $\gamma : R \to H^1$ given by

$$\gamma(r) = \text{homology class of } r^{p-1}\delta r ,$$

whence a homomorphism of graded R-algebras (the "inverse Cartier operator")

$$C^{-1} : \Omega^{\bullet} \to H^{\bullet} .$$

In particular we have an R-homomorphism

$$C^{-1} : \Omega_R \to \Omega_R/\partial\Omega^{d-1}_{R/k}$$

inducing

$$C_{\mathfrak{m}}^{-1} : H_{\mathfrak{m}}^{d}(\Omega_R) \to H_{\mathfrak{m}}^{d}(\Omega_R/\partial\Omega_{R/k}^{d-1}) = \overline{H}_{\mathfrak{m}}^{d}(\Omega_R) .$$

(For the last equality, use the fact that ∂ is R^p-linear, and that $H_{\mathfrak{m}}^{d+1}$ vanishes on R^p-modules.) One checks, for $\beta \in K$ and for non-negative integers $e_1,\ldots,e_d$, that

$$(7.3.5) \qquad C_{\mathfrak{m}}^{-1}[\beta\delta\underset{\sim}{u}/(u_1^{e_1+1},\ldots,u_d^{e_d+1})] \equiv \beta^p\delta\underset{\sim}{u}/(u_1^{e_1p+1},\ldots,u_d^{e_dp+1}) \quad (\text{mod. image of } \partial_{\mathfrak{m}})$$

and consequently that

$$(7.3.6) \qquad \overline{\text{res}}_{\underset{\sim}{u}}(C_{\mathfrak{m}}^{-1}[z]) = (\text{res}_{\underset{\sim}{u}}[z])^p \quad \text{for all } z \in H_{\mathfrak{m}}^{d}(\Omega_R) .$$

Now set

$$\nu_i = \sup\{n \mid p^n \text{ divides } a_i-1\}$$

and

$$\nu = \min_{1\le i\le d} \nu_i$$

so that $\nu < \infty$ if some $a_i > 1$. If $\nu = 0$, then some a_i is $\not\equiv 1 \pmod{p}$, and as above (7.3.2) holds. Then, using (7.3.5) with t in place of u, and (7.3.6), we see by induction that (7.3.2) holds for any value of ν

* * *

It makes sense now to *define*

$$\mathrm{res}_R : H^d_{\mathfrak{m}}(\Omega_R) \to k$$

by $\mathrm{res}_R = \mathrm{res}_{\underset{\sim}{t}}$ for any regular parameter system $\underset{\sim}{t}$.

We have then the following form of *local duality*:

THEOREM (7.4). *Let* R *be a* d-*dimensional complete regular local* k-*algebra, with maximal ideal* $\mathfrak{m}$, *such that* $[R/\mathfrak{m}:k] < \infty$; *and let* $\Omega_R = \Omega^d_{R/k}$ *be as above. Then the pair* $(\Omega_R, \mathrm{res}_R)$ *represents the functor*

$$H'(G) = \mathrm{Hom}_k(H^d_{\mathfrak{m}}(G), k)$$

of finitely generated R-*modules* G.

Proof. Since H' is left exact, we have a natural functorial isomorphism

$$H'(G) \xrightarrow{\sim} \mathrm{Hom}_R(G, H'(R))$$

(cf. [G4, p.44, Proposition 1.1]). In particular, corresponding to $\mathrm{res}_R \in H'(\Omega_R)$ we have an R-homomorphism

$$\sigma : \Omega_R \longrightarrow H'(R) ;$$

and by running through definitions we find that (7.4) simply asserts that σ *is an isomorphism*.

If $\underset{\sim}{t}$ is any regular parameter system in R, then one checks that σ is the inverse limit (cf. (7.2.1)) of the maps

$$\sigma_n : \Omega_R/\underset{\sim}{t}^n\Omega_R \to \mathrm{Hom}_k(R/\underset{\sim}{t}^n R, k) \qquad (n \geq 1)$$

associated with the pairings

$$\rho_n : \Omega_R/\underset{\sim}{t}^n\Omega_R \times R/\underset{\sim}{t}^n R \to k$$

given by

$$\rho_n(\nu+\underset{\sim}{t}^n\Omega_R,\ r+\underset{\sim}{t}^n R) = \operatorname{res}_{\underset{\sim}{t}}(r\nu/\underset{\sim}{t}^n).$$

Using the k-bases

$$\{t_1^{a_1}t_2^{a_2}\cdots t_d^{a_d}\delta t_1\delta t_2\cdots\delta t_d\} \qquad 0 \le a_i < n$$

$$\{t_1^{b_1}t_2^{b_2}\cdots t_d^{b_d}\} \qquad 0 \le b_i < n$$

of $\Omega_R/\underset{\sim}{t}^n\Omega_R$, $R/\underset{\sim}{t}^n R$ respectively, we see at once from the definition of $\operatorname{res}_{\underset{\sim}{t}}$ that the pairing ρ_n is non-degenerate, i.e. σ_n is an isomorphism. Q.E.D.

As a formal consequence of (7.4), we have a more general appearing version of local duality:

COROLLARY (7.5). Let R be any d-dimensional complete local k-algebra, with maximal ideal $\mathfrak{m}$, such that $[R/\mathfrak{m}:k] < \infty$. Let S be a d-dimensional complete regular local k-subalgebra of R, with maximal ideal $\mathfrak{n}$, such that R is a finite S-module. Let $\omega = \omega_{R,S}$ be the R-module

$$\omega = \operatorname{Hom}_S(R,\Omega_S)$$

and let $e:\omega \to \Omega_S$ be the S-homomorphism given by "evaluation at 1". Let $\rho = \rho_{R,S}$ be the composition

$$H^d_{\mathfrak{m}}(\omega) = H^d_{\mathfrak{n}}(\omega) \xrightarrow[H^d_{\mathfrak{n}}(e)]{} H^d_{\mathfrak{n}}(\Omega_S) \xrightarrow[\operatorname{res}_S]{} k \quad .$$

Then the pair (ω,ρ) represents the functor $\operatorname{Hom}_k(H^d_{\mathfrak{m}}(G),k)$ of finitely generated R-modules G.

§8. The residue theorem for projective space

In this section we establish the "primitive residue theorem" (0.4). For this, we will define the canonical isomorphism (for projective space $\mathbb{P} = \mathbb{P}^d_k$)

$$\int_{\mathbb{P}} : H^d(\mathbb{P}, \Omega_{\mathbb{P}}) \xrightarrow{\sim} k$$

using the residue map at the vertex of the projecting cone over $\mathbb{P}$ (Proposition (8.4) below), and then the theorem will come out of Lemma (8.6) which describes a canonical cohomology map via Čech cocycles. In essence this proof is closely related to the one sketched in [RD,p.200,Proposition 10.1]. Another proof, based on the Cousin complex, can be found in [K2,pp.186-187].

Using res_R as defined in §7 for local rings of the form $R = K[[t_1,\ldots,t_d]] \in \mathcal{R}^0_d$, and then (a) of (0.4) as a <u>definition</u> of res_R for non-complete $R \in \mathcal{R}^0_d$, we can reduce (b) of (0.4) to the complete case, which is readily handled (details left to the reader), because any étale $K[[t_1,\ldots,t_d]]$-algebra is of the form $K'[[t_1,\ldots,t_d]]$ with K' finite and separable over K, and the trace is transitive:

$$\mathrm{trace}_{K'/k}(\alpha) = \mathrm{trace}_{K/k}(\mathrm{trace}_{K'/k}(\alpha)) \qquad \alpha \in K'.$$

The rest of this section is devoted to a proof of (0.4)(c). We begin by recalling some explicit descriptions of differentials and cohomology on d-dimensional projective space $\mathbb{P} = \mathbb{P}^d_k$ $(d \geq 1)$, leading up to the definition of $\int_{\mathbb{P}}$.

For any graded module M over the polynomial ring $k[X] = k[X_0,X_1,\ldots,X_d]$, let $\tilde{M}$ be the corresponding quasi-coherent sheaf on $\mathbb{P}$. In particular we consider the module of Kähler differentials $\Omega^1_{k[X]/k}$, graded so that the free generators DX_i (D = universal k-derivation, $0 \leq i \leq n$) have degree 1. There is a canonical exact sequence of $\mathcal{O}_{\mathbb{P}}$-modules [AK,p.11], [H,p.176]

$$(8.1) \qquad 0 \longrightarrow \Omega^1_{\mathbb{P}/k} \xrightarrow[\xi]{} \tilde{\Omega}^1_{k[X]/k} \xrightarrow[\eta]{} \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

where ξ comes from the derivation of $\mathcal{O}_{\mathbb{P}} = (k[X])^{\sim}$ into $\tilde{\Omega}^1_{k[X]/k}$

induced by the universal derivation $D: k[X] \to \Omega^1_{k[X]/k}$ (which is homogeneous, of degree zero); and η is induced by the "Euler derivation" of $k[X]$ into itself, i.e. the derivation taking any homogeneous f of degree n to nf. If $Y \in k[X]$ is a linear form, and $A_Y = k[X_0/Y, \ldots, X_d/Y]$, then over $\mathrm{Spec}(A_Y)$, the complement in $\mathbb{P}$ of the hyperplane $Y = 0$, we can describe (8.1) as follows: we continue to denote by D the natural extension of D to the ring of fractions $k[X]_Y$, and let $\delta: \mathcal{O}_{\mathbb{P}} \to \Omega^1_{\mathbb{P}/k}$ be the universal derivation; the module of sections

$$\tilde{\Omega}^1(Y) = \Gamma(\mathrm{Spec}(A_Y), \tilde{\Omega}^1_{k[X]/k})$$

consists of elements of degree zero in the graded $k[X]_Y$-module

$$\Omega^1_{k[X]_Y/k} = \Omega^1_{A_Y[Y,Y^{-1}]/k} = (A_Y[Y,Y^{-1}] \otimes_{A_Y} \Omega^1_{A_Y/k}) \oplus A_Y[Y,Y^{-1}]DY,$$

and hence is the direct sum of the A_Y-module generated by $\{D(X_i/Y)\}_{0 \le i \le d}$ and the (free, rank one) A_Y-module generated by $Y^{-1}DY$; and, over A_Y, (8.1) corresponds to the split exact sequence of A_Y-modules

$$(8.2) \qquad 0 \longrightarrow \Omega^1_{A_Y/k} \xrightarrow{\xi_Y} \tilde{\Omega}^1(Y) \xrightarrow{\eta_Y} A_Y \longrightarrow 0$$

given by

$$\xi_Y(\delta f) = Df \qquad (f \in A_Y)$$

$$\eta_Y(D(X_i/Y)) = 0 \qquad (0 \le i \le d)$$

$$\eta_Y(Y^{-1}DY) = 1 \,.$$ [1]

From (8.1) and (8.2) we obtain a <u>canonical isomorphism</u>

$$(8.3) \qquad \psi: \Omega_{\mathbb{P}} = \Lambda^1 \mathcal{O}_{\mathbb{P}} \otimes \Lambda^d \Omega^1_{\mathbb{P}/k} \xrightarrow{\sim} \Lambda^{d+1} \tilde{\Omega}^1_{k[X]/k} = \tilde{\Omega}^{d+1}_{k[X]/k}$$

which is given over A_Y by

---

[1] The equivalence class of (8.1) in $\mathrm{Ext}^1_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}, \Omega^1_{\mathbb{P}/k}) = H^1(\mathbb{P}, \Omega^1_{\mathbb{P}/k})$ is the cohomology class of $\mathcal{O}_{\mathbb{P}}(1)$ [H, p.367, Ex. 1.8].

$$\psi(\delta f_1\delta f_2\ldots\delta f_d) = Y^{-1}DYDf_1Df_2\ldots Df_d \qquad (f_i \in A_Y)$$

* * *

Now let R be the localization of $k[X]$ at the maximal ideal $\mathfrak{M} = (X_0,X_1,\ldots,X_d)k[X]$ and let $m = \mathfrak{M}R$ be the maximal ideal of R. For a graded $k[X]$-module M, let M^* be the quasi-coherent sheaf on $\mathbb{A}^* = \mathrm{Spec}(k[X]) - \{\mathfrak{M}\}$ corresponding to M. There exist canonical surjective homomorphisms

$$\theta^p\colon \bigoplus_{n\in\mathbb{Z}} H^p(\mathbb{P},\widetilde{M}(n)) \longrightarrow H^{p+1}_m(M_{\mathfrak{M}}) \qquad (p \geq 0)$$

(bijective if $p > 0$), arising as follows: any finite sequence $\underset{\sim}{f} = (f_0,f_1,\ldots,f_m)$ of positive degree homogeneous elements in $k[X]$ such that $k[X]/(f_0,\ldots,f_m)$ is a finite-dimensional vector space over k defines an affine open covering $\mathcal{U}^* = (U_0^*,\ldots,U_m^*)$ of $\mathbb{A}^*$, with

$$U_i^* = \{x \in \mathbb{A}^* \mid f_i(x) \neq 0\} \qquad (0 \leq i \leq m);$$

and an affine open covering $\mathcal{U} = (U_0,U_1,\ldots,U_m)$ of $\mathbb{P}$, with

$$U_i = \{y \in \mathbb{P} \mid f_i(y) \neq 0\} \qquad (0 \leq i \leq m).$$

Then there is a natural identification of Čech complexes

$$\mathcal{C}^\bullet(\mathcal{U},\bigoplus_{n\in\mathbb{Z}} \widetilde{M}(n)) = \mathcal{C}^\bullet(\mathcal{U}^*,M^*)$$

(cf. [EGA III, §2]). So we have an isomorphism

$$\theta^!_{\underset{\sim}{f}}\colon \bigoplus_{n\in\mathbb{Z}} H^\bullet(\mathbb{P},\widetilde{M}(n)) \xrightarrow{\sim} H^\bullet(\mathbb{A}^*,M^*).$$

This $\theta^!_{\underset{\sim}{f}}$ does not depend on the choice of $\underset{\sim}{f}$: for if $\underset{\sim}{g} = (g_0,g_1,\ldots,g_{m'})$ is another such sequence, and

$$\underset{\sim}{fg} = (f_ig_j)_{1\leq i\leq m,\,1\leq j\leq m'}$$

is the product sequence (ordered in some way), then the coverings of $\mathbb{A}^*$ and $\mathbb{P}$ associated to $\underset{\sim}{fg}$ refine those associated respectively to $\underset{\sim}{f}$ and $\underset{\sim}{g}$, and the standard way of mapping the

Čech cohomology of a covering to the Čech cohomology of a refinement (applied over both $\mathbb{A}^*$ and $\mathbb{P}$) gives

$$\theta'_{\underset{\sim}{f}} = \theta'_{\underset{\sim}{fg}} = \theta'_{\underset{\sim}{g}} \quad .$$

Composing the isomorphism θ' with the natural surjective maps

$$H^p(\mathbb{A}^*, M^*) \to H^{p+1}_{\mathfrak{m}}(M_{\mathfrak{M}})$$

(bijective if $p > 0$) we obtain the above θ^p.

Ordinarily one would compute θ by taking $\underset{\sim}{f}$ to be a "system of coordinates" on $\mathbb{P}$, (i.e. the f_i are linearly independent forms of degree 1, and $m = d$). The point to note is that θ is then <u>independent of the choice of coordinates</u>.

* * *

In particular (cf. (8.3)) there is a <u>natural isomorphism</u>

$$\bigoplus_{n \in \mathbb{Z}} H^d(\mathbb{P}, \Omega_{\mathbb{P}}(n)) \xrightarrow{\sim} H^{d+1}_{\mathfrak{m}}(\Omega^{d+1}_{R/k}) \qquad (d \geq 1).$$

Moreover, we have

$$\begin{aligned} H^d(\mathbb{P}, \Omega_{\mathbb{P}}(n)) &\cong H^d(\mathbb{P}, \tilde{\Omega}^{d+1}_{k[X]/k}(n)) \\ &\cong H^d(\mathbb{P}, \Lambda^{d+1}[\mathcal{O}_{\mathbb{P}}(-1)^{d+1}](n)) \\ &\cong H^d(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n-d-1)) \end{aligned}$$

which <u>vanishes when</u> $n > 0$, while for $n \leq 0$, following through definitions, and with notation as in §7, we find that $H^d(\mathbb{P}, \Omega_{\mathbb{P}}(n)) \subset H^{d+1}_{\mathfrak{m}}(\Omega^{d+1}_{R/k})$ <u>is the</u> k-<u>vector space with basis consisting of all elements of the form</u>

$$DX_0 DX_1 \ldots DX_d / (X_0^{a_0}, \ldots, X_d^{a_d})$$

<u>where</u> $a_i > 0$ $(0 \leq i \leq d)$, <u>and</u>

$$\sum_{i=0}^{d} a_i = d + 1 - n.$$

Now the completion $\hat{\Omega}^1_{R/k}$ is naturally isomorphic to the universal finite differential module $\Omega^1_{\hat{R}/k}$, so that

$$H^{d+1}_{m}(\Omega^{d+1}_{R/k}) = H^{d+1}_{\hat{m}}(\Omega^{d+1}_{\hat{R}/k})$$

and as in §7 we can set

$$\text{res}_R = \text{res}_{\hat{R}}\colon H^{d+1}_{m}(\Omega^{d+1}_{R/k}) \to k .$$

From the definition of $\text{res}_R = \text{res}_{(X_0,X_1,\ldots,X_d)}$, we conclude that:

PROPOSITION (8.4). _With preceding notation, the residue map_ res_R _annihilates_ $H^d(\mathbb{P},\Omega_{\mathbb{P}}(n))$ _for_ $n \neq 0$, _and induces a canonical isomorphism_

$$\int_{\mathbb{P}}\colon H^d(\mathbb{P},\Omega_{\mathbb{P}}) \xrightarrow{\sim} k .$$

Remarks. We do not of course actually need the residue map to define $\int_{\mathbb{P}}$: we can simply say that $\int_{\mathbb{P}}$ _is the unique_ k-_linear map under which the generator_

$$DX_0\ldots DX_d/(X_0,\ldots,X_d)$$

of the one-dimensional k-_vector space_ $H^d(\mathbb{P},\Omega_{\mathbb{P}})$ _goes to_ $1 \in k$.

This description of $\int_{\mathbb{P}}$ is _independent of the "coordinate system_ $(X_0,\ldots,X_d)$"; because if $Y_0,Y_1,\ldots,Y_d$ are linear forms such that

$$k[X_0,\ldots,X_d] = k[Y_0,\ldots,Y_d]$$

then (cf. (7.3.1)'):

$$DX_0\ldots DX_d/(X_0,\ldots,X_d) = DY_0\ldots DY_d/(Y_0,\ldots,Y_d). \tag{8.4.1}$$

By the way, the equality (8.4.1) is equivalent to the following statement, which is also a corollary of (0.2B): _if_ $f\colon \mathbb{P} \to \mathbb{P}$ _is a_ k-_automorphism, then the map_ $H^d(\mathbb{P},\Omega_{\mathbb{P}}) \to H^d(\mathbb{P},f_*\Omega_{\mathbb{P}})$ _induced by the natural isomorphism_ $\Omega_{\mathbb{P}} \to f_*\Omega_{\mathbb{P}}$ _is inverse to the natural map_ $H^d(\mathbb{P},f_*\Omega_{\mathbb{P}}) \to H^d(\mathbb{P},\Omega_{\mathbb{P}})$.

*   *   *

For any closed point $x \in \mathbb{P}$, $H_x^\bullet$ will denote cohomology supported at x. As at the end of §7, we define the k-linear map

$$\mathrm{res}_x\colon H_x^d(\Omega_{\mathbb{P}}) \to k$$

by passing to the completion of the local ring $\mathcal{O}_{\mathbb{P},x}$. Then part (c) of the "primitive residue theorem" is:

PROPOSITION (8.5) (Residue theorem for $\mathbb{P}$). <u>For any closed point</u> $x \in \mathbb{P}$, <u>the following diagram commutes</u>:

$$\begin{array}{ccc} H_x^d(\Omega_{\mathbb{P}}) & \xrightarrow{\text{canonical}} & H^d(\mathbb{P},\Omega_{\mathbb{P}}) \\ & {\scriptstyle \mathrm{res}_x}\searrow \quad \swarrow{\scriptstyle \int_{\mathbb{P}}} & \\ & k & \end{array}$$

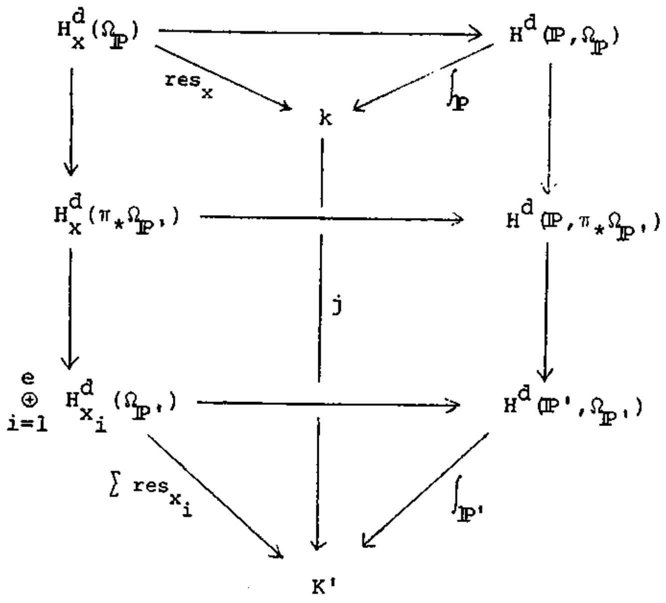
<u>Proof</u>. We first reduce to the case where x is a k-rational point.

Let K be the residue field of $\mathcal{O}_{\mathbb{P},x}$, so that K/k is a finite field extension, of degree, say, e. Let $K' \supset K$ be a finite Galois extension of k, and let

$$\pi\colon \mathbb{P}' = \mathbb{P}^d_{K'} = \mathbb{P}^d_k \times_k K' \to \mathbb{P}^d_k = \mathbb{P}$$

be the projection map. Then the fibre $\pi^{-1}(x)$ has e members $x_1,\ldots,x_e$, each of them K'-rational; and for $i = 1,2,\ldots,e$, the natural map of completions $\hat{\mathcal{O}}_{\mathbb{P},x} \to \hat{\mathcal{O}}_{\mathbb{P}',x_i}$ is étale.

Consider the following diagram, where $j\colon k \hookrightarrow K'$ is the inclusion, and the unlabelled arrows represent canonical maps.



The rectangles in this diagram are clearly commutative. *Once* (8.4) *is known for the* K'-*rational points* x_i, then the lower triangle commutes, and one can deduce (8.4) (details left to the reader) from the following two statements:

(8.5.1) *If* $\underset{\sim}{t} = (t_1,\ldots,t_d)$ *is a regular parameter system in* $\mathcal{O}_{\mathbb{P},x}$, *so that the image* $\underset{\sim}{t}_i$ *of* $\underset{\sim}{t}$ *in* $\mathcal{O}_{\mathbb{P}',x_i}$ *is also a regular parameter system, if* a *is a positive integer, and if* $\nu \in \Omega_{\mathbb{P},x}$ *has image* ν_i *in* $\Omega_{\mathbb{P}',x_i}$, *then*

$$j(\mathrm{res}_x[\nu/\underset{\sim}{t}^a]) = \sum_{i=1}^{e} \mathrm{res}_{x_i}[\nu_i/\underset{\sim}{t}_i^a] ;$$

and:

(8.5.2) *For any* $\mu \in H^d(\mathbb{P},\Omega_{\mathbb{P}})$, *with image* μ' *in* $H^d(\mathbb{P}',\Omega_{\mathbb{P}'})$, *we have*

$$\int_{\mathbb{P}'}\mu' = j\left(\int_{\mathbb{P}}\mu\right) .$$

Proof of (8.5.1). To find $\mathrm{res}_x[\nu/\underset{\sim}{t}^a]$, work in the completion $S = \hat{\mathcal{O}}_{\mathbb{P},x}$ and proceed as in §7: write

$$\nu = \sum_I \alpha_I t^I \delta t ,$$

so that $\mathrm{res}_x[\nu/\underset{\sim}{t}^a]$ is the coefficient of $t_1^{a-1}\ldots t_d^{a-1}\delta t$ in

$$\mathrm{trace}_{S/k[[t]]}(\nu) = \sum_I \mathrm{trace}_{K/k}(\alpha_I)t^I\delta t.$$

Now just note that "trace" = "sum of conjugates": more precisely, using the fact that trace is compatible with base change, and that, with $S' = K'[[t]]$,

$$S\otimes_{k[[t]]}S' = S\otimes_k K' = \prod_{i=1}^{e} \hat{\mathcal{O}}_{\mathbb{P}',x_i}$$

with $\mathcal{O}_{\mathbb{P}',x_i} \cong S'$ for each i, one finds that

$$\mathrm{trace}_{S/k[[t]]}(\nu) = \mathrm{trace}_{(S\otimes_{k[[t]]}S')/S'}(\nu\otimes 1)$$
$$= \sum_{i=1}^{e} \nu_i$$

(where $\Omega_{k[[t]]/k}$ is naturally identified with a subgroup of $\Omega_{S'/k}$). From these observations, (8.5.1) follows.

<u>Proof of (8.5.2)</u>. Check that if μ is the canonical generator of $H^d(\mathbb{P},\Omega_{\mathbb{P}})$, then μ' is the canonical generator of $H^d(\mathbb{P}',\Omega_{\mathbb{P}'})\ldots$

* * *

It remains now to prove (8.5) when x is k-rational. The maps involved do not depend on coordinates, so we may choose a coordinate system $(X_0,\ldots,X_d)$ on $\mathbb{P}$ such that x is the point $(1,0,0,\ldots,0)$. Let U_i $(0 \le i \le d)$ be the complement of the hyperplane $X_i = 0$. Then $\mathcal{U} = \{U_i\}_{0\le i\le d}$ is an open covering of $\mathbb{P}$,

$$\{x\} = \mathbb{P} - \bigcup_{i>0} U_i ,$$

and

$$\mathcal{U}' = \{U_0 \cap U_i\}_{i>0}$$

is an open covering of $U_0 - \{x\}$. Set $t_i = X_i/X_0$, which is a rational function on U_0. An examination of the definitions of

res_x and $\int_{\mathbb{P}}$ shows that it suffices to do the following : consider in the Čech complex $\mathcal{C}^\bullet(\mathcal{U}',\Omega_{\mathbb{P}})$ a (d-1)-cocycle of the form

$$\xi = \delta t_1 \ldots \delta t_d / t_1^{a_1} \ldots t_d^{a_d} \in \Gamma((U_0 \cap U_1) \cap \ldots \cap (U_0 \cap U_d), \Omega_{\mathbb{P}}) \qquad (a_i > 0);$$

the corresponding cohomology class in $H^{d-1}(U_0 - \{x\}, \Omega_{\mathbb{P}})$ has a canonical image in $H^d_x(\Omega_{\mathbb{P}})$ and hence in $H^d(\mathbb{P},\Omega_{\mathbb{P}})$; then <u>prove that this last image is represented by the</u> d-<u>cocycle</u> $\tilde{\xi}$ <u>in</u> $\mathcal{C}^\bullet(\mathcal{U},\Omega_{\mathbb{P}})$ <u>given by</u>

$$\tilde{\xi} = \delta t_1 \ldots \delta t_d / t_1^{a_1} \ldots t_d^{a_d} \in \Gamma(U_0 \cap U_1 \cap \ldots \cap U_d, \Omega_{\mathbb{P}})$$

(which maps, under the canonical isomorphism (8.3), to the d-cocycle

$$DX_0 DX_1 \ldots DX_d / X_0^{d+1-a_1-\ldots-a_d} X_d^{a_1} \ldots X_d^{a_d} \in \Gamma(U_0 \cap \ldots \cap U_d, \tilde{\Omega}^{d+1}_{k[X]/k})$$

which is a coboundary if $d + 1 - a_1 - \ldots - a_d \le 0$, so that its cohomology class vanishes unless $a_1 = a_2 = \ldots = a_d = 1$, in which case the cohomology class is the canonical generator of $H^d(\mathbb{P},\Omega_{\mathbb{P}})$).

Thus we need to explicate the canonical map $H^d_x(\cdot) \to H^d(\mathbb{P},\cdot)$ in terms of Čech cohomology. This is carried out in a more general context in the following discussion, whose principal conclusion (Lemma (8.6)) provides a solution to the preceding problem.

Let U be any topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of U. We assume without loss of generality that the index set I is <u>totally ordered</u>, and has a <u>least element</u> 0. Set

$$Y = U - \bigcup_{i>0} U_i;$$

then $Y \subset U_0$ is a closed subset of U, and

$$\mathcal{U}' = \{U_0 \cap U_i\}_{i>0}$$

is an open covering of $U_0 - Y$.

Let $\mathcal{F}$ be a sheaf of abelian groups on U, and consider the alternating Čech complexes

$$\mathcal{C}_{\mathcal{F}} = \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \qquad \mathcal{C}'_{\mathcal{F}} = \mathcal{C}^\bullet(\mathcal{U}', \mathcal{F}|U_0 - Y)$$

so that for any $p \geq 0$ we have

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<i_1<\ldots<i_p} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p})$$

with the usual differential $d:\mathcal{C}^p \to \mathcal{C}^{p+1}$, and similarly for $\mathcal{C}^p(\mathcal{U}', \mathcal{F}|U_0 - Y)$. Recall that for any complex $\underset{\sim}{C}$:

$$\ldots \longrightarrow C^{m-1} \xrightarrow{d} C^m \xrightarrow{d} C^{m+1} \longrightarrow \ldots$$

$\underset{\sim}{C}[1]$ is the complex such that for all integers n

$$(\underset{\sim}{C}[1])^n = \underset{\sim}{C}^{n+1} ,$$

and whose differential $d[1]$ is $-d$. We define a homomorphism of complexes

$$\varphi = \varphi_{\mathcal{F}}: \mathcal{C}'_{\mathcal{F}} \longrightarrow \mathcal{C}_{\mathcal{F}}[1]$$

as follows:

For any $\xi \in \mathcal{C}'^p_{\mathcal{F}}$, let

$$\tilde{\xi} = \varphi\xi \in (\mathcal{C}_{\mathcal{F}}[1])^p = \mathcal{C}^{p+1}_{\mathcal{F}}$$

be given (for $i_0 < i_1 < \ldots < i_p \in I$) by

$$\begin{aligned} \tilde{\xi}_{i_0 i_1 \ldots i_p} &= \xi_{i_1 \ldots i_p} \quad (\in \mathcal{F}(U_0 \cap U_{i_1} \cap \ldots \cap U_{i_p})) && \text{if } i_0 = 0 \\ &= 0 && \text{if } i_0 > 0. \end{aligned}$$

That φ is a homomorphism of complexes, i.e.

$$(d\xi)^{\sim} = -d\tilde{\xi}$$

is easily checked. Passing to cohomology we deduce a homomorphism, functorial in $\mathcal{F}$:

$$\phi^p_{\mathcal{F}}: H^p(\mathcal{U}', \mathcal{F}|U_0 - Y) = H^p(\mathcal{C}'_{\mathcal{F}}) \to H^p(\mathcal{C}_{\mathcal{F}}[1]) = H^{p+1}(\mathcal{U},\mathcal{F}).$$

LEMMA (8.6). The following diagram - in which unlabelled arrows represent canonical maps, and $H^{\bullet}_Y$ is cohomology with supports in Y- commutes:

$$\begin{array}{ccccc} H^p(\mathcal{U}', \mathcal{F}|U_0 - Y) & & \xrightarrow{\quad \phi^p_{\mathcal{F}} \quad} & & H^{p+1}(\mathcal{U},\mathcal{F}) \\ \downarrow & & & & \downarrow \\ H^p(U_0 - Y,\mathcal{F}) & \longrightarrow & H^{p+1}_Y(\mathcal{F}) & \longrightarrow & H^{p+1}(U,\mathcal{F}) \end{array}$$

Proof. Let

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \varepsilon\ } \underset{\sim}{F}^{\cdot} = F^0 \xrightarrow{\ \partial\ } F^1 \xrightarrow{\ \partial\ } F^2 \longrightarrow \cdots$$

be a flasque resolution of $\mathcal{F}$. As in [Go,p.213] we can describe a p-cocycle ξ^* in $\Gamma(U_0 - Y, F^p)$ representing the canonical image of the homology class of a p-cocycle $\xi \in \mathcal{C}'^p_{\mathcal{F}}$ as follows:

$\varepsilon\xi \in \mathcal{C}'^p_{F^0}$ is the coboundary of some $\xi^{p-1} \in \mathcal{C}'^{p-1}_{F^0}$

$(-1)^{p-2}\partial\xi^{p-1} \in \mathcal{C}'^{p-1}_{F^1}$ is the coboundary of some $\xi^{p-2} \in \mathcal{C}'^{p-2}_{F^1}$

$(-1)^{p-3}\partial\xi^{p-2} \in \mathcal{C}'^{p-2}_{F^2}$ is the coboundary of some $\xi^{p-3} \in \mathcal{C}'^{p-3}_{F^2}$

$\vdots$

$\partial\xi^1 \in \mathcal{C}'^1_{F^{p-1}}$ is the coboundary of some $\xi^0 \in \mathcal{C}'^0_{F^{p-1}}$

$\partial\xi^0$ <u>is a 0-cocycle in</u> $\mathcal{C}'^0_{F^p}$, <u>hence the image of some</u> $\xi^* \in \Gamma(U_0 - Y, F^p)$.

Operating similarly with $\tilde{\xi} \in \mathcal{C}^{p+1}_{\mathcal{F}}$, we find that we can actually take

$$\tilde{\xi}^{p+1-i} = -\varphi_{F^{i-1}}\xi^{p-i} \in \mathcal{C}^{p+1-i}_{F^{i-1}} \qquad 1 \le i \le p .$$

To get $\tilde{\xi}^*$, we still need $\tilde{\xi}^0 \in \mathcal{C}^0_{F^p}$, which we construct as follows: let $\bar{\xi} \in F^p(U_0)$ be such that

$$\bar{\xi}|U_0 - Y = \xi^*$$

($\bar{\xi}$ exists because F^p is flasque); and define $\tilde{\xi}^0$ by

$$(\tilde{\xi}^0)_i = \bar{\xi} \qquad i = 0$$

$$= 0 \qquad i > 0 .$$

Then we check that $\partial\tilde{\xi}^1 \in \mathscr{C}^1_{F^p}$ is indeed the coboundary of $\tilde{\xi}^0$.
Thus $\tilde{\xi}^* \in \Gamma(U,F^{p+1})$ is such that

$$\tilde{\xi}^*|U_0 = \partial\bar{\xi}$$

$$\tilde{\xi}^*|U_i = 0 \qquad i > 0.$$

On the other hand, calculating the canonical map

$$\Psi^p_{\mathscr{F}}: H^p(U_0 - Y,\mathscr{F}) \to H^{p+1}_Y(\mathscr{F}) \to H^{p+1}(U,\mathscr{F})$$

by applying the exact sequence of functors

$$0 \longrightarrow \Gamma_Y \longrightarrow \Gamma(U_0,\cdot) \longrightarrow \Gamma(U_0 - Y,\cdot)$$

to the flasque resolution $\underset{\sim}{F}^\bullet$, we find directly that $\Psi^p_{\mathscr{F}}$ takes the cohomology class of ξ^* to that of the above $\tilde{\xi}^*$.

This completes the proof of Lemma (8.6), and hence of (8.5) and the primitive residue theorem.

§9. Compatibility of local and global duality.

The main result of this section, Theorem (9.1), shows how a dualizing $\mathcal{O}$-module induces local duality. This result may be viewed as a special case of [RD,p.386, Proposition 3.5]. The proof is by reduction to the case of projective space, where the results of §§7-8 are immediately applicable. Theorem (9.1) enables us to give in (9.3) a quick proof of Proposition (6.3), thereby completing the proof of (0.3) (cf. §§4,6) hence of (0.2) (cf. (0.3.2)); and then - as indicated in (0.6.1) - the Residue Theorem (0.6) follows.

As in (1.4), we consider a d-dimensional local k-algebra R which is a localization of a k-algebra of finite type having no non-zero zerodivisors. We assume furthermore that, $\mathfrak{m}$ being the maximal ideal of R, the residue field $R/\mathfrak{m}$ is a finite extension of k. Let V be a proper k-variety, and let

$$\varphi\colon \mathrm{Spec}(R) \to V$$

be a k-morphism such that the corresponding map

$$\mathcal{O}_{V,v} \to R \qquad (v = \varphi(\mathfrak{m}))$$

is an isomorphism (Such V, φ, with say V projective, clearly exist.) Note that v is a closed point of V, since $\mathcal{O}_{V,v}$ is residually finite over k. Let $\omega = (\{\omega_W\},\{\theta_W\})$ be a dualizing $\mathcal{O}$-module (cf. §4). Define $\rho_R = \rho_R(\omega)$ to be the composition

$$H^d(\omega_R) \xrightarrow{\ \mu\ } H^d_v(\omega_V) \xrightarrow{\ \nu\ } H^d(V,\omega_V) \xrightarrow{\ \theta_V\ } k$$

where μ is defined via the isomorphisms $\mathcal{O}_{V,v} \xrightarrow{\sim} R$, $R\otimes_{\mathcal{O}_{V,v}}\omega_{V,v} \xrightarrow{\sim} \omega_R$ induced by φ (cf. (1.4)(iii)), and ν is the natural map from local to global cohomology.

THEOREM (9.1) (a) The map ρ_R depends only on ω and R (not on φ).

(b) If ^ denotes $\mathfrak{m}$-adic completion, so that in particular $H^d_{\mathfrak{m}}(\omega_R) = H^d_{\hat{\mathfrak{m}}}(\hat{\omega}_R)$, then the pair $(\hat{\omega}_R, \rho_R)$ represents the functor $\mathrm{Hom}_k(H^d_{\hat{\mathfrak{m}}}(G),k)$ of finitely generated $\hat{R}$-modules G.

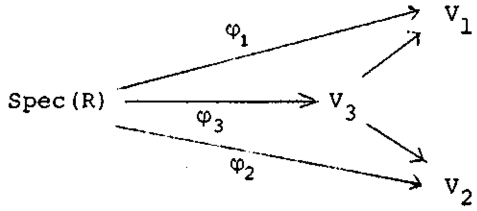
Proof. (a) Suppose we have maps $\varphi_i\colon \mathrm{Spec}(R) \to V_i$ (i = 1,2) as above. By [EGA 0I, pp.311-312], there exist open neighborhoods V_i' of $\varphi_i(\mathfrak{m})$ in V_i, and an isomorphism $\psi\colon V_1' \to V_2'$ such that $\psi\varphi_1 = \varphi_2$. Let $V_3 \subset V_1 \times_k V_2$ be the closure of the graph

$$\Gamma_\psi \subset V_1' \times_k V_2' \subset V_1 \times_k V_2$$

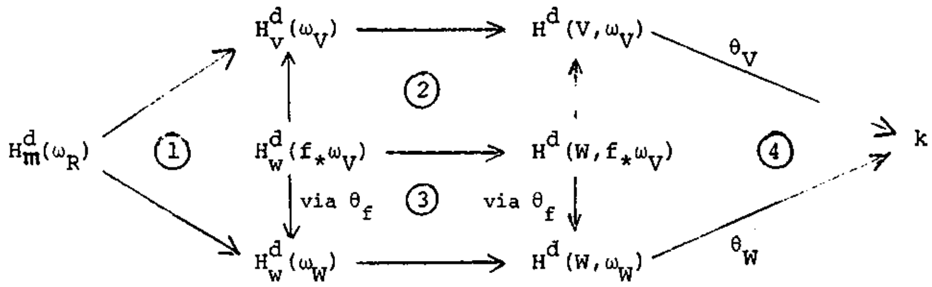
of ψ[1]. Then, as is easily seen, there is a commutative diagram

---

(1) V_3 is also the join of V_1 and V_2, i.e. the closed image of the canonical map $\mathrm{Spec}(K) \to V_1 \times_k V_2$, where K is the fraction field of R.



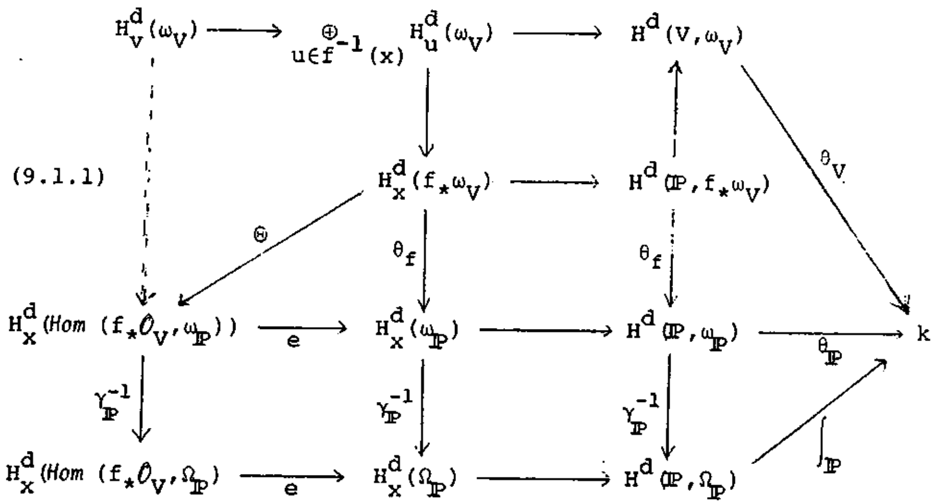
such that the map $\mathcal{O}_{V_3,\varphi_3(\mathfrak{m})} \longrightarrow R$ induced by φ_3 is an isomorphism. Thus we may assume without loss of generality that there exists a map $f:V_2 \to V_1$ such that $f\varphi_2 = \varphi_1$. For convenience we set $V = V_2$, $W = V_1$, $v = \varphi_2(\mathfrak{m})$, $w = \varphi_1(\mathfrak{m})$, so that f is a local isomorphism at v and $f(v) = w$. We have then the map $\theta_f:f_*\omega_V \to \omega_W$ of (5.1), and the assertion (~~8.6~~ 4.1)(a) results from the commutativity of the following diagram (where the unlabelled maps are the obvious ones):



The commutativity of ④ follows from (5.1)(i), and of ① from (5.1)(ii) and (5.3); the commutativity of ② and ③ is left to the reader.

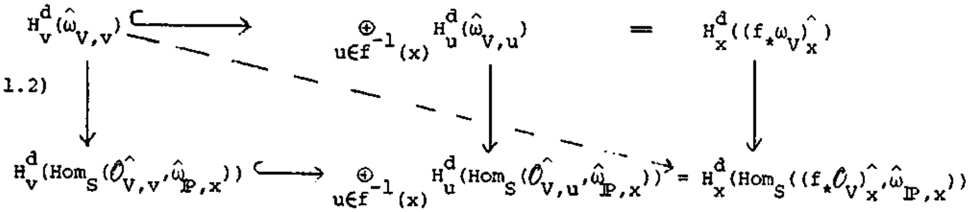
(b). We can choose $\varphi:\mathrm{Spec}(R) \to V$ as above, with V <u>projective</u>, and then by Noether normalization choose a finite map $f:V \to \mathbb{P} = \mathbb{P}^d_k$. Let $x = f(v)$. Then we have the map $\theta_f:f_*\omega_V \to \omega_{\mathbb{P}}$ and the isomorphism $\Theta = \Theta_{f,\mathcal{O}_V}:f_*\omega_V \to \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(f_*\mathcal{O}_V,\omega_{\mathbb{P}})$ of (4.3), and as in §6, the isomorphism $\gamma_{\mathbb{P}}:\Omega_{\mathbb{P}} \xrightarrow{\sim} \omega_{\mathbb{P}}$ corresponding to the canonical isomorphism $\int_{\mathbb{P}}$ of (8.4); and there is a commutative diagram

(9.1.1)



where the maps labelled "e" are induced by evaluation at 1. With S the completion $\hat{\mathcal{O}}_{\mathbb{P},x}$, we also have a commutative diagram

(9.1.2)



where the broken arrows in (9.1.1) and (9.1.2) represent the same map.

In view of (8.5) we deduce from (9.1.1) and (9.1.2) the following commutative diagram (where $\mathfrak{n}$ is the maximal ideal of S):

$$\begin{array}{ccc} H^d_{\mathfrak{m}}(\hat{\omega}_R) & \xrightarrow{\rho_R} & k \\ \downarrow & & \uparrow \mathrm{res}_S \\ H^d_{\mathfrak{m}}(\mathrm{Hom}_S(\hat{R}, \Omega_S)) & \xrightarrow[e]{} & H^d_{\mathfrak{n}}(\Omega_S) \end{array}$$

and the conclusion follows from (7.5). Q.E.D.

COROLLARY (9.2). *With notation as in* (9.1), *assume further that* R *is regular, so that* v *has a neighborhood* V_α *which admits an étale map* h *to* $\mathbb{P}_k^d$. *If* γ_{V_α} *is the isomorphism of* (6.3), *then the following diagram commutes*:

$$\begin{array}{ccc} H_m^d(\Omega_R) & \xrightarrow{\text{via } \varphi \text{ and } \gamma_{V_\alpha}} & H_m^d(\omega_R) \\ & {\scriptstyle \text{res}_R} \searrow \quad \swarrow {\scriptstyle \rho_R} & \\ & k & \end{array}$$

Proof. In (9.1.1) we can put $V = \bar{V}$, $f = \bar{h}$ (cf. (6.1)). From the definition of γ_{V_α}, and in view of (8.5), we obtain a commutative diagram

$$\begin{array}{ccc} H_v^d(\Omega_V) & \longrightarrow & H_v^d(\omega_V) \\ {\scriptstyle t} \downarrow & \swarrow {\scriptstyle \text{cf. } (9.1.1)} & \downarrow {\scriptstyle \rho_{\mathcal{O}_{V,v}}} \\ H_x^d(\Omega_{\mathbb{P}}) & \xrightarrow{\text{res}_x} & k \end{array}$$

where t is induced by the trace map for differential forms. Since $\mathcal{O}_{V,v}$ is étale over $\mathcal{O}_{\mathbb{P},x}$, the inclusion $\hat{\mathcal{O}}_{\mathbb{P},x} \hookrightarrow \hat{\mathcal{O}}_{V,v}$ can be identified with the inclusion

$$k'[[X_1,\ldots,X_d]] \subset k''[[X_1,\ldots,X_d]]$$

where $k' \subset k''$ are the residue fields at x and v respectively. It then follows easily from definitions that

$$\text{res}_v = \text{res}_x \circ t \, ,$$

and (9.2) results. Q.E.D.

Now, finally we can give:

(9.3) *Proof of (6.3)*.

In (6.3), for any $v \in V_\alpha \cap V_\beta$, with $R = \mathcal{O}_{V,v}$ we see by (9.2) (for α and β both) and by (9.1)(b) that the germs at v of γ_{V_α} and γ_{V_β} have the same composition with the (injective) completion map $\omega_R \to \hat{\omega}_R$; so γ_{V_α} and γ_{V_β} agree in a neighborhood of v, hence everywhere on $V_\alpha \cap V_\beta$. Q.E.D.

Remarks (9.4). We may now consider Theorems (0.1), (0.2), (0.3), (0.4) and – above all – (0.6) of the Introduction to be proved.

(9.5) We know now that $\tilde{\omega}$ has a natural dualizing structure (Theorem (0.2)); and, from Remark (0.6.1) we see that

$$\mathrm{res}_R^{\sim} = \rho_R(\tilde{\omega})$$

(notation as in (9.1)).

(9.6) Let v be a closed point on a proper d-dimensional k-variety V, and let $\mathcal{G}$ be a coherent $\mathcal{O}_V$-module. Then the natural map

$$H_v^d(\mathcal{G}) \to H^d(V,\mathcal{G})$$

is surjective. This follows at once from (9.1), since the dual map

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{G},\omega_V) \to \mathrm{Hom}_{\mathcal{O}_{V,v}}(\mathcal{G}_v,\hat{\omega}_v)$$

is clearly injective. (Cf. [G3,p.100,Theorem 6.9]; and also [Kml] for a simpler proof which avoids duality theory.)

§10. A relative residue theorem

In this section we prove an expanded relative version (10.2) of Theorem (9.1). In view of (9.5), this gives a generalization of the Residue Theorem (0.6).

Let $\mathcal{R} = \bigcup_{d\geq 0} \mathcal{R}_d$ (cf. (0.6)) be the collection of all local domains R which are localizations of finitely generated k-algebras, and whose residue field $R/\mathfrak{m}_R$ ($\mathfrak{m}_R$ = maximal ideal of R) is finite over k.

DEFINITION (10.1). We say that a k-homomorphism $\varphi: R \to S$ ($R, S \in \mathcal{R}$) is admissible if φ is injective and if for every height one prime ideal $\mathfrak{p}$ in S the localization $R_{\varphi^{-1}(\mathfrak{p})}$ satisfies the Serre condition (S_2).

Examples. (i) If R itself satisfies (S_2) (in particular if $\dim R \le 1$) then every injective φ is admissible.

(ii) If $\dim R \ge 2$ and for every height one $\mathfrak{p}$ in S, $\varphi^{-1}(\mathfrak{p})$ has height ≤ 1 in R, then φ is admissible. In particular if φ is flat then φ is admissible.

THEOREM (10.2). Let $\omega = (\{\omega_V\},\{\theta_V\})$ be a dualizing $\mathcal{O}$-module. There exists a unique family of R-linear maps

$$\rho_\varphi : H^s_{\mathfrak{m}_S}(\omega_S) \to H^r_{\mathfrak{m}_R}(\omega_R)$$

indexed by admissible maps $\varphi: R \to S$, with $R, S \in \mathcal{R}$, $r = \dim R$, $s = \dim S$, and satisfying the following conditions (a) and (b):

(a) If $R = k$ and φ is the obvious map (which is admissible) then - after naturally identifying $H^0_{\mathfrak{m}_k}(\omega_k)$ with k - we have

$$\rho_\varphi = \rho_S \qquad \text{(cf. (9.1)).}$$

(b) If $\varphi: R \to S$, $\psi: S \to T$, and $\psi\varphi$ are all admissible, then

$$\rho_{\psi\varphi} = \rho_\varphi \circ \rho_\psi \quad .$$

Furthermore:

(c) With $\varphi: R \to S$, r, s as above, if ^ denotes completion, so that

$$H^s_{\mathfrak{m}_S}(\omega_S) = H^s_{\hat{\mathfrak{m}}_S}(\hat{\omega}_S),$$

then the pair $(\hat{\omega}_S, \rho_\varphi)$ represents the functor $\mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G), H^r_{\mathfrak{m}_R}(\omega_R))$ of finitely generated $\hat{S}$-modules G.

(d) Let $f: V \to W$ be a proper map of k-varieties, $s = \dim V$, $r = \dim W$, and suppose that f is "equidimensional in codimension 1", i.e. the subvariety

$$\{v \in V \mid \dim_v(f^{-1}f(v)) > s - r\}$$

of V has codimension ≥ 2. Let $w \in W$ be a closed point. Then for each closed point $v \in E = f^{-1}(w)$, the map $\varphi_{w,v}: \mathcal{O}_{W,w} \to \mathcal{O}_{V,v}$

induced by f *is admissible; and there is a unique* $\mathcal{O}_{W,w}$*-linear map*

$$\theta_{[f,w]}: H^s_E(\omega_V) \to H^r_w(\omega_W)$$

such that for each such v *the following diagram commutes:*

$$\begin{array}{ccc} H^s_v(\omega_V) & \xrightarrow{\text{canonical}} & H^s_E(\omega_V) \\ & \searrow^{\rho_{\varphi_{w,v}}} \quad\quad {}^{\theta_{[f,w]}}\swarrow & \\ & H^r_w(\omega_W) & \end{array}$$

(e) *With assumptions as in* (d), *if* $\hat{V}$ *is the formal completion of* V *along* E, *then the pair* $(\hat{\omega}_V, \theta_{[f,w]})$ *represents the functor*

$$H^*(\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{W,w}}(H^s_{\hat{E}}(\mathcal{G}), H^r_w(\omega_W))$$

of coherent $\mathcal{O}_{\hat{V}}$*-modules* $\mathcal{G}$ (cf. following explanation).

Explanation. We *define* $H^s_{\hat{E}}(\mathcal{G})$ by

$$H^s_{\hat{E}}(\mathcal{G}) = \varinjlim_{n>0} \mathrm{Ext}^s_{\mathcal{O}_{\hat{V}}}(\mathcal{O}_{\hat{V}}/\mathfrak{m}^n_w\mathcal{O}_{\hat{V}}, \mathcal{G})$$

where $\mathfrak{m}_w$ is the maximal ideal of $\mathcal{O}_{W,w}$. Then if $\mathcal{G} = \hat{\mathcal{F}}$ for some coherent $\mathcal{O}_V$-module $\mathcal{F}$, we find, using [EGA III, (4.5.1)] and [G3,p.30,Th.2.8] (= [G4,p.22,Th.6]) that

$$H^s_{\hat{E}}(\hat{\mathcal{F}}) = \varinjlim_{n>0} \mathrm{Ext}^s_{\mathcal{O}_V}(\mathcal{O}_V/\mathfrak{m}^n_w\mathcal{O}_V, \mathcal{F}) = H^s_E(\mathcal{F}).$$

In particular we can consider the map

$$\begin{array}{c} \theta_{[f,w]}: H^s_E(\omega_V) \to H^r_w(\omega_W) \\ \| \\ H^s_{\hat{E}}(\hat{\omega}_V) \end{array}$$

to be an element of $H^*(\hat{\omega}_V)$.

Proof of (10.2). We consider a k-homomorphism $\varphi:R \to S$ $(R,S \in \mathcal{R})$, which, for the moment, need not be admissible. Set

$$I_R = \varinjlim_{n>0} \mathrm{Hom}_k(R/\mathfrak{m}_R^n,k) \subset \mathrm{Hom}_k(R,k).$$

By local duality (cf. (9.1)(b)) the map

$$\rho_S: H^s_{\mathfrak{m}_S}(\omega_S) \longrightarrow k \qquad (s = \dim S)$$

induces an isomorphism of functors of finitely generated $\hat{S}$-modules G:

$$(10.2.1) \qquad \mathrm{Hom}_{\hat{S}}(G,\hat{\omega}_S) \xrightarrow{\sim} \mathrm{Hom}_k(H^s_{\hat{\mathfrak{m}}_S}(G),k)$$
$$= \mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G),\mathrm{Hom}_k(R,k))$$
$$= \mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G),I_R)$$

where the last equality holds because each element of $H^s_{\hat{\mathfrak{m}}_S}(G)$ is annihilated by some power of $\hat{\mathfrak{m}}_S$, hence by some power of $\mathfrak{m}_R$. (Note that $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, since $S/\mathfrak{m}_S$ is finite over k.) So there is a unique R-linear map

$$\sigma_\varphi: H^s_{\mathfrak{m}_S}(\omega_S) = H^s_{\hat{\mathfrak{m}}_S}(\hat{\omega}_S) \longrightarrow I_R$$

corresponding to the identity map of $\hat{\omega}_S$, i.e. such that the composition (evaluation at 1) $\circ\, \sigma_\varphi$ is ρ_S. In particular, the identity map $R \to R$ gives us an R-homomorphism

$$\sigma_R: H^r_{\mathfrak{m}_R}(\omega_R) \to I_R \qquad (r = \dim R);$$

and then a), b), and c) of (10.2) follow easily (details left to the reader) from:

LEMMA (10.3). The preceding map σ_R is surjective. Moreover if $\varphi:R \to S$ is admissible and J is the kernel of σ_R, then for any finitely generated $\hat{S}$-module G,

$$\mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G),J) = \mathrm{Ext}^1_R(H^s_{\hat{\mathfrak{m}}_S}(G),J) = 0\ ,$$

whence σ_R *induces an isomorphism of functors*

$$\mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G), H^r_{\mathfrak{m}_R}(\omega_R)) \xrightarrow{\sim} \mathrm{Hom}_R(H^s_{\hat{\mathfrak{m}}_S}(G), I_R).$$

The proof of (10.3) will be based on Corollary (10.5) below.

LEMMA (10.4). *Let* B *be a normal noetherian domain with fraction field* K, *let* $L \supset K$ *be a finite field extension, and let* $C \supset B$ *be a module-finite* B-*subalgebra of* L. *For any* B-*module* M, *set* $M^* = \mathrm{Hom}_B(M,B)$. *Then, with* C^*, C^{**} *considered as* C-*modules in the obvious way, we have a natural commutative diagram of* C-*linear maps*

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Hom}_C(C^*,C^*) \\ {\scriptstyle\alpha}\downarrow & & \| \\ C^{**} & = = = & \mathrm{Hom}_B(C^*,B) \end{array}$$

with α *injective, and for any prime ideal* $\mathfrak{q}$ *in* C, *the localization* $\alpha_{\mathfrak{q}}$ *is an isomorphism if and only if the local ring* $C_{\mathfrak{q}}$ *satisfies* (S_2).

Proof. Only the last assertion is not straightforward. First of all, since B satisfies (S_2), so therefore does the B-module $C^{**} = \mathrm{Hom}_B(C^*,B)$. Hence [EGA IV, (5.7.11)] so does the C-module C^{**}, as does the $C_{\mathfrak{q}}$-module $(C^{**})_{\mathfrak{q}}$. So if $\alpha_{\mathfrak{q}}$ is an isomorphism, then $C_{\mathfrak{q}}$ satisfies (S_2).

Conversely, suppose that $C_{\mathfrak{q}}$ satisfies (S_2). Since C is a torsion-free finitely-generated B-module, we may identify $C_{\mathfrak{q}}$ with a $C_{\mathfrak{q}}$-submodule of $(C^{**})_{\mathfrak{q}} \subset L$. Since $B_{\mathfrak{q}'\cap B}$ is a discrete valuation ring for every height one prime $\mathfrak{q}' \subset C$, it follows easily that $(C^{**}/C)_{\mathfrak{q}'} = (0)$, so that the annihilator $I_{\mathfrak{q}}$ of $(C^{**})_{\mathfrak{q}}/C_{\mathfrak{q}}$ is not contained in any height one prime of C. Hence, by (S_2), there is a $C_{\mathfrak{q}}$-regular sequence (x,y) contained in $I_{\mathfrak{q}}$. So if $\xi \in (C^{**})_{\mathfrak{q}}$, then $x\xi \in C_{\mathfrak{q}}$, $y\xi \in C_{\mathfrak{q}}$, and $yx\xi \in xC_{\mathfrak{q}}$, whence $x\xi \in xC_{\mathfrak{q}}$ and $\xi \in C_{\mathfrak{q}}$. Thus $(C^{**})_{\mathfrak{q}} = C_{\mathfrak{q}}$. Q.E.D.

COROLLARY (10.5) (cf. [Hc], Proposition (4.1)). *Let* $R \in \mathcal{K}$, *and let* $R^{\#} = \mathrm{Hom}(\omega_R,\omega_R)$. *Then the natural injective map* $\alpha_R : R \to R^{\#}$

localizes to an isomorphism precisely at those prime ideals $\mathfrak{q} \subset R$ such that $R_{\mathfrak{q}}$ satisfies (S_2).

Proof. From (4.4) and (1.4) we see that ω_R is isomorphic to a localization of a C-module C^* (for suitable B,C); so (10.5) follows immediately from (10.4).

Proof of (10.3). With $R^{\#}$ as in (10.5), local duality (9.1)(b) gives an isomorphism

$$(R^{\#})\hat{} \xrightarrow{\sim} \mathrm{Hom}_k(H^n_{\mathfrak{m}_R}(\omega_R),k) \; .$$

We also have the natural isomorphisms

$$\hat{R} \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_k(\mathrm{Hom}_k(R/\mathfrak{m}_k^n,k),k)$$

$$\xrightarrow{\sim} \mathrm{Hom}_k(\varinjlim_n \mathrm{Hom}_k(R/\mathfrak{m}_R^n,k),k)$$

$$= \mathrm{Hom}_k(I_R,k) \; .$$

One checks modulo these isomorphisms that the map α_R of (10.5) completes to the k-dual of σ_R:

$$(\alpha_R)\hat{} = \mathrm{Hom}_k(\sigma_R,k),$$

whence, $(\alpha_R)\hat{}$ being injective, σ_R is surjective (as asserted in (10.3)); and there is a natural isomorphisms of R-modules

$$(R^{\#}/R)\hat{} \xrightarrow{\sim} \mathrm{Hom}_k(J,k).$$

Clearly, then, for $a \in R$, we have that

$$[a(R^{\#}/R) = (0)] \Leftrightarrow [\mathrm{Hom}_k(aJ,k) = (0)] \Leftrightarrow [aJ = (0)].$$

Hence, by (10.5), the prime ideals $\mathfrak{q} \subset R$ containing the annihilator of J are precisely those for which $R_{\mathfrak{q}}$ does not satisfy (S_2).

We can therefore choose $0 \neq a \in R$ annihilating J. For convenience we will write "H^s" for "$H^s_{\mathfrak{m}_S}$". If φ is admissible, hence injective, then G/aG has support of dimension $< s$, and it follows that multiplication by a in $H^s(G)$ is surjective, i.e. we have an exact sequence

$$0 \longrightarrow P \longrightarrow H^s(G) \xrightarrow{a} H^s(G) \longrightarrow 0 .$$

Applying the functor $\operatorname{Hom}_R(\cdot,J)$, and using $aJ = 0$, we find that

$$\operatorname{Hom}_R(H^s(G),J) = 0$$

$$\operatorname{Ext}^1_R(H^s(G),J) \cong \operatorname{Hom}_R(P,J) .$$

It remains then to show that $\operatorname{Hom}_R(P,J) = 0$. If G_a is the kernel of multiplication by a in G, then from the exact sequence

$$0 \longrightarrow G_a \longrightarrow G \xrightarrow{a} aG \longrightarrow 0$$

we obtain an exact sequence

$$H^s(G_a) \longrightarrow P \longrightarrow P' \longrightarrow 0$$

where P' is the kernel of the natural map $H^s(aG) \to H^s(G)$. Since, as we have just seen, $\operatorname{Hom}_R(H^s(G_a),J) = 0$, therefore there is an isomorphism

$$\operatorname{Hom}_R(P',J) \xrightarrow{\sim} \operatorname{Hom}_R(P,J) .$$

Since P' is a homomorphic image of $H^{s-1}(G/aG)$, it will suffice to prove that

$$\operatorname{Hom}_R(H^{s-1}(G/aG),J) = 0.$$

Arguing as above, we need only show that there is an element $b \in R$ with $bJ = 0$ and such that $G/(a,b)G$ has support of dimension $< s - 1$. But this follows from the admissibility of φ, which implies that any height one prime ideal of $\hat{S}$ containing $a\hat{S}$ has an inverse image (say $\mathfrak{q}$) in R such that $R_{\mathfrak{q}}$ satisfies (S_2), so that, as noted above, $\mathfrak{q}$ does not contain the annihilator of J; and since $a\hat{S}$ is contained in only finitely many height one primes, there is a $b \in R$ with $bJ = 0$ and such that $(a,b)S$ is not contained in any height one prime; this b is as desired.

This completes the proof of (10.3), and so of (a), (b) and (c) in (10.2).

* * *

We proceed with the proof of (d). We have the map

$$\theta_f : R^{s-r} f_* \omega_V \to \omega_W$$

of (5.1), whence a map

$$\mu : H_w^r(R^{s-r} f_* \omega_V) \to H_w^r(\omega_W) \ .$$

Now if $p > r$ then $H_w^p(R^q f_* \omega_V) = 0$ for all q; and if $q > s - r$ then, as in (4.3.3), the support of $R^q f_* \omega_V$ has dimension $< s - 1 - q$, whence for $p + q \geq s - 1$ we have again $H_w^p(R^q f_* \omega_V) = 0$. Therefore the Leray spectral sequence gives a natural isomorphism

$$\text{(10.2.2)} \qquad \nu : H_w^r(R^{s-r} f_* \omega_V) \xrightarrow{\sim} H_E^s(\omega_W) ;$$

and we set

$$\theta_{[f,\omega]} = \mu \circ \nu^{-1} \ .$$

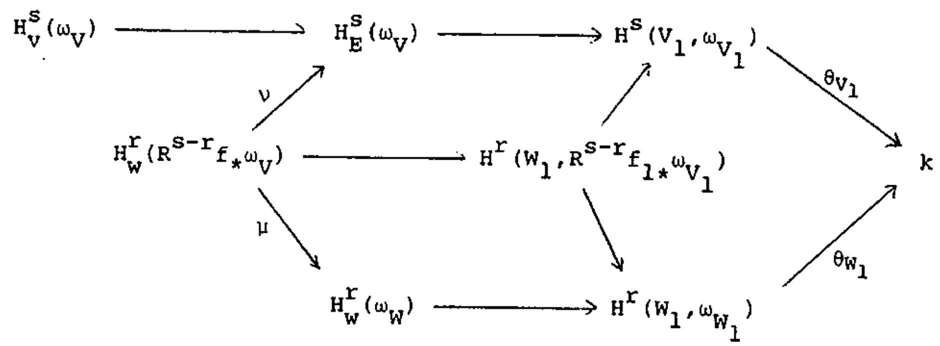
As in (4.3.3), f takes codimension one closed subvarieties of V to codimension one subvarieties of W, and therefore the maps $\varphi_{w,v}$ in (d) are all admissible (cf. example (ii) following (10.1)).

In view of the preceding proof of (a), (b), (c) (Lemma (10.3) plus the uniqueness of σ_φ), to complete the proof of (d) it will suffice to show that for closed $v \in E$ the composed map

$$H_v^s(\omega_V) \longrightarrow H_E^s(\omega_V) \xrightarrow[\theta_{[f,w]}]{} H_w^r(\omega_W) \xrightarrow[\rho_{\mathcal{O}_{W,w}}]{} k$$

is equal to $\rho_{\mathcal{O}_{V,v}}$.

For this purpose, choose a compactification $f_1 : V_1 \to W_1$ of f (cf. (5.4)), and consider the diagram



where unlabelled arrows represent natural maps. Using (5.1), we see that this diagram commutes; and the desired conclusion then results from the definition of ρ (cf. (9.1)).

The uniqueness assertion in (d) follows from Proposition (10.6) below.

As for (e), we recall from (5.1) (and (4.3.3)) that θ_f induces an isomorphism

$$f_* \mathcal{H}om_{\mathcal{O}_V}(\mathcal{F},\omega_V) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_W}(R^{s-r}f_*\mathcal{F},\omega_W) \tag{10.2.3}$$

for any quasi-coherent $\mathcal{O}_V$-module $\mathcal{F}$. We need a similar result after making a base change to the completion $\hat{R}$ of $R = \mathcal{O}_{W,w}$, i.e. after replacing W by $W' = \mathrm{Spec}(\hat{R})$, V by $V' = V\times_W W'$, f by the projection $f':V' \to W'$, and ω_V (resp. ω_W) by $\omega'_V = \omega_V \otimes_V V'$ (resp. $\omega'_W = \omega_W \otimes_W W'$); and this can be established in the same way as (i) in [Km 2, pp.44-45, Theorem (5)], once we note that $\mathcal{H}om_{\mathcal{O}_{W'}}(R^{s-r}f'_*\mathcal{H},\omega_{W'})$ is a <u>left-exact</u> contravariant functor of quasi-coherent $\mathcal{O}_{V'}$-modules $\mathcal{H}$ (as follows readily from (3.1.2) and the fact that for all $\mathcal{H}$ the support of $R^{s-r+1}f'_*\mathcal{H}$ has codimension ≥ 2 in W' [EGA III,(4.2.2)]).

Suppose now that $\mathcal{F}$ is a <u>coherent</u> $\mathcal{O}_{V'}$-module. Completing along $E' = E\times_V V'$ and applying [EGA III,(4.5.3)] we deduce from (10.2.3) (for f') <u>isomorphisms</u>

$$(10.2.4) \qquad \mathrm{Hom}_{\mathcal{O}_{\hat{V}}}(\hat{\mathcal{F}},\hat{\omega}_V) \xrightarrow{\sim} \mathrm{Hom}_{\hat{R}}(R^{s-r}f'_*\mathcal{F},(\omega_{W,w})\hat{\ })$$

$$\xrightarrow{\sim} \mathrm{Hom}_R(H^r_{\hat{m}_R}(R^{s-r}f'_*\mathcal{F}),H^r_w(\Omega_W))$$

(cf. (10.2.1) and (10.3), with $S = R = \mathcal{O}_{W,w}$)

$$\xrightarrow{\sim} \mathrm{Hom}_R(H^s_{E'}(\mathcal{F}),H^r_w(\omega_W))$$

(via ν - for f' - cf. (10.2.2) noting that f' is also equidimensional in codimension 1 [EGA IV, (13.3.8)]). As in the "explanation" following (10.2), we have a natural identification

$$H^s_{\hat{E}}(\hat{\mathcal{F}}) = H^s_{E'}(\mathcal{F})$$

and via this, we can check that the composition of the isomorphisms (10.2.4) is equal to the composition

$$\mathrm{Hom}_{\mathcal{O}_{\hat{V}}}(\hat{\mathcal{F}},\hat{\omega}_V) \longrightarrow \mathrm{Hom}_R(H^s_{\hat{E}}(\hat{\mathcal{F}}),H^s_{\hat{E}}(\hat{\omega}_V))$$

$$\xrightarrow[\text{via } \theta_{[f,w]}]{} \mathrm{Hom}_R(H^s_{\hat{E}}(\hat{\mathcal{F}}),H^r_w(\omega_W))$$

This proves (e) for $\mathcal{G} = \hat{\mathcal{F}}$, and hence, by [EGA III,(5.1.6)], for every coherent $\mathcal{G}$. Q.E.D.

PROPOSITION (10.6). _Let V be an n-dimensional separated scheme (not necessarily reduced or irreducible) over a field k (not necessarily perfect), and let E be a locally closed subset of V. Then there exists a finite set $E' \subset E$ consisting of closed points and depending only on the pair of reduced schemes $E \subset V_{red}$, such that for any quasi-coherent $\mathcal{O}_V$-module $\mathcal{G}$ we have_

$$H^n_{E-E'}(V,\mathcal{G}) = 0 ,$$

i.e. the canonical map $H^n_{E'}(\mathcal{G}) \to H^n_E(\mathcal{G})$ is surjective.

Proof. This is basically a corollary of [Km 1]. First of all it is enough to consider coherent $\mathcal{O}_V$-modules $\mathcal{G}$, since every quasi-coherent $\mathcal{O}_V$-module is the direct limit of its coherent submodules [EGA 01,p.319,(6.9.9)], and cohomology with supports commutes with direct limits (the proof of [H, p.209,Prop. 2.9] carries over). Next, if V_i $(1 \le i \le m)$ are the irreducible components of V, considered as reduced k-schemes, $E_i = E \cap V_i$, and $E_i' \subset E_i$ is a closed subset of V_i such that $H^n_{E_i - E_i'}(\mathcal{G}_i) = 0$ for every coherent $\mathcal{O}_{V_i}$-module $\mathcal{G}_i$, then with $E' = \bigcup_{i=1}^m E_i'$ we see, using Lemma 1 of [Km 1], that $H^n_{E-E'}(\mathcal{G}) = 0$ for every coherent $\mathcal{O}_V$-module $\mathcal{G}$. (Note that for any $\mathcal{G}_i$, $H^n_{E-E'}(\mathcal{G}_i) = H^n_{E_i - E' \cap E_i}(\mathcal{G}_i)$ is a homomorphic image of $H^n_{E_i - E_i'}(\mathcal{G}_i)$.) Thus we may assume that V is reduced and irreducible.

Now if $\dim E = \dim V$, then we can let E' be a finite set of closed points, one from each component of E. Then

$$H^n_{E-E'}(V,\mathcal{G}) = H^n(E-E',\mathcal{G}|E-E')$$

which vanishes [Km 1, Theorem]. So assume $\dim E < \dim V$, and induct on $\dim V$, the case $\dim V \le 1$ being trivial. Let U be any affine open subset of $V - E$, with inclusion map $i: U \to V$, and set $Y = V - U$. Both the kernel $\mathcal{K}$ and the cokernel $\mathcal{C}$ of the natural map $\mathcal{G} \to i_* i^* \mathcal{G}$ are supported in Y. Therefore by induction there exists $E' \subset E$ (not depending on $\mathcal{G}$) such that $H^{n-1}_{E-E'}(V,\mathcal{C}) = 0$. Also, we have

$$H^n_{E-E'}(V,\mathcal{K}) = 0$$

and

$$H^n_{E-E'}(V,i_* i^* \mathcal{G}) = H^n_{U \cap (E-E')}(U,i^* \mathcal{G}) = 0$$

since i is an affine morphism and $U \cap E$ is empty. It follows easily that $H^n_{E-E'}(\mathcal{G}) = 0$. Q.E.D.

§11. Reformulation via holomorphic differentials

In this section we first give a weaker version (11.2) of the Residue Theorem (0.6), in terms of holomorphic differential forms. The nice thing about this version is that it lends itself to generalization, say to the "relative" situation of flat morphisms of noetherian schemes, where $\tilde{\omega}$ may not be definable.(1) Unlike (0.6), it does not contain (0.3A) (existence of a dualizing $\mathcal{O}$-module); but together with (0.3A), it does imply (0.6), as we will see in (11.3).

We then give, in (11.4), some intrinsic local descriptions of $\tilde{\omega}$ via holomorphic differentials and residues.

We keep the notation of (0.6), and as before set

$$\Omega_R = \Lambda_R^d(\Omega_{R/k}^1) \qquad (R \in \mathcal{R}_d).$$

As in [Kl,p.15, Satz 5.5], if R is regular and $S \supset R$ is a domain which is a finite R-module, with fraction field separable over that of R, then there is a unique R-homomorphism

$$\tau' : \Omega_{S/k}^d \to \Omega_{R/k}^d = \Omega_R$$

whose localization at the prime ideal (0) of R is the trace map $\tau : \Omega_{L/k}^d \to \Omega_{K/k}^d$ (L = fraction field of S, K = fraction field of R). Hence if S_i $(1 \le i \le n)$ are the localizations of S at its maximal ideals, m_i is the maximal ideal of S_i, and m_R of R, then we have the map

$$(11.1) \qquad \bigoplus_{i=1}^{n} H_{m_i}^d(\Omega_{S_i}) = H_{m_R}^d(\Omega_{S/k}^d) \xrightarrow{\text{via } \tau'} H_{m_R}^d(\Omega_R)$$

THEOREM (11.2). There exists a unique family of k-linear maps

$$\mathrm{res}_R : H_{m_R}^d(\Omega_R) \to k \qquad (R \in \mathcal{R}_d)$$

satisfying the following conditions (a)' and (b)':

(a)' (Normalization). If $R \in \mathcal{R}_d$ is regular then res_R is as in the "primitive residue theorem" (0.4).

(1) We will not deal with such generalizations here, but cf. [E].

(b)' (Trace property). For any R, S_i as above, with R regular, the following diagram commutes:

$$\begin{array}{ccc} H^d_{\mathfrak{m}_i}(\Omega_{S_i}) & \xrightarrow{\text{cf. (11.1)}} & H^d_{\mathfrak{m}_R}(\Omega_R) \\ & {\scriptstyle \mathrm{res}_{S_i}} \searrow \quad \swarrow {\scriptstyle \mathrm{res}_R} & \\ & k & \end{array}$$

Furthermore:

(c)' (Local duality). If ^ denotes m_R-adic completion, so that

$$H^d_{\mathfrak{m}_R}(\Omega_R) = H^d_{\hat{\mathfrak{m}}_R}(\hat{\Omega}_R) \ ,$$

and if R is regular, then the pair $(\hat{\Omega}_R, \mathrm{res}_R)$ represents the functor $\mathrm{Hom}_k(H^d_{\hat{\mathfrak{m}}_R}(G), k)$ of finitely generated $\hat{R}$-modules G.

(d)' (Globalization). There exists for each proper d-dimensional k-variety V a unique k-linear map

$$\int_V : H^d(V, \Omega_V) \to k$$

such that for each closed point $v \in V$, the following diagram (with $\mathrm{res}_v = \mathrm{res}_{\mathcal{O}_{V,v}}$) commutes:

$$\begin{array}{ccc} H^d_v(\Omega_V) & \xrightarrow{\text{canonical}} & H^d(V, \Omega_V) \\ & {\scriptstyle \mathrm{res}_v} \searrow \quad \swarrow {\scriptstyle \int_V} & \\ & k & \end{array}$$

(e)' (Global duality). For each smooth V as in (d)', the pair $(\Omega_V, \int_V)$ is dualizing, i.e. represents the functor $\mathrm{Hom}_k(H^d(V, \mathcal{G}), k)$ of coherent $\mathcal{O}_V$-modules $\mathcal{G}$.

Remarks. (i). The uniqueness statements are easy consequences of Noether normalization and the fact that $H^d_v(\Omega_V) \to H^d(V, \Omega_V)$ is surjective (cf. (9.6)). In particular, if V is projective space $\mathbb{P}^d_k$, then $\int_V$ is the canonical isomorphism.

(ii). Existence in (11.2) follows at once from (0.6): just let res_R be the composition

$$H^d_{\mathfrak{m}_R}(\Omega_R) \xrightarrow{\text{natural}} H^d_{\mathfrak{m}_R}(\tilde{\omega}_R) \xrightarrow{\text{res}^{\sim}_R} k$$

and let $\int_V$ be the composition

$$H^d(V,\Omega_V) \xrightarrow{\text{natural}} H^d(V,\tilde{\omega}_V) \xrightarrow{\tilde{\theta}_V} k.$$

Incidentally, since the kernel and cokernel of $\Omega \to \tilde{\omega}$ are supported on the singular locus of V, therefore the corresponding local or global cohomology maps in degree $d = \dim V$ are surjective, and even bijective if V is smooth in codimension one.

(iii). In the same way that [(0.6) without (c)] and (0.2B) imply each other (cf. Remark (0.6.1)), also [(11.2) without (c)'] and (0.1) imply each other.

(iv). Analogues of (7.3.4) and (7.3.6) can be established for any $R \in \mathcal{R}_d$, basically because τ' commutes with exterior differentiation and with the inverse Cartier operator. Details are left to the reader.

(11.3). Deduction of (0.2A) and the Residue Theorem (0.6) from (11.2) and (0.3A).

First of all, let $B \subset C$ be as in (0.2A), let $R = C_{\mathfrak{p}}$ ($\mathfrak{p}$ a prime ideal in C), $\mathfrak{m} = \mathfrak{p}C_{\mathfrak{p}}$, $\mathfrak{q} = \mathfrak{p} \cap B$, and let $\tilde{\omega}_{R/B}$ be the localization

$$\tilde{\omega}_{R/B} = (\tilde{\omega}_{C/B})_{\mathfrak{p}}.$$

Then $\tilde{\omega}_{R/B}$ is an R-submodule of $\Omega^d_{K/k}$ (K = fraction field of R), and (0.2A) is equivalent to the assertion - which we will now deduce from (11.2) - that $\tilde{\omega}_{R/B}$ depends only on R (not on C or B).

From (11.2) and the trace map for differential forms we obtain a composed map

$$\operatorname{res}^{\sim}_{R/B}: H^d_{m}(\tilde{\omega}_{R/B}) \longrightarrow H^d_{q}(\Omega^d_{B/k}) \xrightarrow[\operatorname{res}_{B_q}]{} k \;;$$

and, after completing, we deduce from (c)' of (11.2) that *the pair* $(\hat{\tilde{\omega}}_{R/B}, \operatorname{res}^{\sim}_{R/B})$ *represents the functor* $\operatorname{Hom}_k(H^d_{\hat{m}}(G), k)$ *of finitely generated* R-*modules* G (cf. Corollary (7.5)). Moreover from (b)' we see that

(11.3.1)
$$\begin{array}{ccc} H^d_{m}(\Omega_R) & \xrightarrow{\text{natural}} & H^d_{m}(\tilde{\omega}_{R/B}) \\ & {\scriptstyle \operatorname{res}_R}\searrow \quad \swarrow{\scriptstyle \operatorname{res}^{\sim}_{R/B}} & \\ & k & \end{array}$$

commutes. Hence if B, C are replaced by B', C', then there is a unique $\hat{R}$-isomorphism

$$\alpha : \hat{\tilde{\omega}}_{R/B} \xrightarrow{\sim} \hat{\tilde{\omega}}_{R/B'}$$

such that

$$\operatorname{res}^{\sim}_{R/B} = \operatorname{res}^{\sim}_{R/B'} \circ H^d_{\hat{m}}(\alpha)$$

and the resulting natural diagram

$$\begin{array}{ccccc} & & \tilde{\omega}_{R/B} & \hookrightarrow & \hat{\tilde{\omega}}_{R/B} \\ & \nearrow & & & \downarrow \alpha \\ \Omega_R & & & & \\ & \searrow & & & \\ & & \tilde{\omega}_{R/B'} & \hookrightarrow & \hat{\tilde{\omega}}_{R/B'} \end{array}$$

commutes. But clearly

$$\nu/r \in \Omega^d_{K/k} \qquad\qquad (\nu \in \Omega_R,\ 0 \neq r \in R)$$

lies in $\tilde{\omega}_{R/B}$ if and only if the canonical image of ν in $\tilde{\omega}_{R/B}$ is divisible by r, i.e. (by faithful flatness of completion) if and only if the canonical image of ν in $\hat{\tilde{\omega}}_{R/B}$ is divisible by r; and similarly with B' in place of B; whence

$$\tilde{\omega}_{R/B} = \tilde{\omega}_{R/B'} = (\text{say})\,\tilde{\omega}_R \quad .$$

This proves (0.2A).

Furthermore, since $\Omega \to \tilde{\omega}$, being bijective at smooth points, has cokernel with support of dimension < d, the natural map $H^d_{\mathfrak{m}}(\Omega_R) \to H^d_{\mathfrak{m}}(\tilde{\omega}_R)$ is <u>surjective</u>; so in the commutative diagram

$$\begin{array}{ccccc} & \nearrow & H^d_{\mathfrak{m}}(\tilde{\omega}_R) = H^d_{\mathfrak{m}}(\hat{\tilde{\omega}}_R) & \overset{\text{res}^{\sim}_{R/B}}{\searrow} & \\ H^d_{\mathfrak{m}}(\Omega_R) & & \downarrow H^d_{\mathfrak{m}}(\alpha) & & k \\ & \searrow & H^d_{\mathfrak{m}}(\tilde{\omega}_R) = H^d_{\mathfrak{m}}(\hat{\tilde{\omega}}_R) & \underset{\text{res}^{\sim}_{R/B'}}{\nearrow} & \end{array}$$

we have $H^d_{\mathfrak{m}}(\alpha)$ = identity, whence α = identity and

$$\text{res}^{\sim}_{R/B} = \text{res}^{\sim}_{R/B'} = (\text{say})\ \text{res}^{\sim}_R \ .$$

This gives (a), (b) and (c) of (0.6).

Now for the global statements (d) and (e) of (0.6), we let $\{\omega,\theta\}$ be a dualizing module (cf. (0.3A)). Then we have the composition

$$H^d_v(\omega_V) \xrightarrow{\text{canonical}} H^d(V,\omega_V) \xrightarrow{\theta_V} k$$

whence, by the local dualizing property of $\tilde{\omega}$, an $\hat{\mathcal{O}}_{V,v}$-homomorphism

(11.3.2) $$\hat{\omega}_{V,v} \longrightarrow \hat{\tilde{\omega}}_{V,v} \quad .$$

<u>This map depends only on the local ring</u> $\mathcal{O}_{V,v}$ (not on V). To see this, note that we have a map of sheaves $c_V\colon\Omega_V \to \omega_V$ corresponding to the map $\int_V$ of (d)', whence (cf. (11.3.1)) a commutative diagram

$$\begin{array}{ccccccc} H^d_v(\Omega_V) & & \longrightarrow & & H^d(V,\Omega_V) & & \\ & \searrow & & & & \overset{\int_V}{\searrow} & \\ \downarrow \; (1) & & H^d_v(\tilde{\omega}_V) & \xrightarrow{\quad\text{res}^{\sim}\quad} & \downarrow & \longrightarrow & k \\ & \nearrow & & & & \underset{\theta_V}{\nearrow} & \\ H^d_v(\omega_V) & & \longrightarrow & & H^d(V,\omega_V) & & \end{array}$$

(Note that by the dualizing property of $\tilde{\omega}$, commutativity of (1) can be checked after composing with $\mathrm{res}^{\sim}$.) It therefore suffices to check that the cokernel of c_V has support of dimension $< d$ (since then $H^d_v(\Omega_V) \to H^d_v(\omega_V)$ is surjective, so the map $H^d_v(\omega_V) \to H^d_v(\tilde{\omega}_V)$ is uniquely determined, and hence, by the dualizing property of $\tilde{\omega}$, so is (11.3.2)). Now we remarked before that $H^d_v(\Omega_V) \to H^d_v(\tilde{\omega}_V)$ is surjective, and $\mathrm{res}^{\sim}_v = \mathrm{res}^{\sim}_{\mathcal{O}_{V,v}} \neq 0$ (otherwise the pair $(\tilde{\omega}_{V,v}, \mathrm{res}^{\sim}_v)$ would represent the zero-functor, and $\tilde{\omega}_{V,v}$ would vanish), hence $\int_V \neq 0$ and therefore c_V is not the zero map; but using (e)' we see as in (4.4) that ω_V is generically free, of rank one; and the conclusion follows.

Next one shows that for each affine open $U \subset V$ there exists an isomorphism $\lambda_U : \omega_U \xrightarrow{\sim} \tilde{\omega}_U$ inducing (11.3.2) for every closed point v in U. This can easily be done (details left to the reader) by choosing a projective closure $\bar{U}$ of U, and a finite separable $\pi : \bar{U} \to \mathbb{P} = \mathbb{P}^d_k$, and using the natural isomorphisms

$$\pi_*\omega_{\bar{U}} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\pi_*\mathcal{O}_{\bar{U}}, \Omega_{\mathbb{P}}) \qquad \text{(cf. (4.4) and (e)')}$$

$$\pi_*\omega_{\bar{U}} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\pi_*\mathcal{O}_{\bar{U}}, \Omega_{\mathbb{P}}) \qquad \text{(via trace)}$$

Basically, this argument is just a variant of (9.3).

Finally, by the uniqueness of (11.3.2), we can patch all the isomorphisms λ_U to obtain a natural global isomorphism $\lambda : \omega_V \xrightarrow{\sim} \tilde{\omega}_V$. Defining $\tilde{\theta}_V$ to be the composition

$$H^d(V, \tilde{\omega}_V) \xrightarrow{\lambda^{-1}} H^d(V, \omega_V) \xrightarrow{\theta_V} k$$

we obtain (d) and (e) of (0.6). Q.E.D.

Remark (11.3.3). In the preceding argument, we needed (11.2)(e)' only for $V = \mathbb{P}$.

* * *

(11.4). Some descriptions of $\tilde{\omega}$ via res.

From (0.6)(c) we obtain, for $R \in \mathcal{R}_d$, an isomorphism

$$\hat{\tilde{\omega}}_R = \mathrm{Hom}_{\hat{R}}(\hat{R}, \hat{\tilde{\omega}}_R) \xrightarrow{\sim} \mathrm{Hom}_k(H^d_{\hat{m}}(\hat{R}), k) = \text{(say) } H^d_m(R)',$$

whose composition with $\Omega_R \to \tilde{\omega}_R \to \hat{\omega}_R$ is, in view of (11.3.1), just the map

$$\xi : \Omega_R \to H^d_m(R)'$$

corresponding canonically to the bilinear form

$$\begin{array}{c} res_R : \Omega_R \otimes_R H^d_m(R) \longrightarrow k \\ \| \\ H^d_m(\Omega_R) \end{array} .$$

As in (11.3) (proof of (0.2A)), we conclude that:

(11.4.1) $\tilde{\omega}_R = \{\nu/r \mid \nu \in \Omega_R,\ 0 \neq r \in R,$ _and_ $\xi(\nu)$ _is divisible by_ $r\}$.

We can rephrase this description as follows: since multiplication by $r \neq 0$ in R has cokernel with support of dimension $< d$, therefore the map

$$\mu_r = \{\text{multiplication by } r \text{ in } H^d_m(R)\}$$

is _surjective_, whence $\lambda \in H^d_m(R)'$ is divisible by r if and only if λ vanishes on the kernel of μ_r. Thus:

(11.4.2) $\tilde{\omega}_R = \{\nu/r \mid \nu \in \Omega_R,\ 0 \neq r \in R,$ _and for every_ $h \in H^d_m(R)$ _such that_ $rh = 0$, _we have_ $res_R[\nu \otimes h] = 0\}$.

In particular (cf. (7.2(a))):

(11.4.3) _If_ R _is Cohen-Macaulay_, _then_

$\tilde{\omega}_R = \{\nu/r \mid \nu \in \Omega_R,\ 0 \neq r \in R,$ _and for every system of parameters_ $\underset{\sim}{t} = (t_1, \ldots, t_d)$ _and every_ $s \in R$ _such that_ $rs \in \underset{\sim}{t}R$, _we have_ $res_R[s\nu/\underset{\sim}{t}] = 0\}$.

Finally, for any $R \in \mathcal{R}_d$:

PROPOSITION (11.4.4). <u>With notation for local cohomology as in §7, we have (for $\nu \in \Omega_R$ and $0 \neq r \in R$) that $(\nu/r) \in \tilde{\omega}_R$ if and only if, for every sequence $r_2,\ldots,r_d$ such that $(r,r_2,\ldots,r_d)$ is a system of parameters in R and for all $s \in R$:</u>

$$\operatorname{res}_R[s\nu/(r,r_2,\ldots,r_d)] = 0$$

<u>Proof</u>. After canonically identifying $H^d_{\mathfrak{m}}(\Omega_R)$ and $\Omega_R \otimes_R H^d_{\mathfrak{m}}(R)$, we have

$$s\nu/(r,r_2,\ldots,r_d) = \nu \otimes [s/(r,r_2,\ldots,r_d)] = (\text{say})\ \nu \otimes h.$$

But by (7.2)(a) $rh = 0$, so (11.4.2) shows that

$$\nu/r \in \tilde{\omega}_R \Rightarrow \operatorname{res}_R[s\nu/(r,r_2,\ldots,r_d)] = 0 .$$

For the converse, we may assume that r is a non-unit, and choose $r_2,\ldots,r_d$ such that $(r,r_2,\ldots,r_d)$ is a system of parameters. Let h be as in (11.4.2). Then for some integer $n > 0$ and some $t \in R$, we have

$$h = t/(r^n,r_2^n,\ldots,r_d^n) .$$

Since $rh = 0$, (7.2) allows us to assume (enlarging n if necessary) that

$$rt \in (r^n,r_2^n,\ldots,r_d^n)R ,$$

say

$$rt = s_1 r^n + \sum_{i=2}^{d} s_i r_i^n \qquad (s_i \in R).$$

Then, (cf. (7.2)):

$$\begin{aligned} \nu \otimes h &= rt\nu/(r^{n+1},r_2^n,\ldots,r_d^n) \\ &= (s_1 r^n + \Sigma\, s_i r_i^n)\nu/(r^{n+1},r_2^n,\ldots,r_d^n) \\ &= s_1\nu/(r,r_2^n,\ldots,r_d^n). \end{aligned}$$

Hence

$$\operatorname{res}_R[\nu \otimes h] = \operatorname{res}_R[s_1\nu/(r,r_2^n,\ldots,r_d^n)] = 0. \qquad \text{Q.E.D.}$$

§12. Sums of residues; Koszul complexes of vector bundles.

The basic result in this section is (12.2), which is just a reformulation of (11.2)(d') (or (0.6)(d)). Examples (12.3), (12.4) and (12.5) are special cases of (12.2), giving some more familiar statements which have gone by the name "residue theorem". Example (12.6) relates some recent results of Akyildiz and Carrell to the formalism developed here.

(12.1) As always, V is a d-dimensional variety over the perfect field k. To avoid annoying trivialities, we assume $d \geq 1$.

Let

$$C: 0 \to C_d \to C_{d-1} \to \cdots \to C_2 \to C_1 \to \mathcal{O}_V \to 0$$

be a sequence of $\mathcal{O}_V$-modules and let $F \subset V$ be a finite set of closed points such that the restriction $C|(V-F)$ is exact. Suppose further that we are given an $\mathcal{O}_{V-F}$ homomorphism

$$\psi: C_d|(V-F) \to \Omega_{V-F} \quad .$$

Then the exact sequence $C|(V-F)$ represents an element

$$[C]_F \in \mathrm{Ext}^{d-1}_{V-F}(\mathcal{O}_{V-F}, C_d|(V-F)) = H^{d-1}(V-F, C_d) \quad ,$$

from which we obtain an element

$$\psi_*[C]_F \in H^{d-1}(V-F, \Omega_V)$$

by applying the map $\psi_* = H^{d-1}(V-F, \psi)$ induced by ψ.

Note that when $d = 1$,

$$[C]_F \in \mathrm{Hom}_{V-F}(\mathcal{O}_{V-F}, C_1|V-F)$$

is the inverse of the isomorphism $C_1|V-F \xrightarrow{\sim} \mathcal{O}_{V-F}$ given by C.

We denote by

$$[\psi/C] \in \bigoplus_{v \in V} H^d_v(\Omega_V)$$

the image of $\psi_*[C]_F$ under the natural map

$$H^{d-1}(V-F, \Omega_V) \xrightarrow{\alpha} H^d_F(\Omega_V) \subset \bigoplus_{v \in V} H^d_v(\Omega_V).$$

If we consider two pairs (ψ_1,F_1), (ψ_2,F_2) as above to be equivalent if ψ_1 and ψ_2 agree outside a finite set $F_3 \supset (F_1 \cup F_2)$, then it is easily checked that $[\psi/C]$ depends only on the equivalence class of (ψ,F).

[The point is that if C is exact at $v \in F$, and if ψ extends across v, say to $\bar{\psi}$, then the component $[\psi/C]_v \in H^d_v(\Omega_V)$ vanishes since $\psi_*[C]_F$ lifts back to the element $\bar{\psi}_*[C]_{F-v} \in H^{d-1}((V-F)\cup\{v\},\Omega_V)$.]

We set

$$\mathrm{res}_v[\psi/C] = \mathrm{res}_v([\psi/C]_v)$$

where, again, $[\psi/C]_v \in H^d_v(\Omega_V)$ is the v-component of $[\psi/C]$, and $\mathrm{res}_v = \mathrm{res}_{\mathcal{O}_{V,v}}$ (cf. (11.2)).

PROPOSITION (12.2). *If V is proper over k then*

$$\sum_{v\in V} \mathrm{res}_v[\psi/C] = 0.$$

Proof. The natural composition

(12.2.1) $$H^{d-1}(V-F),\Omega_V) \xrightarrow{\alpha} H^d_F(\Omega_V) \xrightarrow{\beta} H^d(V,\Omega_V)$$

is the zero map, whence (cf. (11.2)(d'))

$$\sum_{v\in V} \mathrm{res}_v[\psi/C] = \int_V \beta\alpha\psi_*[C]_F = 0 .$$

Remarks (12.2.2) Throughout, we can replace Ω_V by $\tilde{\omega}_V$, res by $\mathrm{res}^\sim$, and (11.2)(d') by (0.6)(d).

(12.2.3) Conversely, (11.2)(d') can be deduced from (12.2). For, in view of (9.6), (11.2)(d') just says that for any ξ in the kernel of the natural map $\bigoplus_{v\in V} H^d_v(\Omega_V) \to H^d(V,\Omega_V)$ we have $\sum_v \mathrm{res}_v(\xi_v) = 0$. But this map is the direct limit of maps $\beta = \beta_F$ as in (12.2.1) (where the finite sets F form a directed system under the order relation given by inclusion); and since (12.2.1) is exact, we need only see that every element $\eta \in \mathrm{Ext}^{d-1}(\mathcal{O}_{V-F},\Omega_V)$ is of the form $[C]_F$, where C is a sequence as before, with $C_d = \Omega_V$ (and ψ = identity). Now, with $U = V - F$, η corresponds to an exact sequence of $\mathcal{O}_U$-modules

$$0 \to \Omega_U \to C'_{d-1} \to \cdots \to C'_2 \to C'_1 \to \mathcal{O}_U \to 0 ,$$

to which we can apply the functor i_* ($i: U \to V$ being the inclusion), then replace

$$i_*\Omega_U \longrightarrow i_*C'_{d-1}$$

by its composition with the natural map

$$\Omega_V \longrightarrow i_*\Omega_U$$

and replace

$$i_*C'_2 \xrightarrow{\varphi} i_*C'_1 \longrightarrow i_*\mathcal{O}_U$$

by

$$i_*C'_2 \xrightarrow{(\varphi,0)} i_*C'_1 \times_{i_*\mathcal{O}_U} \mathcal{O}_V \longrightarrow \mathcal{O}_V$$

to get the desired C (which is, in fact, a complex).

Example (12.3). Let V be a proper curve $(d=1)$, and let $\nu \in \Omega^1_{k(V)/k}$ be a non-zero meromorphic differential. Let $F \subset V$ be the finite set consisting of zeros and poles of ν, so that with $U = V - F$ there is an isomorphism $\Omega_U \xrightarrow{\sim} \mathcal{O}_U$ taking ν to 1. Let $i:U \to V$ be the inclusion, and let C_ν be the sequence

$$0 \to C_1 = i_*\Omega_U \times_{i_*\mathcal{O}_U} \mathcal{O}_V \xrightarrow{\text{projection}} \mathcal{O}_V \to 0 \ .$$

Straightforward checking verifies that

$$\text{res}_v(\nu) = \text{res}_v[1/C_\nu]$$

is the good old-fashioned residue of ν at v; and (12.2) becomes the classical residue theorem:

$$\sum_{v \in V} \text{res}_v(\nu) = 0 \ .$$

Example (12.4). Let $\mathcal{E}$ be a locally free $\mathcal{O}_V$-module of rank d, and let σ be a global section of $\mathcal{E}$, having isolated zeros. We can identify σ with an $\mathcal{O}_V$-homomorphism

$$\sigma:\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{E},\mathcal{O}_V) \to \mathcal{O}_V \ ,$$

which is surjective outside a zero-dimensional subset F of V, and then build the Koszul complex

$$C_\sigma : 0 \to \Lambda^d\mathcal{E}^* \to \Lambda^{d-1}\mathcal{E}^* \to \dots \to \Lambda^1\mathcal{E}^* = \mathcal{E}^* \xrightarrow{\sigma} \mathcal{O}_V \to 0$$

where the maps $\sigma^i:\Lambda^i\mathcal{E}^* \to \Lambda^{i-1}\mathcal{E}^*$ are given locally by

$$\sigma^i(e_1 \wedge e_2 \wedge \ldots \wedge e_i) = \sum_{j=1}^{i} (-1)^{j-1}\sigma(e_j)e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_i$$

(where $e_1,\ldots,e_i \in \mathcal{E}^*$, and "$\hat{e}_j$" means "omit e_j"). Then each $v \in V$ has a punctured neighborhood on which σ is surjective and C_σ is exact.

Now if $e_1, e_2, \ldots, e_d$ is a basis of the stalk $\mathcal{E}^*_v$, and we set $\sigma(e_i) = \sigma_i$, then $\mathcal{O}_{V,v}/(\sigma_1,\ldots,\sigma_d)$ has dimension ≤ 0, and from the definitions in §7 we find that <u>the natural image of</u>

$$[C_\sigma]_F \in H^{d-1}(V-F, \Lambda^d\mathcal{E}^*)$$

<u>in</u> $H^d_V(\Lambda^d\mathcal{E}^*)$ <u>is</u>

$$e_1 \wedge \ldots \wedge e_d/(\sigma_1,\ldots,\sigma_d)$$

(which we define to be 0 if $(\sigma_1,\ldots,\sigma_d)\mathcal{O}_{V,v} = \mathcal{O}_{V,v}$). Thus if we have an $\mathcal{O}_V$-homomorphism

$$\psi: \Lambda^d\mathcal{E}^* \to \Omega_V$$

and if, at v,

(12.4.1) $$\psi(e_1 \wedge \ldots \wedge e_d) = h\delta\sigma_1 \ldots \delta\sigma_d \in \Omega_{V,v},$$

then

(12.4.2) $$[\psi/C_\sigma]_v = h\delta\sigma_1 \ldots \delta\sigma_d/(\sigma_1,\ldots,\sigma_d);$$

and (12.2) becomes the <u>residue theorem for vector bundles</u> [GH, p.731] (with no assumption on the singularities of V).

<u>Example</u> (12.5). As a special case of (12.4), let $D_1,\ldots,D_d$ be effective divisors on V such that $D_1 \cap D_2 \cap \ldots \cap D_d$ is zero-dimensional; let

$$\mathcal{E} = \mathcal{O}_V(D_1) \oplus \mathcal{O}_V(D_2) \oplus \ldots \oplus \mathcal{O}_V(D_d);$$

and let $\sigma: \mathcal{E}^* \to \mathcal{O}_V$ be the map whose restriction to $\mathcal{O}_V(-D_i) \subset \mathcal{O}_V$ $(1 \leq i \leq d)$ is just the inclusion map. We have then

$$\Lambda^d\mathcal{E}^* = \mathcal{O}_V(-D_1 - D_2 - \ldots - D_d)$$

and any

$$\psi \in \mathrm{Hom}_{\mathcal{O}_V}(\Lambda^d\mathcal{E}^*, \Omega_V) = H^0(V, \Omega_V(D_1 + \ldots + D_d))$$

can be identified with a <u>meromorphic differential</u> d-<u>form</u> with poles no worse than $D_1 + \ldots + D_d$. With C_σ the Koszul complex of (12.4), we can set

$$[\psi/C_\sigma] = [\psi/(D_1,\ldots,D_d)]^{(1)}$$

and rewrite (12.2) as

$$\sum_{v\in V} \mathrm{res}_v[\psi/(D_1,\ldots,D_d)] = 0.$$

When $d = 1$, $[\psi/D_1]$ depends only on ψ (not on D_1), and we have, again, the classical residue theorem for (possibly singular) curves.

Example (12.6). This example is inspired by [AC], where a particularly interesting application to Gysin homomorphisms is given.

Let $\mathcal{E}$, σ, ψ be as in (12.4), so that for any closed point $v \in V$ we have in $H^d_v(\Omega_V)$

$$[\psi/C_\sigma]_v = \psi(e_1 \wedge\ldots\wedge e_d)/(\sigma_1,\ldots,\sigma_d)$$

(cf. (12.4.1), (12.4.2)), an element which is annihilated by $(\sigma_1,\ldots,\sigma_d)\mathcal{O}_{V,v}$ (cf. (7.2)(a)). Thus for any $\lambda \in (\mathrm{coker}\ \sigma)_v$, the element

$$\lambda[\psi/C_\sigma]_v \in H^d_v(\Omega_V)$$

is well-defined.

Now suppose we have a map $f:V \to W$ with W smooth. Let

$$\tau:\Omega^1_{W/k} \to \mathcal{O}_W$$

be an $\mathcal{O}_W$-homomorphism which is surjective outside a finite set of closed points (i.e. τ is a vector field on W, with isolated zeros). There is then a corresponding element

$$[1/C_\tau] \in \bigoplus_{w\in W} H^r_w(\Omega_W) \qquad (r = \dim W).$$

As before, if $x_1,x_2,\ldots,x_r$ is a regular system of parameters in $\mathcal{O}_{W,w}$, then

$$[1/C_\tau]_w = \delta x_1\ldots\delta x_r/(\tau\delta x_1,\ldots,\tau\delta x_r) \in H^r_w(\Omega_W);$$

and for any $\mu \in (\mathrm{coker}\ \tau)_w$, the element

$$\mu[1/C_\tau]_w \in H^r_w(\Omega_W)$$

is well-defined.

Assume further that there exists an $\mathcal{O}_V$-homomorphism $f^*(\Omega^1_{W/k}) \to \mathcal{E}^*$ such that

(1) coming from the image in $\check{H}^{d-1}(\{V-D_i\},\Omega_V)$ of the Čech cocycle ψ, cf. §7.

$$\begin{array}{ccc} f^*(\Omega^1_{W/k}) & \longrightarrow & \mathcal{E}^* \\ & {\scriptstyle f^*(\tau)}\searrow \quad \swarrow{\scriptstyle \sigma} & \\ & \mathcal{O}_V & \end{array}$$

commutes. Let $w = f(v)$. Then f induces a k-algebra homomorphism $(\operatorname{coker} \tau)_w \to (\operatorname{coker} \sigma)_v$; and for each $\lambda \in (\operatorname{coker} \sigma)_v$, we have a k-linear map

$$r_\lambda: \{h \in H^r_w(\Omega_W) \mid (\tau\delta x_i)h = 0, 1 \le i \le r\} = \Omega_{W,w}/(\tau\delta x_1, \ldots, \tau\delta x_r)\Omega_{W,w} \to k$$

given by

$$r_\lambda(\mu[1/C_\tau]) = \operatorname{res}_v(\mu\lambda[\psi/C_\sigma]) \qquad (\mu \in (\operatorname{coker} \tau)_w).$$

From the proof of (7.4) (local duality), we now find easily that:

There exists a unique $\mathcal{O}_{W,w}$-homomorphism

$$t: (\operatorname{coker} \sigma)_v \to (\operatorname{coker} \tau)_w$$

such that for each $\lambda \in (\operatorname{coker} \sigma)_v$, we have

$$\operatorname{res}_w(t(\lambda)[1/C_\tau]) = \operatorname{res}_v(\lambda[\psi/C_\sigma])$$

Remark (12.6.1). If the map $\varphi: \mathcal{O}_{W,w} \to \mathcal{O}_{V,v}$ induced by f is admissible (cf. (10.1)), and if ρ is the composition

$$H^d_v(\Omega_V) \xrightarrow[\text{natural}]{} H^d_v(\tilde{\omega}_V) \xrightarrow[\rho_\varphi]{} H^r_w(\tilde{\omega}_W) = H^r_w(\Omega_W)$$

(with ρ_φ as in (10.2)), then (cf. (9.4) and (10.2))

$$\operatorname{res}_v = \operatorname{res}_w \circ \rho$$

and consequently the above map t is the unique one for which

$$t(\lambda)[1/C_\tau] = \rho(\lambda[\psi/C_\sigma]) \qquad (\lambda \in (\operatorname{coker} \sigma)_w).$$

§13. Adjunction

A common strategy in studying duality on a k-variety V is to relate V to a non-singular variety X, and then to deduce results on V from corresponding results on X via this relation. (The main problem usually is to show that what is obtained in this way is <u>intrinsic</u> to V.) Up to now we have used Noether normalization to get such a relation. In this section we use instead a <u>closed immersion</u> $V \to X$. This is the approach used by Grothendieck in [G1], by El Zein in [E,part III], and by Kunz in a recent preprint on regular differentials.

We consider, then, an n-dimensional variety X, a d-dimensional closed subvariety V of X, and the (prime) $\mathcal{O}_X$-ideal $\mathcal{P}$ of functions vanishing on V. We assume throughout that V <u>is not entirely contained in the singular locus of</u> X (i.e. X is smooth almost everywhere along V).

The main result (Theorem (13.5)) connects $\tilde{\omega}_X$ and $\tilde{\omega}_V$ via the <u>fundamental local homomorphism</u> ([G1,p.149-05])

$$(13.1)\qquad \varphi\colon \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X) \longrightarrow \mathcal{H}_{V,X} = \mathcal{H}om_{\mathcal{O}_V}(\Lambda^{n-d}\mathcal{P}/\mathcal{P}^2,\tilde{\omega}_X/\mathcal{P}\tilde{\omega}_X)$$

which can be described locally as follows:

If $Y = \mathrm{Spec}(A)$ is an affine open subset of X, and $P \subset A$ is the prime ideal corresponding to $V \cap Y$, then, with $\omega = \Gamma(Y,\tilde{\omega}_X)$ we have a natural map

$$\mathrm{Ext}^{n-d}_A(A/P,\omega) \to \mathrm{Ext}^{n-d}_A(A/P,\omega/P\omega)$$
$$\to \mathrm{Hom}_{A/P}(\mathrm{Tor}^A_{n-d}(A/P,A/P),\omega/P\omega)$$

[the second arrow being given by the natural maps

$$H^{n-d}(\mathrm{Hom}_A(F_\bullet,\omega/P\omega)) \xrightarrow{\sim} H^{n-d}(\mathrm{Hom}_{A/P}(F_\bullet\otimes_A A/P,\omega/P\omega)) \longrightarrow \mathrm{Hom}_{A/P}(H_{n-d}(F_\bullet\otimes_A A/P),\omega/P\omega)$$

where $F_\bullet$ is an A-projective resolution of A/P ...] which can be combined with the natural map

$$\Lambda^{n-d}_A(P/P^2) \to \mathrm{Tor}^A_{n-d}(A/P,A/P)$$

[arising from the canonical isomorphism $P/P^2 \xrightarrow{\sim} \mathrm{Tor}^A_1(A/P,A/P)$ plus the natural anticommutative graded A/P-algebra structure on $\oplus_{m\geq 0}\mathrm{Tor}^A_m(A/P,A/P)$] to give (13.1.) over Y.

We need to identify $\mathcal{H}_{V,X}$ (modulo torsion) with a module of meromorphic d-forms on V.

There is a natural exact sequence of $\mathcal{O}_V$-modules

$$\mathcal{P}/\mathcal{P}^2 \xrightarrow{\ \gamma\ } \Omega^1_{X/k}/\mathcal{P}\Omega^1_{X/k} \longrightarrow \Omega^1_{V/k} \longrightarrow 0\ .$$

Over the open subset U of V where X is smooth, $\Omega^1_{X/k}/\mathcal{P}\Omega^1_{X/k}$ is locally free of rank n and $\tilde{\omega}_X/\mathcal{P}\tilde{\omega}_X = \Omega_X/\mathcal{P}\Omega_X$ is invertible; moreover on the open subset U_0 of U where V too is smooth, $\mathcal{P}/\mathcal{P}^2$ is locally free of rank $n-d$ and γ is injective. It follows that there is a natural map (over U)

$$\psi:\Omega_U \to \mathcal{H}om_{\mathcal{O}_U}(\Lambda^{n-d}\mathcal{P}/\mathcal{P}^2, \Omega_X/\mathcal{P}\Omega_X) = \mathcal{K}_{V,X}|U$$

such that, locally,

(13.2) $\psi(\delta_V\bar{x}_1\delta_V\bar{x}_2\dots\delta_V\bar{x}_d)[f'_1\wedge f'_2\wedge\dots\wedge f'_{n-d}]$

$$= \delta_X f_1\dots\delta_X f_{n-d}\delta_X x_1\dots\delta_X x_d + \mathcal{P}\Omega_X$$

where the x_i are functions on X with respective restrictions $\bar{x}_i$ to V, the f_j are functions vanishing on V, with natural images f'_j in $\mathcal{P}/\mathcal{P}^2$, and δ_V (resp. δ_X) is the universal derivation. (ψ is well-defined because, on the smooth part of X, if $f \in \mathcal{P}$ then $\delta f_1\dots\delta f_{n-d}\delta f \in \mathcal{P}\Omega^{n-d+1}_{X/k}$, as can be seen by restricting further to any open set whose intersection with V is U_0.) It is easily seen that ψ is an isomorphism over U_0. Hence there is an isomorphism of constant sheaves

(13.2.1) $$\psi_{k(V)}:\Omega^d_{k(V)/k} \xrightarrow{\sim} \mathcal{K}_{V,X}\otimes_{\mathcal{O}_V} k(V)$$

via which the image ($= \mathcal{K}_{V,X}$/torsion) of the natural map $\mathcal{K}_{V,X} \to \mathcal{K}_{V,X}\otimes k(V)$ gets identified with an $\mathcal{O}_V$-submodule $\tilde{\mathcal{K}}_{V,X}$ of $\Omega^d_{k(V)/k}$.

<u>For example</u>, at any $v \in V$ where X is <u>smooth</u>, the stalk $(\tilde{\mathcal{K}}_{V,X})_v$ can be described as follows:

We can choose $(x_1, x_2, \dots, x_n) \in R = \mathcal{O}_{X,v}$ such that $(\delta x_1, \delta x_2, \dots, \delta x_n)$ is a basis of $\Omega^1_{R/k}$, the subscripts being arranged so that $\delta\bar{x}_1, \dots, \delta\bar{x}_d$ form a basis of $\Omega^1_{k(V)/k}$. Let $f_1, \dots, f_t$ be generators of the stalk $\mathcal{P}_v \subset R$, and let J_v be the $\mathcal{O}_{V,v}$-ideal generated by the restrictions to V of all the $(n-d)\times(n-d)$ minors of the $t\times(n-d)$ matrix

$$(\partial f_i/\partial x_j)_{1\le i\le t,\, d+1\le j\le n}$$

(where $\partial f/\partial x_i$ is defined by the equation $\delta f = \sum_{i=1}^{n}(\partial f/\partial x_i)\delta x_i$ in $\Omega^1_{R/k}$). Then

$$(13.3)\qquad (\mathcal{H}^{\sim}_{V,X})_v = J_v^{-1}\delta\bar{x}_1\ldots\delta\bar{x}_d$$

$$= \{\lambda\delta\bar{x}_1\delta\bar{x}_2\ldots\delta\bar{x}_d \mid \lambda \in k(V), \lambda J_v \subset \mathcal{O}_{V,v}\}\ ^{(1)}$$

In any case, we have the composed map

$$(13.4)\qquad \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V, \tilde{\omega}_X) \xrightarrow[(13.1)]{} \mathcal{H}_{V,X} \xrightarrow[(13.2.1)]{\text{via } \psi^{-1}_{k(V)}} \tilde{\mathcal{H}}_{V,X} \subset \Omega^d_{k(V)/k} .$$

THEOREM (13.5). <u>The image of the map</u> (13.4) <u>is contained in</u> $\tilde{\omega}_V$; <u>and moreover at any point where</u> X <u>is locally Cohen-Macaulay</u>, (13.4) <u>maps</u> $\mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V, \tilde{\omega}_X)$ <u>isomorphically onto</u> $\tilde{\omega}_V$.

<u>Remark</u>. The statement is essentially <u>local</u>. In fact it will be transformed below into Theorem (13.12), which is a generalized local version of the property (R3) of residues given in [RD,p.197]. Lacking, however, a good local theory of residues (cf. remark (ii) following (0.1.3)) we will argue, as in §6, in a roundabout way, using a <u>global</u> statement (13.8) to reduce the proof of (13.12) to the easily disposed of case of smooth points (the only case, by the way, covered by (R3) of <u>loc</u>. <u>cit</u>.)

But first here are some corollaries of (13.5).

COROLLARY (13.6). <u>If at</u> $v \in V$, X <u>is smooth and</u> V <u>is locally a complete intersection</u>, $\mathcal{P}_v$ <u>being generated by</u> $f_1, f_2, \ldots, f_{n-d}$, <u>then, with notation as in</u> (13.3) <u>we have that</u> $\tilde{\omega}_{V,v}$ <u>is invertible, generated by</u>

$$\delta\bar{x}_1\ldots\delta\bar{x}_d/[\partial(f_1,\ldots,f_{n-d})/\partial(x_{d+1},\ldots,x_n]^{-}$$

(<u>where the denominator is the restriction to</u> V <u>of a Jacobian determinant</u>).

Indeed, using the Koszul complex on $(f_1,\ldots,f_{n-d})$ to calculate Ext's and Tor's, one shows that (13.4) is an isomorphism at v, whence by (13.5)

---

(1) It follows that $(\mathcal{H}^{\sim}_{V,X})_v$ depends only on $\mathcal{O}_{V,v}$: in fact if A is any commutative domain, with fraction field F, M any finitely generated A-module, and $e_1, e_2, \ldots, e_d$ a linearly independent sequence in M such that $M/(Ae_1+\ldots+Ae_d)$ is a torsion module, with 0-th Fitting ideal J, then the submodule $J^{-1}e_1 \wedge e_2 \wedge \ldots \wedge e_d$ of $\Lambda^d_F(M \otimes_A F)$ depends only on M (cf. [LS, p.211, Proposition]).

$$\tilde{\omega}_{V,v} = (\tilde{\mathcal{K}}_{V,X})_v$$

and then (13.3) gives (13.6).

COROLLARY (13.7). *Suppose that* X *is smooth everywhere along* V. *Assume further that* V *is locally* (S_2), *and that moreover* V *is locally a complete intersection outside a subvariety of codimension two in* V (*these conditions hold for example if* V *is normal*). *Then*

$$\tilde{\omega}_V = \mathcal{K}_{V,X}^{\sim} \cong \mathcal{K}_{V,X} \,.$$

Proof of (13.7). As in (2.1.3), we see that both $\tilde{\omega}_V$ and $\mathcal{K}_{V,X}$ satisfy (S_2) (in particular, $\mathcal{K}_{V,X}$ is torsion free, i.e. $\mathcal{K}_{V,X}^{\sim} \cong \mathcal{K}_{V,X}$). Thus (cf. (3.1.2)) in checking that $\tilde{\omega}_V = \mathcal{K}_{V,X}^{\sim}$, we may remove from V any subvariety of codimension ≥ 2, and so we may assume that V is a local complete intersection and argue as in (13.5).
Q.E.D.

* * *

The *proof of (13.5)* will occupy the rest of this section.

PROPOSITION (13.8). (a) *There exists a unique* $\mathcal{O}_V$-*homomorphism*

$$\eta : \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V, \tilde{\omega}_X) \to \tilde{\omega}_V$$

such that for each closed point $v \in V$, *the following diagram commutes* (*where* $R = \mathcal{O}_{X,v}$, $S = \mathcal{O}_{V,v}$, *and* m *is the maximal ideal of* R):

$$\begin{array}{ccccc} H^d_v(\mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) & \xrightarrow{\text{via } \eta} & H^d_v(\tilde{\omega}_{V,v}) & \searrow^{\widetilde{\text{res}}_S} & \\ \| & & & & k \\ H^d_m(S) \otimes_S \mathrm{Ext}^{n-d}_R(S,\tilde{\omega}_{X,v}) & \xrightarrow[\text{Yoneda}]{} & H^n_m(\tilde{\omega}_{X,v}) & \nearrow_{\widetilde{\text{res}}_R} & \end{array}$$

(b) *If* $\mathcal{O}_{X,v}$ *is Cohen-Macaulay*, *then* η *is an isomorphism at* v.

Recall: $\mathrm{Ext}^q_R(S,G)$ is a universal δ-functor of R-modules G (cf. [H,pp.205-206]), and the Yoneda pairings

$$H^d_m(S) \otimes_S \mathrm{Ext}^q_R(S,G) \longrightarrow H^{d+q}_m(G) \qquad (q \geq 0)$$

can be defined by the following condition:

(13.9) <u>For fixed</u> $\mu \in H^d_{\mathfrak{m}}(S)$, <u>the family of maps</u>

$$Y^q_\mu : \mathrm{Ext}^q_R(S,G) \to H^{d+q}_{\mathfrak{m}}(G)$$

<u>given by</u>

$$Y^q_\mu(\lambda) = \mathrm{Yoneda}(\mu \otimes \lambda)$$

<u>is the unique homomorphism of</u> δ-<u>functors such that</u>

$$Y^0_\mu(\lambda) = [H^d_{\mathfrak{m}}(\lambda)](\mu)$$

(i.e. the image of μ under the map $H^d_{\mathfrak{m}}(S) \to H^d_{\mathfrak{m}}(G)$ induced by $\lambda \in \mathrm{Hom}_R(S,G)$).

We will prove (13.8) below. Given (13.8), it is clear that (13.5) follows from:

PROPOSITION (13.10). <u>The following diagram commutes</u>:

$$\begin{array}{ccc} \mathcal{E} = \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X) & \xrightarrow{(13.4)} & \mathcal{K}_{V,X} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \text{inclusion}} \\ \tilde{\omega}_V & \xrightarrow[\text{inclusion}]{} & \Omega^d_{k(V)/k} \end{array}$$

If Proposition (13.10) is true at one point $v \in V$, then it is clearly true everywhere. If X is Cohen-Macaulay at v, so that η_v is an isomorphism (cf. (13.8)(b)), it follows then from definitions that to prove (13.10) at v (hence everywhere) it suffices to show that the following diagram - with $x \in X$ the generic point of V (x smooth on X), and c the natural map - commutes:

(13.10.1)

$$\begin{array}{ccccccc} \Omega_{V,v} & \xrightarrow{c} & \tilde{\omega}_{V,v} & \xrightarrow{\eta_v^{-1}} & \mathcal{E}_v & \xrightarrow{(13.1)} & (\mathcal{K}_{V,X})_v \\ \downarrow{\scriptstyle \text{natural}} & \swarrow{\scriptstyle \text{inclusion}} & & & & & \downarrow{\scriptstyle \text{natural}} \\ \Omega^d_{k(V)/k} & & & \xrightarrow[(13.2.1)]{} & & & \mathrm{Hom}_{k(V)}(\Lambda^{n-d}\mathcal{P}_x/\mathcal{P}_x^2, \Omega_{X,x}/\mathcal{P}_x\Omega_{X,x}) \end{array}$$

We proceed then as follows.

As before we denote by $\mathcal{R}$ the collection of all local domains R which are localizations of finitely generated k-algebras at maximal ideals (so that the residue field R/m_R is finite over k). We will prove below:

LEMMA (13.11). *Let* $R \in \mathcal{R}$ *be Cohen-Macaulay, of dimension* n; *let* P *be a prime ideal in* R *such that* R_P *is regular, of dimension* $n - d$; *and set* $S = R/P$. *Let* $g = (g_1, g_2, \ldots, g_{n-d})$ *be an* R-*regular sequence of elements in* P *such that* $gR_P = PR_P$ (such sequences exist - cf. e.g. [LS, p.213, Lemma (3.8)]). *Set* $\tilde{\omega} = \tilde{\omega}_R$. *Then there is an isomorphism*

$$\alpha : \mathrm{Ext}_R^{n-d}(S, \tilde{\omega}) \xrightarrow{\sim} \mathrm{Hom}_R(S, \tilde{\omega}/g\tilde{\omega}) = (g\tilde{\omega} : P)/g\tilde{\omega}$$

such that, if for any $\nu \in g\tilde{\omega} : P \subset \tilde{\omega}$ *we set*

$$\nu_* = \alpha^{-1}(\nu + g\tilde{\omega}) \in \mathrm{Ext}_R^{n-d}(S, \tilde{\omega})$$

then:

(a) *with the "fundamental local homomorphism"*

$$\varphi : \mathrm{Ext}_R^{n-d}(S, \tilde{\omega}) \to \mathrm{Hom}_R(\Lambda^{n-d} P/P^2, \tilde{\omega}/P\tilde{\omega})$$

as in (13.1), *and*

$$\bar{g}_i = (g_i + P^2) \in P/P^2$$

we have

$$\varphi(\nu_*)[\bar{g}_1 \wedge \bar{g}_2 \wedge \ldots \wedge \bar{g}_{n-d}] = \nu + P\tilde{\omega};$$

and

(b) *for any sequence* $s = (s_1, \ldots, s_d)$ *in* R *whose image in* R/gR *is a system of parameters, if*

$$\bar{s}_i = s_i + P \in R/P = S \qquad (1 \le i \le d)$$

then the Yoneda pairing

$$H_m^d(\mathrm{Ext}_R^{n-d}(S, \tilde{\omega})) = H_m^d(S) \otimes_S \mathrm{Ext}_R^{n-d}(S, \tilde{\omega}) \to H_m^n(\tilde{\omega})$$

(where m *is the maximal ideal of* R) *satisfies*

$$\text{Yoneda}[\nu_*/(\bar{s}_1,\dots,\bar{s}_d)] = \text{Yoneda}([1/(\bar{s}_1,\dots,\bar{s}_d)]\otimes\nu_*)$$
$$= \nu/(g_1,\dots,g_{n-d},s_1,\dots,s_d)$$

(For notation cf. beginning of §7.)

Now supposing (13.11) to have been shown, look again at (13.10.1), and set $R = \mathcal{O}_{X,v}$ (assumed to be Cohen-Macaulay), $\tilde{\omega} = \tilde{\omega}_R$, and $S = \mathcal{O}_{V,v}$. Let $c_R:\Omega_R \to \tilde{\omega}$, $c_S:\Omega_S \to \tilde{\omega}_S$, be the natural maps. For each $\nu \in \Omega^d_{R/k}$ let $\bar{\nu}$ be its natural image in $\Omega^d_{S/k} = \Omega_{V,v}$, and choose $\nu' \in g\,\tilde{\omega}:P$ such that

$$\eta_v^{-1}c_S(\bar{\nu}) = \nu'_* \ .$$

Then for (13.10.1) to commute it is, in view of (13.2) and (13.11)(a), necessary and sufficient that for all ν, ν' as above

(13.12.1) $$\nu' \equiv c_R(\delta g_1\delta g_2\dots\delta g_{n-d}\wedge\nu) \quad (\text{mod. } P\tilde{\omega}_P \cap \tilde{\omega}) \ .$$

Moreover, with "res" as in (11.2), and notation as in (13.11)(b), we must have, in view of (13.8):

$$\text{res}_S[\bar{\nu}/(\bar{s}_1,\dots,\bar{s}_d)] = \text{res}^{\sim}_S[c_S(\bar{\nu})/(\bar{s}_1,\dots,\bar{s}_d)]$$
$$= \text{res}^{\sim}_R[\text{Yoneda}([1/(\bar{s}_1,\dots,\bar{s}_d)]\otimes\nu'_*)]$$

i.e.

(13.12.2) $$\text{res}_S[\bar{\nu}/(\bar{s}_1,\dots,\bar{s}_d)] = \text{res}^{\sim}_R[\nu'/(g_1,\dots,g_{n-d},s_1,\dots,s_d)]$$

In summary, (13.8), (13.10) and (13.11) imply:

THEOREM (13.12). <u>With notation and assumptions as in</u> (13.11), <u>for any</u> $\nu \in \Omega^d_{R/k}$ <u>there exists a</u> $\nu' \in g\tilde{\omega}:P$ <u>satisfying</u> (13.12.1) <u>above</u>. <u>This</u> ν' <u>is unique modulo</u> $g\tilde{\omega}$; <u>and it also satisfies</u> (13.12.2) (<u>where</u> $\bar{\nu}$ <u>is the natural image of</u> ν <u>in</u> Ω_S, <u>and</u> $(s_1,\dots,s_d)$ <u>is as in</u> (13.11)(b)).

<u>Remarks</u>. (i) The <u>uniqueness</u> of ν' (mod $g\tilde{\omega}$) results from the equality

$$(g\tilde{\omega}:P) \cap (P\tilde{\omega}_P \cap \tilde{\omega}) = (g\tilde{\omega}:P) \cap g\tilde{\omega}_P = g\tilde{\omega}$$

which holds because the natural map

$$(g\tilde{\omega}:P)/g\tilde{\omega} \to \tilde{\omega}_P/g\tilde{\omega}_P = [(g\tilde{\omega}:P)/g\tilde{\omega}]\otimes_S k(V)$$

is injective, the source being isomorphic to $\tilde{\omega}_S$ (cf. (13.11) and (13.8)(b)) which is torsion free (cf. (2.1.3)).

(ii) Conversely, (13.8), (13.11) and (13.12) (for $R = \mathcal{O}_{X,v}$, $S = \mathcal{O}_{V,v}$) imply that (13.10.1) commutes. For (13.12)(a) gives us an S-homomorphism

$$\beta:\Omega_S \to (g\tilde{\omega}:P)/g\tilde{\omega}$$

(viz. $\beta(\bar{\nu}) = \nu' + g\tilde{\omega}$), such that

$$\begin{array}{ccc}
\Omega_S \xrightarrow{\beta} (g\tilde{\omega}:P)/g\tilde{\omega} \xrightarrow{\alpha^{-1}} \mathrm{Ext}_R^{n-d}(S,\tilde{\omega}) \xrightarrow{(13.1)} & & \mathrm{Hom}_R(\Lambda^{n-d}P/P^2,\tilde{\omega}/P\tilde{\omega}) \\
\downarrow & & \downarrow \\
\Omega^d_{k(V)/k} & \xrightarrow[(13.2.1)]{} & \mathrm{Hom}_{k(V)}(\Lambda^{n-d}PR_P/P^2R_P,\tilde{\omega}_P/P\tilde{\omega}_P)
\end{array}$$

commutes. (Note that $\tilde{\omega}_P = \Omega_{R_P}$ since R_P is regular. Note also that β is well-defined, since if $\bar{\nu} = 0$, then we have (via (13.2.1)) that $\delta g_1\ldots\delta g_{n-d} \wedge \nu \in P\tilde{\omega}_P$, whence - by the uniqueness in (13.12) - $\nu' = 0$.) So we need to show that $\alpha^{-1}\beta = \eta_V^{-1}c_S$. But, since the pair $(\tilde{\omega}_S, \mathrm{res}_S^{\sim})$ is locally dualizing (cf. (0.6)(c)), therefore, by (13.8) so is the pair $(\mathrm{Ext}_R^{n-d}(S,\tilde{\omega}), \mathrm{res}_R^{\sim}\circ\mathrm{Yoneda})$; and we can verify that $\alpha^{-1}\beta = \eta_V^{-1}c_S$ by first applying the functor $H^d_{\mathfrak{m}}$, and then using (13.11)(b) and (13.12.2) to show that

$$\mathrm{res}_R^{\sim}\circ\mathrm{Yoneda} \circ H^d_{\mathfrak{m}}(\alpha^{-1}\beta) = \mathrm{res}_S = \mathrm{res}_R^{\sim}\circ\mathrm{Yoneda}\circ H^d_{\mathfrak{m}}(\eta_V^{-1}c_S)$$

where the last equality comes from (13.8).

(iii). Now, given (13.8) and (13.11), we can prove (13.10) and (13.12) as follows. First note that (13.12) is practically trivial when both X and V are smooth at v, g is part of a regular system of parameters $(g_1,\ldots,g_n)$ in R (so that $gR = P$), and for some integer a

$$s_i = (g_{n-d+i})^a \qquad (1 \le i \le d).$$

(Just take $\nu' = \delta g_1\ldots\delta g_{n-d} \wedge \nu$, and use the definition of res given in §7). As in remark (ii), (13.10.1) then commutes at such a v, and hence (13.10) holds, and hence, as we have seen, (13.12) follows in full generality!

* * *

It remains to prove (13.8) and (13.11).

Proof of (13.8). The uniqueness of η follows easily from local duality ((0.6)(c)).

For the existence of η, we may replace X by a compactification $\bar{X}$ and V by its closure in $\bar{X}$; so we may assume that X and V are proper over k.

Consider then the standard spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X).$$

Since $E_2^{pq} = 0$ for $p > d = \dim V$, we have edge homomorphisms

$$H^d(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) \to \mathrm{Ext}^{d+q}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X) \qquad (q \geq 0)$$

which compose with the natural maps

$$\mathrm{Ext}^{d+q}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X) \to \mathrm{Ext}^{d+q}_{\mathcal{O}_X}(\mathcal{O}_X,\tilde{\omega}_X) = H^{d+q}(X,\tilde{\omega}_X)$$

to give maps

$$(13.8.1) \qquad H^d(V,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) = H^d(X,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) \to H^{d+q}(X,\tilde{\omega}_X) \quad (q \geq 0).$$

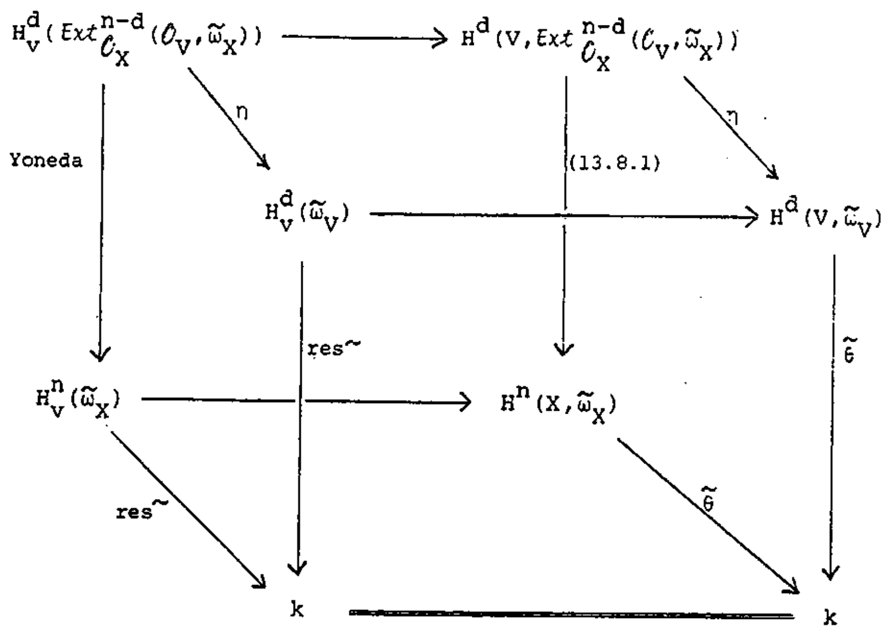
X being proper, of dimension n, we can take $q = n - d$ and obtain a composed map

$$H^d(V, \mathcal{E}xt^{n-d}_{\mathcal{O}_V}(\mathcal{O}_V,\tilde{\omega}_X)) \to H^n(X,\tilde{\omega}_X) \xrightarrow{\tilde{\theta}_X} k.$$

Since $\tilde{\omega}_V$ is dualizing, we have then a corresponding map

$$\eta: \mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X) \to \tilde{\omega}_V .$$

Now for any closed point $v \in X$ consider the cube



with horizontal arrows representing natural maps. It is clear that the four faces which do not have "Yoneda" as an edge commute. The assertion in (13.8)(a) is that the face on the left side commutes; to prove this it will be enough to show that the rear face commutes.

For this purpose, in the construction of η replace the functor $\Gamma(X,\cdot)$ by its subfunctor $\Gamma_v(\cdot)$ (sections supported at $v \in X$) to get a composed map

$$\beta: H^d_V(\mathcal{E}xt^{n-d}_{\mathcal{O}_X}(\mathcal{O}_V,\tilde{\omega}_X)) \to \mathrm{Ext}^n_v(\mathcal{O}_V,\tilde{\omega}_X)$$

$$\to \mathrm{Ext}^n_v(\mathcal{O}_X,\tilde{\omega}_X) = H^n_V(\tilde{\omega}_X)$$

(where Ext^n_v is the derived functor of $\Gamma_v \circ \mathcal{H}om$). Clearly the rear face will commute if "Yoneda" is replaced by "β". So we need only show that

(13.8.2) $$\beta = \text{Yoneda}\ .$$

The question is local, so we replace X by $R = \mathcal{O}_{X,v}$ and V by $S = \mathcal{O}_{V,v}$. Also, in the construction of β, we can replace $\tilde{\omega}_R$ by an arbitrary R-module G, and we get G-functorial maps

$$\beta^q(G): H_{\mathfrak{m}}^d(\mathrm{Ext}_R^q(S,G)) \to H_{\mathfrak{m}}^{d+q}(G) \qquad (q \geq 0)$$

(where $\mathfrak{m}$ is the maximal ideal of R). We can then prove (13.8.2) by showing that the conditions in (13.9) hold with "β" in place of "γ".

First of all, then, we have to show that

$$\beta^0(G): H_{\mathfrak{m}}^d(\mathrm{Hom}_R(S,G)) \to H_{\mathfrak{m}}^d(G)$$

is the map induced by the natural inclusion (i.e. "evaluation at 1")

$$e: \mathrm{Hom}_R(S,G) \hookrightarrow G.$$

(Note that the diagram

$$\begin{array}{ccc} & H_{\mathfrak{m}}^d(S) \otimes \mathrm{Hom}_R(S,G) & \\ /\!\!/ & & \searrow \text{ takes } \mu\otimes\lambda \text{ to } [H_{\mathfrak{m}}^d(\lambda)](\mu) \\ H_{\mathfrak{m}}^d(\mathrm{Hom}_R(S,G)) & \xrightarrow[\text{via inclusion}]{} & H_{\mathfrak{m}}^d(G) \end{array}$$

commutes.) For this we let $I^{\bullet}$ be an R-injective resolution of G, then extend the natural (inclusion) map of complexes

$$\mathrm{Hom}_R(S,I^{\bullet}) \longrightarrow \mathrm{Hom}_R(R,I^{\bullet}) = I^{\bullet}$$

to a map of Cartan-Eilenberg resolutions (double complexes) $C^{\bullet\bullet} \to D^{\bullet\bullet}$, and consider the resulting map

$$\Gamma_{\mathfrak{m}}(C^{\bullet\bullet}) \to \Gamma_{\mathfrak{m}}(D^{\bullet\bullet}),$$

which induces a natural homomorphism of spectral sequences; in particular we obtain the commutative diagram

$$\begin{array}{ccccc}
E_2^{d0}(C^{\bullet\bullet}) = H^d_{\mathfrak m}(\mathrm{Hom}_R(S,G)) & \longrightarrow & \mathrm{Ext}^d_{\mathfrak m}(S,G) & & \\
\Big\downarrow \text{via } e & \searrow & \Big\downarrow & & \\
E_2^{d0}(D^{\bullet\bullet}) = H^d_{\mathfrak m}(\mathrm{Hom}_R(R,G)) & = & \mathrm{Ext}^d_{\mathfrak m}(R,G) & = & H^d_{\mathfrak m}(G)
\end{array}$$

which gives us the desired result.

Second, we have to show that β is δ-functorial, i.e. if

(13.8.3) $$0 \to G' \to G \to G'' \to 0$$

is an exact sequence of R-modules, then the resulting diagram

$$\begin{array}{ccccc}
H^d_{\mathfrak m}(\mathrm{Ext}^q_R(S,G'')) & \longrightarrow & \mathrm{Ext}^{d+q}_{\mathfrak m}(S,G'') & \longrightarrow & H^{d+q}_{\mathfrak m}(G'') \\
\Big\downarrow & (1) & \Big\downarrow & (2) & \Big\downarrow \\
H^d_{\mathfrak m}(\mathrm{Ext}^{q+1}_R(S,G')) & \longrightarrow & \mathrm{Ext}^{d+q+1}_{\mathfrak m}(S,G') & \longrightarrow & H^{d+q+1}_{\mathfrak m}(G')
\end{array}$$

commutes for all $q \ge 0$. This is easily checked for the subdiagram (2). As for (1), we choose an exact sequence of injective resolutions.

$$0 \longrightarrow I'^{\bullet} \longrightarrow I^{\bullet} \longrightarrow I''^{\bullet} \longrightarrow 0$$

over (13.8.3); and then, with $J'^{\bullet} = \mathrm{Hom}_R(S,I'^{\bullet})$ etc., we have an exact sequence

$$0 \longrightarrow J'^{\bullet} \xrightarrow{u} J^{\bullet} \longrightarrow J''^{\bullet} \longrightarrow 0 .$$

The horizontal arrows in (1) come from the standard spectral sequences for the hypercohomologies $\mathbb{H}^{\bullet}_{\mathfrak m}(J''^{\bullet})$, $\mathbb{H}^{\bullet}_{\mathfrak m}(J'^{\bullet})$ respectively. There is however a natural map from the <u>mapping cone</u> $K^{\bullet}$ of u [H,p.26] to $J''^{\bullet}$ which induces homology isomorphisms; so we may replace $\mathbb{H}_{\mathfrak m}(J''^{\bullet})$ by $\mathbb{H}_{\mathfrak m}(K^{\bullet})$. But then the vertical arrows in (1) are those associated to the natural projection $K^{\bullet} \to J''[1]$, where [1] denotes "shifting one place left" (cf. remarks preceding (8.6)). The commutativity of (1) results, and that finishes the proof of (13.8)(a).

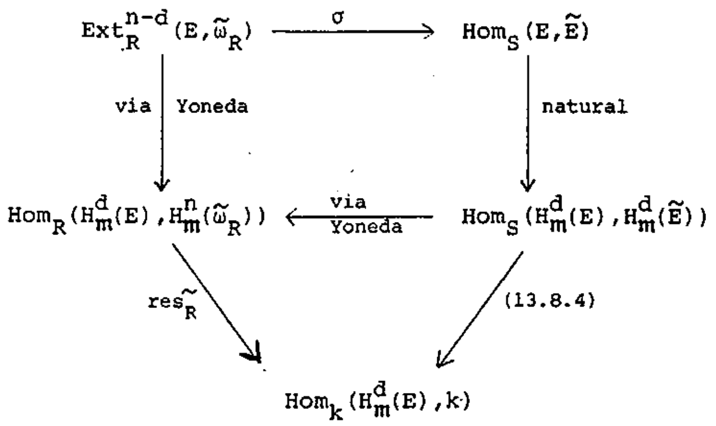
Now we prove (13.8)(b). With R and S as above, (R now being assumed Cohen-Macaulay), to show that η is an isomorphism we may pass to completions; so we keep the same notations as before, but assume that R and S are complete.

It suffices then to show that the pair consisting of $\tilde{E} = \mathrm{Ext}_R^{n-d}(S,\tilde{\omega}_R)$ together with the map

$$(13.8.4)\qquad H_{\mathfrak{m}}^d(\tilde{E}) \xrightarrow{\text{Yoneda}} H_{\mathfrak{m}}^n(\tilde{\omega}_R) \xrightarrow{\mathrm{res}_{\tilde{R}}} k$$

represents the functor $\mathrm{Hom}_k(H_{\mathfrak{m}}^d(E),k)$ of S-modules E. In other words, we need to show that <u>the composition</u> (13.8.4) $\circ$natural <u>in the following diagram is an isomorphism</u>.

(13.8.5)



Here σ corresponds to the natural pairing

$$(13.8.6)\qquad \mathrm{Ext}_R^{n-d}(E,\tilde{\omega}_R)\otimes E = \mathrm{Ext}_R^{n-d}(E,\tilde{\omega}_R)\otimes \mathrm{Hom}_R(S,E) \to \mathrm{Ext}_R^{n-d}(S,\tilde{\omega}_R) = \tilde{E}.$$

But the diagram (13.8.5) <u>commutes</u>: for the lower triangle, this is clear, and for the square it is equivalent to the commutativity of

$$\begin{array}{ccc}
\mathrm{Ext}_R^{n-d}(E,\tilde{\omega}_R) \otimes E \otimes H_{\mathfrak{m}}^d(S) & \xrightarrow{\text{via}(13.8.6)} & \mathrm{Ext}_R^{n-d}(S,\tilde{\omega}_R) \otimes H_{\mathfrak{m}}^d(S) \\
\| & & \downarrow \text{Yoneda} \\
\mathrm{Ext}_R^{n-d}(E,\tilde{\omega}_R) \otimes H_{\mathfrak{m}}^d(E) & \xrightarrow[\text{Yoneda}]{} & H_{\mathfrak{m}}^n(\tilde{\omega}_R)
\end{array}$$

which is readily checked (for example by replacing $\tilde{\omega}_R$ by a variable R-module G and using (13.9)). It will suffice, therefore to show:

(13.8.7) (Full Local Duality). For any R-module G, the composition

$$\mathrm{Ext}_R^{n-d}(G,\tilde{\omega}_R) \xrightarrow{\text{via Yoneda}} \mathrm{Hom}_R(H_{\mathfrak{m}}^d(G), H_{\mathfrak{m}}^n(\tilde{\omega}_R)) \xrightarrow{\widetilde{\mathrm{res}}_R} \mathrm{Hom}_k(H_{\mathfrak{m}}^d(G), k)$$

is an isomorphism;

and, in addition, that:

(13.8.8) for any S-module E, the map σ in (13.8.5) is an isomorphism.

By ((0.6)(c)) we have a natural isomorphism of functors of R-modules G:

$$\mathrm{Hom}_R(G,\tilde{\omega}_R) \xrightarrow{\sim} \mathrm{Hom}_k(H_{\mathfrak{m}}^n(G), k),$$

which then extends uniquely to a homomorphism of δ-functors

$$\mathrm{Ext}_R^i(G,\tilde{\omega}_R) \longrightarrow \mathrm{Hom}_k(H_{\mathfrak{m}}^{n-i}(G), k) \qquad (i \geq 0). \tag{13.8.9}$$

Replacing d by n - i in (13.8.7), and Yoneda by $(-1)^i$ Yoneda, we get a homomorphism of δ-functors, which coincides with (13.8.9) for i = 0, and hence for all i. So (13.8.7) asserts that (13.8.9) is an isomorphism for all i. But this follows from the fact that, R being Cohen-Macaulay, we have $H^i_{\mathfrak{m}}(F) = 0$ for any free R-module F and any integer $i \neq n = \dim R$ [G3,p.13,Prop. 1.12 and p.47, Cor. 3.10], so that the target in (13.8.9) is the derived functor of $\mathrm{Hom}_k(H^n_{\mathfrak{m}}(G), k)$.

In particular, if E is an S-module, then

$$\mathrm{Ext}_R^{n-d-1}(E,\tilde{\omega}_R) \cong \mathrm{Hom}_k(H_{\mathfrak{m}}^{d+1}(E),k) = 0$$

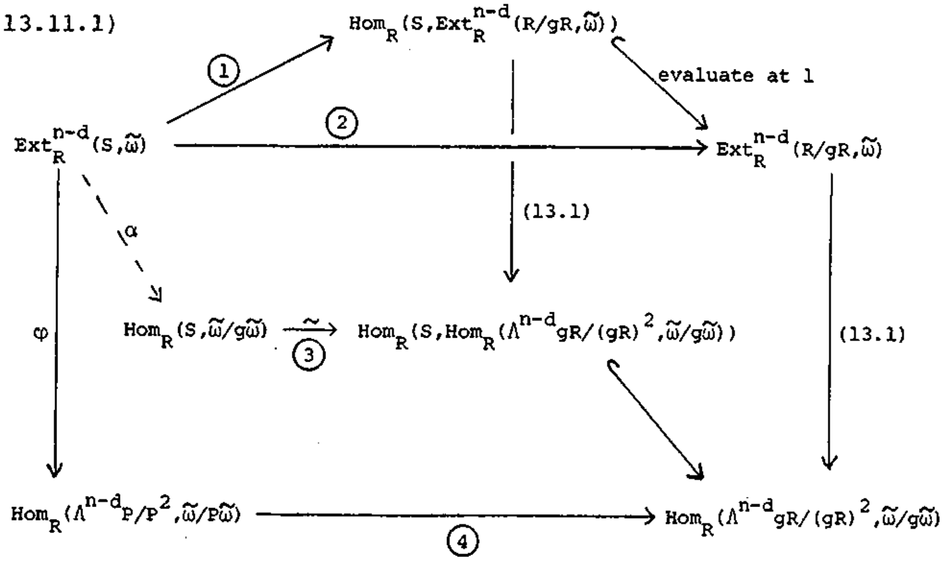
and so the functor $\mathrm{Ext}_R^{n-d}(E,\tilde{\omega}_R)$ of S-modules E is <u>left exact</u>. From this (13.8.8) follows (cf. [G3,p.49,Prop. 4.2]; the case where E is finitely generated would suffice for our purposes, but anyway the general case reduces to this one via direct limits).

This completes the proof of (13.8).

* * *

<u>Proof of (13.11)</u>. Consider the diagram

(13.11.1)



Here the map ② corresponds to the natural map $R/gR \to S$; and the map ① is the unique one making the upper triangle commute. As in the proof of (13.8.8) just above, since the functor $\mathrm{Ext}_R^{n-d}(E,\tilde{\omega})$ of R/gR-modules E is left exact, therefore ① <u>is an isomorphism</u>.

The maps labelled (13.1) are fundamental local homomorphisms; and they are <u>isomorphisms</u> as can be seen by using the Koszul complex $K_R(g)$ determined by the regular sequence g as a projective resolution of R/g to calculate (13.1) explicitly.

The map ③ corresponds to the <u>isomorphism</u> $S \xrightarrow{\sim} \Lambda^{n-d} gR/(gR)^2$ taking 1 to the natural image of $g_1 \wedge g_2 \wedge \ldots \wedge g_{n-d}$.

Thus the map α given by

$$\alpha = \textcircled{3}^{-1} \circ (13.1) \circ \textcircled{1}$$

is an isomorphism.

The assertion (13.11)(a) then follows from the commutativity of the diagram (13.11.1), which we leave to the reader to check. (A variant of all this is given in [LS, p.218, Lemma (A.2)]).

As for (13.11)(b), we first note that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_R^{n-d}(S,\tilde{\omega}) & \xrightarrow{\textcircled{2}} & \mathrm{Ext}_R^{n-d}(R/gR,\tilde{\omega}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_R(H_{\mathfrak{m}}^d(S), H_{\mathfrak{m}}^n(\tilde{\omega})) & \longrightarrow & \mathrm{Hom}_R(H_{\mathfrak{m}}^d(R/gR), H_{\mathfrak{m}}^n(\tilde{\omega}))
\end{array}$$

where the horizontal arrows are induced by the natural map $R/gR \to S$ (so that ② is the same as in (13.11.1)), and the vertical arrows by Yoneda. As before, if we use the Koszul complex $K_R(g)$ to calculate $\mathrm{Ext}_R^{n-d}(R/gR,\tilde{\omega})$, we find that ② takes ν_* to the cohomology class of the (n-d)-cocycle

$$\nu \in g\tilde{\omega}:P \subset \tilde{\omega} = \mathrm{Hom}_R(R,\tilde{\omega}) = \mathrm{Hom}_R(K_R(g)_{n-d},\tilde{\omega}) \;.$$

We see then, after a little thought, that it will suffice to prove the following statement:

(13.11)(b)' *The Yoneda product of*

$$[1/(\bar{\bar{s}}_1,\dots,\bar{\bar{s}}_d)] \in H^d_{\mathfrak{m}}(R/gR) \qquad (\bar{\bar{s}}_i = (s_i + gR) \in R/gR)$$

with the equivalence class $\langle K_R(g)\rangle$ *in* $\mathrm{Ext}^{n-d}_R(R/gR,R)$ *of the exact sequence consisting of* $K_R(g)$ *together with its natural augmentation* $K_R(g)_0 = R \to R/gR$, *is*

$$[1/(g_1,\dots,g_{n-d},s_1,\dots,s_d)] \in H^n_{\mathfrak{m}}(R) \ .$$

Now we have a natural map of δ-functors (of R-modules G)

$$\mathrm{Ext}^i(R/(g,s)R,G) \xrightarrow{\ \tau\ } H^i_{\mathfrak{m}}(G) \qquad (i \geq 0)$$

[which may be thought of as the Yoneda product with $1 \in H^0_{\mathfrak{m}}(R/(s,g)R)$], and, hence a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^d_R(R/(g,s)R,R/gR)\otimes \mathrm{Ext}^{n-d}_R(R/gR,R) & \xrightarrow{\tau\otimes 1} & H^d_{\mathfrak{m}}(R/gR)\otimes \mathrm{Ext}^{n-d}_R(R/gR,R) \\ \text{Yoneda}\downarrow & & \downarrow\text{Yoneda} \\ \mathrm{Ext}^n_R(R/(g,s)R,R) & \xrightarrow[\tau]{} & H^n_{\mathfrak{m}}(R) \end{array} \ .$$

On the other hand, according to the definitions in §7, we find (with $i = d$, $G = R/gR$) that

$$[1/(\bar{\bar{s}}_1,\dots,\bar{\bar{s}}_d)] = \tau\langle K_{R/gR}(\bar{\bar{s}})\rangle \ .$$

Similarly (with $i = n$, $G = R$),

$$[1/(g_1,\dots,g_{n-d},s_1,\dots,s_d)] = \tau\langle K_R(g,s)\rangle \ .$$

So (13.11)(b)' follows from:

(13.11)(b)": The Yoneda product of

$$\langle K_{R/gR}(\bar{\bar{s}})\rangle \in \mathrm{Ext}_R^d(R/(g,s)R, R/gR)$$

and

$$\langle K_R(g)\rangle \in \mathrm{Ext}_R^{n-d}(R/gR, R)$$

is

$$\langle K_R(g,s)\rangle \in \mathrm{Ext}_R^n(R/(g,s)R, R) .$$

As in (13.9), we can characterize the Yoneda pairing (for R-modules E, F, G)

$$\mathrm{Ext}_R^d(E,F) \otimes_R \mathrm{Ext}_R^q(F,G) \longrightarrow \mathrm{Ext}_R^{d+q}(E,G)$$

as follows: for fixed E, F and fixed $\mu \in \mathrm{Ext}_R^d(E,F)$, the family of maps

$$Y_\mu^q : \mathrm{Ext}_R^q(F,G) \longrightarrow \mathrm{Ext}_R^{d+q}(E,G)$$

given by

$$Y_\mu^q(\lambda) = \mathrm{Yoneda}(\mu \otimes \lambda)$$

is the unique homomorphism of δ-functors such that

$$Y_\mu^0(\lambda) = \text{image of } \mu \text{ under the map } \mathrm{Ext}_R^d(E,F) \to \mathrm{Ext}_R^d(E,G) \text{ induced by } \lambda \in \mathrm{Hom}_R(F,G).$$

It follows that this pairing is given by pasting (composition) of exact sequences (cf. [M, Ch.III, §9]).

This being so, it is clear that (13.11)(b)" follows from

(13.11)(b)‴. Let R be any commutative ring, and let $h = (h_1,\ldots,h_n)$ be an R-regular sequence. Then, in $\mathrm{Ext}_R^n(R/hR,R)$, the equivalence class $\langle K_R(h)\rangle$ is the same as the class of the composition $\sigma_1\sigma_2\cdots\sigma_k$ of the following exact sequences:

$$(\sigma_1) \qquad 0 \longrightarrow R \xrightarrow{h_1} R \longrightarrow R/h_1R \longrightarrow 0$$

$$(\sigma_2) \qquad 0 \longrightarrow R/h_1R \xrightarrow{h_2} R/h_1R \longrightarrow R/(h_1,h_2)R \longrightarrow 0$$

$$\vdots$$

$$(\sigma_n) \qquad 0 \longrightarrow R/(h_1,\ldots,h_{n-1})R \xrightarrow{h_n} R/(h_1,\ldots,h_{n-1})R \longrightarrow R/(h_1,\ldots,h_n)R \longrightarrow 0$$

Proof of (13.11)(b)‴. We need only find a map ψ of complexes $K_R(h) \to \sigma_1\sigma_2\ldots\sigma_n$ over the identity map of R/hR, which is the identity map of R in degree n.

Let $(e_1,e_2,\ldots,e_n)$ be the standard basis of R^n, and define ψ in degree j by

$$\psi_j : \Lambda^j(R^n) \to R/(h_1,\ldots,h_{n-j-1})R \qquad (0 \le j \le n)$$

(the target is R for $j = n - 1$ or $j = n$) where

$$\psi_j(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}) = 1 \quad \text{if } (i_1,i_2,\ldots,i_j) = (n-j+1,n-j+2,\ldots,n)$$

$$= 0 \quad \text{otherwise.}$$

This ψ has the required properties.

APPENDIX A

Projective Noether normalization.

In this Appendix we prove:

PROPOSITION (A.1). Let V be a d-dimensional projective variety over a field k, and let $v \in V$ be a smooth point. Then there exists a finite map $f:V \to \mathbb{P} = \mathbb{P}_k^d$ which is étale at v.

Remark. Geometrically (i.e. when k is infinite and v is a k-rational point) (A.1) is clear: project $V \subset \mathbb{P}^N$ from a linear $\mathbb{P}^{N-d-1} \subset \mathbb{P}^N$ which meets neither V nor the tangent space to V at v.

The main difficulty arises when k is finite.

Proof of (A.1). Recall that V is smooth precisely where the differential sheaf $\Omega^1_{V/k}$ is free of rank d, and that a map $f:V \to \mathbb{P}$ is étale precisely where the relative differential sheaf $\Omega^1_{V/\mathbb{P}} = 0$. So the closure of v contains points which are both smooth and closed, and we may assume that v itself is closed. Then if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{V,v}$, $\mathcal{O}_{V,v}/\mathfrak{m}$ is finite and separable over k, and f is étale at v if and only if $\mathfrak{m}$ is generated by the maximal ideal of $\mathcal{O}_{\mathbb{P},f(v)}$. So it will be enough to find a sequence $(g_0', g_1', \ldots, g_d')$ of forms (= homogeneous elements) in the homogeneous coordinate ring $k[V]$ (defined with respect to some embedding $V \hookrightarrow \mathbb{P}^N$), all of the same degree, such that $g_0'(v) \neq 0$, g_i'/g_0' $(1 \le i \le d)$ generate $\mathfrak{m}$, and $k[V]/(g_0', g_1', \ldots, g_d')$ is finite-dimensional over k. (Then we can take f to be the map associated to the inclusion $k[g_0', \ldots, g_d'] \subset k[V]$.)

The case $d = 0$ is trivial, so assume that $d > 0$. Pick an element g_1/h in $\mathfrak{m} - \mathfrak{m}^2$, where g_1 and h are forms in $k[V]$ of the same degree, with $h(v) \neq 0$. Assume inductively that a sequence $(g_1, g_2, \ldots, g_j)$ $(1 \le j < d)$ of forms has been found such that:

(a) for each $i \le j$, there is a form ℓ_i having the same degree as g_i, with $\ell_i(v) \neq 0$, and such that $g_i/\ell_i \in \mathfrak{m}$;

(b) if $\mathfrak{m}_j$ is the ideal $\mathfrak{m}^2 + (g_1/\ell_1, \ldots, g_j/\ell_j)$ in $\mathcal{O}_{V,v}$ (ℓ_i as in (a)), then the $(\mathcal{O}_{V,v}/\mathfrak{m})$-vector space $\mathfrak{m}_j/\mathfrak{m}^2$ has dimension j;

(c) every prime ideal in $k[V]$ containing $g_1, \ldots, g_j$ has height $\ge j$.

Let $M \subset k[V]$ be the prime ideal corresponding to v, let $\gamma \notin M$ be a form of degree 1, and let $M_j \subset M$ be the homogeneous ideal whose elements of degree $e > 0$ are the forms g such that $g/\gamma^e \in \mathfrak{m}_j$ (cf. (b) above). Then $M \not\subset M_j$ (since $\mathfrak{m} \not\subset \mathfrak{m}_j$, j being $< d$); and if $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are prime ideals which are minimal among the ones containing $(g_1, \ldots, g_j)$, then $M \not\subset \mathfrak{p}_i$, because M has height $d > j = \text{height } \mathfrak{p}_i$. I claim then that:

(A.1.1) <u>there is a form</u>

$$g_{j+1} \in M - (M_j \cup \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \ldots \cup \mathfrak{p}_s).$$

Once (A.1.1) is proved, we can continue in the same way, to build up a sequence $(g_1, \ldots, g_d)$ of forms such that the above conditions (a), (b), (c) are satisfied for $j = d$. We can also choose a form g_0 not lying in any of the minimal primes of the ideal $(g_1, \ldots, g_d)k[V]$ (cf. [ZS, p.286]). So $(g_0, g_1, \ldots, g_d)$ has all the required properties, <u>except that</u> the g_i may have different degrees. We will return to this problem below, but first let us prove (A.1.1).

As in [ZS, p.286], there is a form

$$\gamma_0 \in M - (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s)$$

and we may assume that $\gamma_0 \in M_j$ (otherwise take $g_{j+1} = \gamma_0$). A similar argument yields the following Lemma, which we need to complete the proof:

LEMMA (A.1.2). Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ be homogeneous ideals in $k[V]$ such that $k[V]/\mathfrak{a}_i$ has Krull dimension > 0 for $1 \leq i \leq t$. Then there is an integer n_0 such that for every $n \geq n_0$, there exists a form $g \in k[V]$ of degree n with

$$g \notin \bigcup_{i=1}^{t} \mathfrak{a}_i .$$

<u>Proof</u>. The assertion is clear when $t = 1$ (consider the Hilbert polynomial of $k[V]/\mathfrak{a}_1$). Assume then that $t > 1$. After replacing $\mathfrak{a}_i$ by a suitable prime ideal containing it, we may assume that $\mathfrak{a}_i$ is prime. We may also assume that $\mathfrak{a}_i \not\subset \mathfrak{a}_j$ if $i \neq j$. For each pair $i \neq j$ choose a form a_{ij} in $\mathfrak{a}_i - \mathfrak{a}_j$, and set

$$a_j = (\prod_{i \neq j} a_{ij}) \in (\bigcap_{i \neq j} \mathfrak{a}_i) - \mathfrak{a}_j .$$

Let δ_j be the degree of a_j, and let $\delta = \max_{1 \le j \le t}(\delta_j)$. Let δ' be an integer such that for any j with $1 \le j \le t$ and for any $n' \ge \delta'$ there is a form $b_{j,n'}$ of degree n' not contained in $\mathfrak{a}_j$ (cf. the above case $t = 1$). Then, for any $n \ge \delta + \delta'$, the form $\sum_j a_j b_{j,n-\delta_j}$ has degree n and lies in no $\mathfrak{a}_j$. Q.E.D.

Returning now to (A.1.1), we proceed by induction on s, the case $s = 0$ being trivial. We may assume, by the inductive hypothesis, that for each $i = 1,2,\ldots,s$, there exists a form

$$\gamma_i \in M - (M_j \cup \mathfrak{p}_1 \cup \ldots \cup \hat{\mathfrak{p}}_i \cup \ldots \cup \mathfrak{p}_s)$$

(where "$\hat{\mathfrak{p}}_i$" means, as usual, "omit $\mathfrak{p}_i$"). If $\gamma_i \notin \mathfrak{p}_i$ for some i, then we can take $g_{j+1} = \gamma_i$. So assume $\gamma_i \in \mathfrak{p}_i$ for all i. Choose n_0 as in (A.1.2), and let m be an integer large enough that

$$n = m(\text{degree } \gamma_0) - (\text{degree } \gamma_1) + \sum_{i=2}^{s} (\text{degree } \gamma_i) \ge n_0$$

(where, as above, $\gamma_0 \in M_j - (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s)$) so that there is a form g of degree n not lying in $M \cup \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s$. Then consider the form

$$g_{j+1} = g\gamma_1 + \gamma_0^m \gamma_2 \gamma_3 \cdots \gamma_s .$$

Since $g_n \notin M$ and $\gamma_1 \notin M_j$, we find that $g\gamma_1 \notin M_j$. Also, for $i > 2$, $g \notin \mathfrak{p}_i$ and $\gamma_1 \notin \mathfrak{p}_i$ so $g\gamma_1 \notin \mathfrak{p}_i$. Moreover $g\gamma_1 \in \mathfrak{p}_1$ and

$$\gamma_0^m \gamma_2 \cdots \gamma_s \in (M_j \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_s) - \mathfrak{p}_1 .$$

The assertion (A.1.1) follows.

* * *

We consider now the sequence $(g_0,\ldots,g_d)$ found above. We will construct a sequence of forms $(h_0,h_1,\ldots,h_d)$ such that

(i) $h_i(v) \ne 0 \quad (0 \le i \le d)$;

(ii) all the forms $h_i g_i \quad (0 \le i \le d)$ have the same degree; and

(iii) $k[V]/(h_0 g_0, h_1 g_1, \ldots, h_d g_d)$ is finite-dimensional over k.

Then the sequence $(g_i') = (h_i g_i)$ has all the properties needed - as previously explained - for the proof of (A.1).

We choose h_0 to be any form not contained in any of the minimal primes of $(g_1,\ldots,g_d)k[V]$. Assume that we have found forms $h_0,h_1,\ldots,h_j$ $(j < d)$ such that $h_i(v) \neq 0$ $(0 \le i \le j)$, such that all the forms $h_i g_i$ $(0 \le i \le j)$ have the same degree, say δ, and such that

$$k[V]/(h_0 g_0, h_1 g_1, \ldots, h_j g_j, g_{j+1}, \ldots, g_d)$$

is finite-dimensional. Lemma (A.1.2) gives us an n_0 such that for $n \ge n_0$ there exists a form $h_{(n)}$ of degree n, not lying in any of the minimal primes of the ideal

$$(h_0 g_0, h_1 g_1, \ldots, h_j g_j, g_{j+2}, \ldots, g_d)k[V],$$

and such that $h_{(n)}(v) \neq 0$. Let r_i = degree h_i $(0 \le i \le j)$ and let r be any common multiple of the r_i such that

$$n_r = r + \delta - \text{degree}(g_{j+1}) \ge n_0.$$

Then replacing h_i by $h_i^{(r/r_i)+1}$, and setting $h_{j+1} = h_{(n_r)}$, we have the same conditions as above with j replaced by $j + 1$.

Continuing in this way, we complete the construction.

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RÉSUMÉ

Il s'agit d'un exposé plutôt concret et detaillé du rôle des formes différentielles dans la théorie de dualité des variétés propres (non necéssairement lisses) surs un corps parfait. On décrit (d'après Kunz) un faisceau $\tilde{\omega}$ de formes méromorphes de degré maximal qui est un faisceau dualisant canonique. Le morphisme trace correspondant induit en chaque point un morphisme résidu, qu'on caractérise localement; et qui, avec $\tilde{\omega}$, est localement dualisant (compatibilité de dualité globale et locale). On donne aussi des compléments divers, y compris une interprétation de l'adjonction, (i.e. réalisation du faisceau canonique d'une sous-variété d'une variété localement de Cohen-Macaulay) en termes de résidus.