On blowing down projective spaces in singular varieties

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In this article we study some "contractible" projective spaces in singular varieties, especially with regard to the singularities through which they can—or cannot—pass (cf. Corollary (2. 3)). While this may be interesting from the point of view of singularity theory, our motivation was to remove the last obstacle to proving in its natural generality a basic structure theorem connected with the projective classification of varieties. We discuss this application at the end of the introduction.

Proposition (1. 1) describes the behavior of the dualizing sheaf of a hypersurface under blowing up. Corollary (1. 2) shows under certain circumstances that blowing up can make "normal bundles" (of codimension one subvarieties in singular hypersurfaces) more positive. Theorem (1. 3) is a version of Hironaka's resolution theorem in our special case, with the positivity increasing property of (1. 2) built in.

These results are used in § 2 in the proof of the main theorem of the paper:

Theorem (2. 1). *Let V be a reduced irreducible algebraic variety of dimension $d \geqq 2$ over an algebraically closed field, k, of characteristic zero. Assume that V is locally a hypersurface. Let $P \subseteqq V$ be a codimension one subvariety which is isomorphic (as a k-variety) to the projective space $\mathbb{P}_k^{d-1}$ and which is not entirely contained in the singular locus* $\operatorname{Sing}(V)$; *and let $\alpha: P \to A$ be the inclusion. Assume that P "blows down birationally", i.e. that there exists a birational map $\varphi: V^* \to A$ where V^* is an open neighborhood of P in V, and A is an affine k-variety (whence $\varphi(P)$ is a single point). Then the normal bundle $\nu_P = \omega_P \otimes \alpha^* \omega_V^{-1}$ (ω = dualizing sheaf) is isomorphic to $\mathcal{O}_P(m)$ for some $m < 0$. Moreover if $m = -1$ then $P \cap \operatorname{Sing}(V)$ is either empty or of pure dimension $d-2$.*

Corollary (2. 3) restates (2. 1) for threefolds, with somewhat different hypotheses. It makes use of Theorem (0. 1), due to H. Laufer, which refines (2. 1) in the case $d = 2$, $m = -1$, with the "local hypersurface" hypothesis replaced by the assumption that V is normal and Gorenstein: Theorem (0. 1) states then that $P \cap \operatorname{Sing}(V)$ *consists of at most one A_n-type rational singularity.*

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To appreciate the above, it helps to understand the main application it was designed for. Let (V, S) be a pair consisting of a smooth ample divisor, S, on a smooth connected projective threefold, V,—everything defined over $\mathbb{C}$. In [So1], [So2] it was shown that under very weak conditions on the pair, S is of general type and there is a second pair (V', S') satisfying:

a) there is a map $\varphi: V \to V'$ expressing V as the blowup at a finite set of points of a projective manifold V',

b) $S' = \varphi(S)$ is a smooth ample divisor on V' and $\varphi_S: S \to S'$ is the map of S onto its minimal model.

The relation of (V, S) to (V', S') is very close and has many nice properties. In particular, it lets the highly developed theory of minimal models of general type surfaces be used directly to study ample divisors on threefolds; it leads to very precise delicate results, e.g. [So3].

In [So4] a rather elaborate method based on a generalization of the Fano-Morin three-dimensional adjunction process (see [So4], [F+So]) was developed to classify where the main theorem of [So3] broke down. It turned out to be unnecessary to publish [So4] because as N. Shepherd-Barron first pointed out to the second author, the then new results of Mori [M] yielded an easy proof.

Recently versions of the above results of [So3], [So4] when V is singular have become important. For example in the thesis of M. L. Fania [F], a problem about extension of modifications from a smooth ample divisor on a smooth fourfold to the fourfold led to the need for the results for a possibly singular divisor on the fourfold. Mori's work is restricted to smooth varieties but the methods of [So3] combined with [So4] work on Gorenstein varieties. In [F+So] this was carried out, but besides the methods being painfully more elaborate than [So4], the results were incomplete. The pair (V', S') existed but it could only be concluded that V' was Cohen-Macaulay and S' was a Weil divisor. The difficulty came from the lack of Theorem (2.1) above. Besides allowing the results on ample divisors to be proved in full generality, the use of Theorem (2.1) has meant that the proofs have to be only a little more complicated than in the manifold case.

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§ 0. Blowing down and A_n singularities

We give here an unpublished result of Henry Laufer, needed for Corollary (2.3).

By a Gorenstein variety, X, we mean a locally Cohen-Macaulay variety over an algebraically closed field, with an *invertible* dualizing sheaf ω_X. The restriction of ω_X to the open set of smooth points of X is isomorphic to the sheaf of germs of differential n-forms, where $n = \dim X$. (For more details on dualizing sheaves cf. § 1.) We abuse notation by also using "ω_X" to denote any canonical divisor, i.e. a Cartier divisor associated to the dualizing sheaf.

Theorem (0.1). *Let V be a normal Gorenstein surface; and let $P \subseteqq V$ be a smooth projective rational curve such that*

a) *P is contractible, i.e. there is a birational map $\pi: V \to A$ onto a normal surface A such that $\pi(P)$ is a single point and $\pi^{-1}\pi(P) = P$; and*

b) *the intersection number $\omega_V \cdot P = -1$, where ω_V is a canonical divisor on V.*

Then P contains at most one singular point x of V, and such an x must be an A_n-type rational singularity for some $n \geqq 1$.

Remarks. — When P contains no singularity of V, the condition $\omega_V \cdot P = -1$ is equivalent (by adjunction) to $P \cdot P = -1$.

— For an example where P contains an A_n singularity of V, consider the map

$$\pi: V \to A = \mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[X, Y])$$

obtained by blowing up the ideal (X, Y^{n+1}) in $\mathbb{C}[X, Y]$.

Proof of (0.1). Let $\sigma: V' \to V$ be a minimal resolution of V around P, in the sense that if $x_1, \ldots, x_r$ are the singularities of V on P and $E_1, \ldots, E_n$ are the irreducible components of $\sigma^{-1}\{x_1, \ldots, x_r\}$ then:

i) σ is a proper birational map inducing an isomorphism

$$V' - \sigma^{-1}\{x_1, \ldots, x_r\} \approx V - \{x_1, \ldots, x_r\}.$$

ii) No E_i is a smooth rational curve satisfying $E_i \cdot E_i = -1$.

Set $\omega = \omega_V$, $\omega' = \omega_{V'}$. Since ω is invertible, we have

$$\sigma^* \omega = \omega' + \Delta$$

for some divisor $\Delta = \sum_{i=1}^{n} a_i E_i$ $(a_i \in \mathbb{Z})$. *We show first that* $\Delta = 0$.

Since $\sigma^* \omega \cdot E_i = 0$ $(1 \leqq i \leqq n)$, the adjunction formula gives

$$(0.1.1) \qquad -\Delta \cdot E_i = \omega' \cdot E_i = 2p_a(E_i) - 2 - E_i \cdot E_i \geqq 0$$

where the last inequality holds because $E_i \cdot E_i < 0$ and $p_a(E_i) \geqq 0$ with equality only if E_i is a smooth rational curve, and because of the preceding condition ii). Hence *either $\Delta = 0$ or $a_i > 0$ for all i.* (This is standard: write $\Delta = \Delta_1 - \Delta_2$ where Δ_1, Δ_2 are effective divisors without common components; then (0.1.1) implies that

$$0 \geqq \Delta \cdot \Delta_2 = \Delta_1 \cdot \Delta_2 - \Delta_2 \cdot \Delta_2 \geqq -\Delta_2 \cdot \Delta_2 \geqq 0$$

and so $\Delta_2 \cdot \Delta_2 = 0$, i.e. $\Delta_2 = 0$ by negative definiteness; thus $a_i \geqq 0$ for all i. Moreover *if some* $a_i > 0$, and E_i meets E_j, then $\Delta \cdot E_j \leqq 0$ implies $a_j > 0$; using the connectedness of $\bigcup E_i$ we see then that *all* $a_i > 0$.)

Now, let P' be the proper transform of P on V'. Then

$$-1 = \sigma^* \omega \cdot P' = \omega' \cdot P' + \Delta \cdot P' = -2 - P' \cdot P' + \Delta \cdot P'$$

(the last equality by adjunction). Since $\pi\sigma(P')$ is a single point, therefore $P' \cdot P' < 0$; and since $\Delta \cdot P' \geqq 0$, we conclude that $P' \cdot P' = -1$, $\Delta \cdot P' = 0$. Since P' meets E_i for some i, therefore $a_i = 0$ and hence $\Delta = 0$, as asserted.

So we have equality in (0. 1. 1), and therefore every E_i is a smooth rational curve with $E_i \cdot E_i = -2$. Thus *each* x_i *is a rational double point*; and to see that it is of type A_n, we need to show that no E_i meets three others. This is done below.

As above, P' is a smooth rational curve with self-intersection -1. Let $\beta: V' \longrightarrow V''$ be the map blowing down P', and set $E_i' = \beta(E_i)$. Then V'' is smooth along $\bigcup E_i'$, and there is a birational map $V'' \longrightarrow A$ taking $\bigcup E_i'$ to the point $\pi(P)$. If μ_i is the multiplicity of the point $\beta(P')$ on E_i', then

$$0 > E_i' \cdot E_i' = E_i \cdot E_i + \mu_i = -2 + \mu_i$$

whence $\mu_i = 0$ or 1, and so E_i' is a smooth rational curve with $E_i' \cdot E_i' = -2$ or -1. Furthermore, there is at least one i for which P' meets E_i (and hence $\mu_i = 1$), and for this i, $E_i' \cdot E_i' = -1$. We can then blow down E_i' and repeat the argument. Continue in this way until there is nothing left to blow down, at which point there is a birational map $\tau: V^{(n+2)} \longrightarrow A$ with $V^{(n+2)}$ smooth and with $\tau^{-1}\pi(P)$ having dimension zero. Then τ is an isomorphism (by Zariski's main theorem), and so $\pi(P)$ *is a smooth point of* A.

Finally, we note that P' could not meet E_i and E_j for $i \neq j$, since then E_i' and E_j' would meet, and both have self-intersection -1, so that when we blow down E_i', the image of E_j' would have self-intersection $\geqq 0$, which is not possible. Thus *the number* r *of singularities of* V *on* P *is at most* 1. Similarly, if $E_i' \cdot E_i' = -1$, then E_i' can meet no more than one E_j' $(j \neq i)$, and blowing down E_i', we see again that this E_j' can meet no more than one E_k' $(k \neq i, k \neq j)$. Continuing in this way we find that *no* E_i *can meet more than two others.* This completes the proof. □

§ 1. Canonical sheaves and blowing up

Let V be an n-dimensional algebraic variety (reduced and irreducible) over a field k, supposed for simplicity to be algebraically closed. Denote by $k(V)$ the field of rational functions on V. We assume throughout that V is *locally a hypersurface*, i.e. for each closed point $v \in V$ if R is the local ring $\mathcal{O}_{V,v}$ of v on V, then R is k-isomorphic to S/fS, where S is a regular local ring of dimension $n+1$. (It is equivalent to say that the maximal ideal of R can be generated by a sequence of $n+1$ elements.)

There is a *canonical* "dualizing sheaf" ω_V, which is a coherent $\mathcal{O}_V$-submodule of the constant sheaf $\Omega^n_{k(V)/k}$ of meromorphic differential n-forms on V. This ω_V is implicit in Grothendieck duality theory, and is explicitly described by Kunz in [K1] and [K2]. All we have to know about ω_V is that *the stalk* $\omega_{V,v}$ *can be described as follows* (cf. e.g. [B], pp. 196—197).

With $R = S/fS$ as above we choose a regular system of parameters $(X_0, X_1, \ldots, X_n)$ in S (i.e. a sequence of $n+1$ elements generating the maximal ideal of S). Then the differentials $dX_0, dX_1, \ldots, dX_n$ form a free basis of the Kähler differential module $\Omega^1_{S/k}$. If f_{X_i} $(0 \leqq i \leqq n)$ is defined by the equation (in $\Omega^1_{S/k}$)

$$df = \sum_{i=0}^{n} f_{X_i} dX_i,$$

and if $x_i \in R$ (respectively $f_{x_i} \in R$) is the natural image of X_i (respectively f_{X_i}), then the n-dimensional $k(V)$-vector space $\Omega^1_{k(V)/k}$ is generated by $(dx_0, dx_1, \ldots, dx_n)$ subject to the single relation

$$\sum_{i=0}^{n} f_{x_i} d_{x_i} = 0.$$

Hence $(dx_0, \ldots, dx_{i-1}, dx_{i+1}, \ldots, dx_n)$ forms a $k(V)$-basis of $\Omega^1_{k(V)/k}$ if and only if $f_{x_i} \neq 0$; and then $\omega_{V,v}$ satisfies

$$\omega_{V,v} = R(dx_0 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n / f_{x_i}) \subset \Omega^n_{k(V)/k}.$$

Thus ω_V is an *invertible* $\mathcal{O}_V$-module. In particular, *if v is a smooth point of V*, then for some i, f_{x_i} is a unit in R, and consequently

$$\omega_{V,v} = \Omega^n_{R/k} \qquad (R = \mathcal{O}_{V,v}).$$

Our first result has to do with the behavior of ω_V under "permissible" blowups.

Proposition (1.1). *With V as above, let $B \subset V$ be a smooth subvariety of codimension $c > 0$, and such that every point of B has the same multiplicity e on V. Let $\pi\colon V' \longrightarrow V$ be the birational map obtained by blowing up B, so that if I is the $\mathcal{O}_V$-ideal of functions vanishing on B then $\mathrm{I}\mathcal{O}_{V'}$ is invertible. Then the image of the natural map*

$$\pi^* \omega_V \longrightarrow \pi^* \Omega^n_{k(V)/k} = \Omega^n_{k(V')/k}$$

is $(\mathrm{I}\mathcal{O}_{V'})^{c-e} \omega_{V'}$.

Proof. The assertion is local, so consider what happens at a closed point $v' \in V'$. If $v' \notin \pi^{-1}(B)$ then π is an isomorphism around v', $\mathrm{I}\mathcal{O}_{V',v'} = \mathcal{O}_{V',v'}$, and the assertion is obvious.

Assume then that $v = \pi(v') \in B$. Set

$$\mathcal{O}_{V,v} = R = S/fS$$

as before. For a suitable regular system of parameters $(X_0, X_1, \ldots, X_n)$ in S, with image $(x_0, x_1, \ldots, x_n)$ in R, we have that $x_0 \neq 0$ and that the stalk I_v is given by

$$\mathrm{I}_v = (x_0, x_1, \ldots, x_c) R;$$

and furthermore, if

$$Y_i = X_i \quad \text{when} \quad i = 0 \text{ or } i > c,$$
$$Y_i = X_i / X_0 \quad \text{when} \quad 1 \leqq i \leqq c$$

then

$$\mathcal{O}_{V',v'} = R' = S'/f'S'$$

where S' is the localization

$$S' = S[Y_1, \ldots, Y_c]_{(Y_0, Y_1, \ldots, Y_n)}$$

and where

$$f' = Y_0^{-e} f.$$

Then $IR' = x_0 R'$, and we need to show that

$$R' \omega_{V,v} = x_0^{c-e} \omega_{V',v'}.$$

Choose i such that $f_{x_i} \neq 0$, but $f_{x_j} = 0$ for all $j > i$. Let $y_j \in R'$ $(0 \leqq j \leqq n)$ be the natural image of $Y_j \in S'$. Then simple calculations give

$$\begin{aligned} f'_{y_i} &= x_0^{-e} f_{x_i} && \text{if} \quad i = 0 \text{ or } i > c, \\ &= x_0^{-e+1} f_{x_i} && \text{if} \quad 1 \leqq i \leqq c. \end{aligned}$$

So if

$$\xi = dx_0 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n / f_{x_i}$$

and

$$\xi' = dy_0 \cdots dy_{i-1}\, dy_{i+1} \cdots dy_n / f_{y_i}$$

are generators of $\omega_{V,v}$ and $\omega_{V',v'}$ respectively (see above) then, since

$$\begin{aligned} dy_j &= dx_j && \text{if} \quad j = 0 \text{ or } j > c \\ &= x_0^{-2}(x_0\, dx_j - x_j\, dx_0) && \text{if} \quad 1 \leqq j \leqq c \end{aligned}$$

we find easily, when $i > 0$, that

$$\xi = x_0^{c-e} \xi',$$

and the conclusion follows.

In case $i = 0$, then $f_{x_0} \neq 0$ and $f_{x_j} = 0$ for $j > 0$; and since

$$\sum_{j=0}^{n} f_{x_j}\, dx_j = 0$$

therefore $dx_0 = 0$, and we find again that $\xi = x_0^{c-e} \xi'$. □

Corollary (1.2). *Let $B \subset V$, c, e, be as in (1.1), with $c \geqq 2$ and $e \geqq 2$. Let $B \subset P$, where P is a smooth codimension one subvariety of V; and let $P' \subset V'$ be the proper transform of P, so that the restriction $\pi_P \colon P' \to P$ of π can be identified with the blowup of $B \subset P$. Let $\alpha \colon P \to V$ be the inclusion, let $\omega_{V|P}$ be the invertible $\mathcal{O}_P$-module $\alpha^*(\omega_V)$, and set*

$$\nu_P = \omega_P \otimes (\omega_{V|P})^{-1}.$$

Define $\nu_{P'}$ similarly. Then

$$H^0(P', \nu_{P'} \otimes \pi_P^*(\nu_P^{-1})) \neq 0.$$

Proof. Let I be the $\mathcal{O}_V$-ideal defining B. Then, by (1.1),

$$\omega_{V'}^{-1} \otimes \pi^*(\omega_V) \cong (I\mathcal{O}_{V'})^{c-e}$$

and similarly

$$\omega_{P'}^{-1} \otimes \pi^*(\omega_P) \cong (I\mathcal{O}_{P'})^{(c-1)-1}.$$

Hence

$$\nu_{P'} \otimes \pi_P^*(\nu_P^{-1}) \cong (I\mathcal{O}_{P'})^{2-e} \supseteq \mathcal{O}_{P'},$$

and (1.2) follows.

Proposition (1.3). *Assume that k has characteristic zero. Let P be a smooth connected codimension one subvariety of V, containing at least one smooth point of V. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} V_r & \to & V_{r-1} & \to \cdots \to & V_0 & \subset & V \\ \cup & & \cup & & \cup & & \cup \\ P_r & \to & P_{r-1} & \to \cdots \to & P_0 & = & P \end{array}$$

such that:

a) *V_0 is an open neighborhood of P in V;*

b) *for $0 \leqq j < r$, $V_{j+1} \to V_j$ is obtained by blowing up $B_j \subset P_j$, with B_j smooth, and all points of B_j having the same multiplicity $\geqq 2$ on V_j; and P_{j+1} is the proper transform of P_j;*

c) *V_r is smooth;*

d) *if $\sigma_P: P_r \to P$ is the composed map in the diagram, and ν_{P_j} is defined as in* (1.2), *then*

$$H^0(P_r, \nu_{P_r} \otimes \sigma_P^*(\nu_P^{-1})) \neq 0.$$

Proof. Everything but d) follows from Main Theorem I of [H], p. 174 (with $\beta = 0$). (Note that since $P_r \nsubseteq \operatorname{Sing}(V_r)$, therefore V_r smooth in a neighborhood of $P_r \Leftrightarrow V_r$ is normally flat along P_r.) Then d) follows by induction from (1.2).

Remark (1.4). (Not used elsewhere.) Let us say that V is "strongly smooth in codimension e" if there exists a desingularization $\rho: V^* \to V$ of V obtained as a succession of blowups as in (1.1), where the blown up subvarieties always have codimension $> e$. If, in addition, all points of V have multiplicity $\leqq e+1$, then it follows from (1.1) that *the canonical inclusion*

$$g_* \omega_{V^*} \to \omega_V$$

is surjective. In characteristic zero, this means that *the resolution ρ is "rational"* [K], p. 50.

For example, if $e=1$, and if all the singularities of V are double points with reduced Zariski tangent space, then it is not hard to check (at least if k has characteristic $\neq 2$) that V is strongly smooth in codimension 1 (i.e. "strongly normal"); so if the characteristic is zero, then all the singularities of V are rational. (Cf. [T] for examples and references on rational singularities.)

§ 2. The main result

Let k be an algebraically closed field of characteristic zero. We consider, as in § 1, a reduced irreducible algebraic k-variety V of dimension $n \geqq 2$, which is locally a hypersurface; and a codimension one subvariety P which is not entirely contained in the singular locus $\operatorname{Sing}(V)$. We assume further that P is isomorphic, as a k-variety, to the projective space $\mathbb{P}_k^{n-1}$. Then the "normal bundle"

$$\nu_P = \omega_P \otimes (\omega_{V|P})^{-1}$$

(cf. (1.2)) is isomorphic to $\mathcal{O}_P(m)$ for some integer m.

Theorem (2.1). *With preceding notation, assume that P "blows down birationally", i.e. there exists a birational map $\varphi: V^* \to A$ where V^* is an open neighborhood of P in V, and A is an affine k-variety (whence $\varphi(P)$ is a single point). Then, with $\nu_P = \mathcal{O}_P(m)$ as above, we have $m<0$; and if $m=-1$ then $P \cap \mathrm{Sing}(V)$ is either empty or of pure dimension $n-2$.*

Proof. We may assume that $V=V^*$, and choose a desingularization

$$\sigma: V' = V_r \to V_0 \subset V$$

as in (1.3), inducing

$$\sigma_P: P' = P_r \to P.$$

Let $p \in P$, and let $h: A \to k$ be any non-zero regular function such that $h \circ \varphi$ vanishes along some subvariety $W \subset V$ with $p \in W \nsubseteq P$. Let $L \subset P = \mathbb{P}_k^{n-1}$ be a line through p such that L is not contained in the intersection of P with any other component of the subvariety of V where $h \circ \varphi$ vanishes, and such that L is not contained in $P \cap \mathrm{Sing}(V)$; and let L' be the *proper transform* of L on P', i.e. L' is the closure of $\sigma_P^{-1}(L - \mathrm{Sing}(V))$. Write the divisor of zeros of the function $h \circ \varphi \circ \sigma$ on V' as

$$(h)_{V'} = a_0 P' + \sum_{i=1}^{t} a_i E_i + H$$

where $E_1, \ldots, E_t$ are all the irreducible components of $\sigma^{-1}(P \cap \mathrm{Sing}(V))$, and where neither P' nor any E_i is a component of H. Note that

$$a_i > 0 \qquad (i \geqq 0)$$

because

$$(h \circ \varphi \circ \sigma)(\sigma^{-1} P) = h(\varphi(P)) = 0.$$

For any divisor D on V', denote by $(D \cdot L')$ the degree (= Chern class) of the invertible sheaf $\mathcal{O}_{V'}(D) \otimes \mathcal{O}_{L'}$ on L'. It is clear by the choice of L that L' meets H in at most finitely many points, whence $(H \cdot L') \geqq 0$. If $p \notin \mathrm{Sing}(V)$, then σ is an isomorphism over a neighborhood of p, and by our choice of h, some component of H passes through p, so that $(H \cdot L') > 0$.

Similarly $(E_i \cdot L') \geqq 0$ for $1 \leqq i \leqq t$. Moreover if p is singular on V then L' must certainly meet $\bigcup E_i$ since $p \in L = \sigma_P(L')$ and $\sigma_P^{-1}(p) \subset P' \cap (\bigcup E_i)$ [because $\sigma(\bigcup E_i) = P \cap \mathrm{Sing}(V)$, and σ induces an isomorphism outside $\bigcup E_i$]; thus $(E_i \cdot L') > 0$ for some i.

In any case, we must have

$$\sum_{i=1}^{t} a_i (E_i \cdot L') + (H \cdot L') > 0.$$

As for $(P' \cdot L')$, recall that $\omega_{V'} \cong \Omega_{V'}^n$, $\omega_{P'} \cong \Omega_{P'}^{n-1}$, so that the "adjunction formula" gives

$$\nu_{P'} \cong \mathcal{O}_{V'}(P') \otimes \mathcal{O}_{P'};$$

and since the restriction of $v_P \cong \mathcal{O}_P(m)$ to L has degree m, therefore the restriction of $\sigma_P^*(v_P)$ to L' also has degree m; so we conclude from (1.3) d) that if L is in sufficiently general position[1]) then the restriction of $v_{P'}$ to L' has degree $\geqq m$, i.e.

$$(2.1.1) \qquad (P' \cdot L') \geqq m.$$

It follows now that

$$(2.1.2) \qquad 0 = ((h)_{V'} \cdot L') = a_0 (P' \cdot L') + \sum_{i=1}^{t} a_i (E_i \cdot L') + (H \cdot L')$$
$$> a_0 (P' \cdot L') \geqq a_0 m.$$

Thus $m < 0$.

Suppose next that $m = -1$, and that $p \in P \cap \operatorname{Sing}(V)$. As we saw above, there is an i such that $(E_i \cdot L') > 0$, so that (cf. (2.1.1), (2.1.2)):

$$0 \geqq a_0 (P' \cdot L') + a_i \geqq -a_0 + a_i$$

i.e. $a_0 \geqq a_i$.

Now from Lemma (2.2) below (as applied to the local ring $R = \mathcal{O}_{V,p}$, the prime ideal $\mathfrak{p} \subset R$ corresponding to P, and the valuations v, $v_\mathfrak{p}$ corresponding respectively to E_i and P') it follows that *if* $P \cap \operatorname{Sing}(V)$ *has dimension* $< n-2$ *near* $\mathfrak{p}$, then also $a_i \geqq a_0$, so that $a_i = a_0$. This leads to a *contradiction* (whence $P \cap \operatorname{Sing}(V)$ must be empty!), as follows. Choose a rational function ρ on V' whose divisor of zeros has P', but no E_i, as a component. (The existence of such a ρ follows easily e.g. from [Bo], p. 134, Cor. 2.) We can write

$$\rho = (h_1 \circ \varphi \circ \sigma)/(h_2 \circ \varphi \circ \sigma)$$

where h_1 and h_2 are regular functions on A, which we can assume (after multiplying both by a suitable function) are non-constant and vanishing at $\varphi(P)$. Choose a line $L \subset P$ as above, with h replaced by the product $h_1 h_2$. Then as before, there is an i such that $(E_i \cdot L') > 0$ and $h_1 \circ \varphi \circ \sigma$ vanishes to the same order along P' and E_i as does $h_2 \circ \varphi \circ \sigma$. But this is impossible, since ρ vanishes everywhere on P' but not everywhere on E_i.

To complete the proof, then, we need the following lemma.

Recall that for any integer s, the symbolic power $\mathfrak{p}^{(s)}$ of a prime ideal $\mathfrak{p}$ in an integral domain R is defined by

$$\mathfrak{p}^{(s)} = \mathfrak{p}^s R_\mathfrak{p} \cap R = \{x \in R \mid \exists y \in R,\ y \notin \mathfrak{p},\ xy \in \mathfrak{p}^s\}.$$

Thus, when $R_\mathfrak{p}$ is a discrete valuation ring, with valuation $v_\mathfrak{p}$, then

$$\mathfrak{p}^{(s)} = \{x \in R \mid v_\mathfrak{p}(x) \geqq s\}.$$

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[1]) Let D be an effective divisor on P' such that $\mathcal{O}_{P'}(D) \cong v_{P'} \otimes \sigma_P^{-1}(v_P)$; and choose L as before, but also not contained in $\sigma_P(D)$, so that $L' \not\subset D$ and $(D \cdot L') \geqq 0$.

Lemma (2.2). *Let S be a regular local ring, f an irreducible non-unit in S, and $R=S/fS$. Let $\mathfrak{p}$ be a prime ideal in R such that the local ring $R/\mathfrak{p}$ is regular and the localization $R_{\mathfrak{p}}$ is a discrete valuation ring, and such that if $\mathfrak{q}\supset\mathfrak{p}$ is a prime ideal with* $\dim R_{\mathfrak{q}}=2$ *then the local ring $R_{\mathfrak{q}}$ is regular. Then for all $s>0$ we have*

$$\mathfrak{p}^{(s)}=\mathfrak{p}^{s}.$$

Hence if v is any discrete rank one valuation of the fraction field of R such that $v(h)\geqq 1$ for all $h\in\mathfrak{p}$, then for every such h we have

$$v(h)\geqq v_{\mathfrak{p}}(h).$$

Proof. For the last assertion, note that if $v_{\mathfrak{p}}(h)=s$, then $h\in\mathfrak{p}^{(s)}=\mathfrak{p}^{s}$, whence $v(h)\geqq s$.

The equality $\mathfrak{p}^{(s)}=\mathfrak{p}^{s}$ means that if $y\notin\mathfrak{p}$ and $xy\in\mathfrak{p}^{s}$, then $x\in\mathfrak{p}^{s}$; in other words, the natural image of y in $R/\mathfrak{p}$ is not a zero-divisor in the graded ring

$$G=\bigoplus_{s\geqq 0}\mathfrak{p}^{s}/\mathfrak{p}^{s+1}.$$

So it will be more than enough to show that *G is an integral domain.*

Let $\mathfrak{P}$ be the inverse image of $\mathfrak{p}$ in S. Then we have a natural surjective homomorphism of graded rings

$$\gamma\colon\Gamma=\bigoplus_{s\geqq 0}\mathfrak{P}^{s}/\mathfrak{P}^{s+1}\rightarrow G.$$

Since $S/\mathfrak{P}=R/\mathfrak{p}$ is regular, and $\dim R_{\mathfrak{p}}=1$, it follows that $\mathfrak{P}$ is generated by two elements, and that Γ is isomorphic to a polynomial ring in two variables over $S/\mathfrak{P}$:

$$\Gamma\cong(S/\mathfrak{P})\,[X,Y].$$

Since $R_{\mathfrak{p}}=S_{\mathfrak{P}}/fS_{\mathfrak{P}}$ is regular, we see that $f\in\mathfrak{P}$, $f\notin\mathfrak{P}^{2}$; and the kernel of γ is generated by the image $\bar{f}$ of f in $\mathfrak{P}/\mathfrak{P}^{2}$, which is an element of the form

$$\bar{f}=\alpha X+\beta Y\qquad(\alpha,\beta\in S/\mathfrak{P}).$$

We need then to show that $\bar{f}$ is irreducible, i.e. that *α and β are relatively prime in $S/\mathfrak{P}$* (which, being regular, is a unique factorization domain).

Suppose that α and β are not relatively prime, so that both are contained in a height one prime ideal of $S/\mathfrak{P}$. If $\mathfrak{Q}$ is the inverse image in S of this prime ideal, then clearly $f\in\mathfrak{Q}^{2}$. So if $\mathfrak{q}=\mathfrak{Q}/fS$, a prime ideal in R containing $\mathfrak{p}$, then $\dim R_{\mathfrak{q}}=2$ and

$$R_{\mathfrak{q}}=S_{\mathfrak{Q}}/fS_{\mathfrak{Q}}$$

is not regular, contradicting our assumptions.

Thus α and β are relatively prime, and the proof is complete.

The following is the main form in which we will make use of Theorem (2. 1).

Corollary (2. 3). *Let V be a three-dimensional irreducible normal Gorenstein variety over an algebraically closed field, k, of characteristic zero. Let $P \subsetneqq V$ be a subvariety which is k-isomorphic to the projective plane $\mathbb{P}_k^2$, and which meets the singular set* $\mathrm{Sing}(V)$ *in at most finitely many points. Assume that $\nu_P \approx \mathcal{O}_P(-1)$, (cf. (2. 1)) and that there exists a map $\varphi: V^* \to A$ as in* (2. 1) *such that furthermore φ induces an isomorphism $V^* - P \approx A - \varphi(P)$. Assume also that there is an ample line bundle H on V spanned by its global sections and such that $H|_P \approx \mathcal{O}_P(1)$. Then $P \cap \mathrm{Sing}(V)$ is empty.*

Proof. By (2. 1) it is enough to show that V is locally a hypersurface at any $x \in P \cap \mathrm{Sing}(V)$.

Let $\Psi: V \to \mathbb{P}^N$ be the map defined by the linear system $|H|$. Then the restriction of H to the fibre $\Psi^{-1}\Psi(x)$ is both ample and trivial, whence $\Psi^{-1}\Psi(x)$ is a finite set, i.e. the linear system $|H - x|$ of divisors in H containing x has only a finite set of base points (viz. $\Psi^{-1}\Psi(x)$). By Bertini, a generic $D \in |H - x|$ has only isolated singularities; and moreover the Cartier divisor D is Gorenstein (since V is), and therefore normal.

Now $D \cap P$ is a line L containing x; and since $\nu_P \approx \mathcal{O}_P(-1)$, therefore

$$\mathcal{O}_L(-1) = \nu_{P|L} \approx \omega_{P|L} \otimes (\omega_{V|L})^{-1} = \mathcal{O}_L(-3) \otimes (\omega_{V|L})^{-1}$$

whence $\omega_{V|L} \approx \mathcal{O}_L(-2)$, and

$$\begin{aligned} \omega_{D|L} &\approx (\omega_{V|D})|_L \otimes (H|_D)|_L \approx \omega_{V|L} \otimes (H|_P)|_L \\ &\approx \mathcal{O}_L(-2) \otimes \mathcal{O}_L(1) = \mathcal{O}_L(-1). \end{aligned}$$

By Theorem (0. 1) (applied to $L \subsetneqq D$ and the map π induced by φ from D onto the normalization of $\varphi(D)$), x is an A_n singularity of D, and so D is a hypersurface locally at x. Since the divisor D is locally principal, therefore V is a hypersurface at x. □

Remark (2. 4). It is possible (but tedious) to remove the characteristic zero assumption in (2. 3) by explicitly constructing a resolution of singularities as in (1. 3), starting with the fact that D has an A_n singularity at x.

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