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On Complete Ideals in Regular Local Rings

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Introduction.

We present here an approach to unique factorization of complete (=integrally closed) ideals in two-dimensional regular local rings, based on a decomposition theorem ((2.5) below) which is valid in all dimensions.

The theory of complete ideals in two-dimensional regular local rings was founded by Zariski [Z], [ZS_2, appendix 5], and furthur developed in [Ho], [D], [L], and [G]. Recent interest in this subject (and related ones) has been shown in [R], [Sp], [C] and [Hy].

Zariski's work was motivated by the birational theory of linear systems on smooth surfaces [Z′, Chapter 2]. Roughly speaking, the monoid of complete ideals in a normal noetherian ring R (with product $I * J = \{\text{completion of } IJ\}$) generates the group of locally principal divisors on the Zariski-Riemann space of R; these divisors can also be interpreted more concretely as $\varinjlim$ of divisors on schemes birational over R, and thereby one connects to classical situations involving divisors and linear systems on such schemes (cf. [Z′, appendix to ch.2] for more details).

Zariski first raised the question of higher-dimensional generalizations in [Z, p.152], but not much has happened in this respect during the intervering fifty years. Perhaps the lack of progress is understandable in view of the complexity of birational geometry in dimension ≥ 3. It *is* possible to extend the *definitions* in [Z′, appendix to Ch.2] to higher dimension (cf. [W],[Sn],[ZS_2, pp.356–361], and §1 below) but *not the main results* in [*ibid*, §4]. In itself what this yields is little more than a convenient language for discussing linear systems with base

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conditions, a theory in search of theorems. Zariski himself eventually concluded [ZS_2, p.362]: "It is almost certain that the theory ... cannot be generalized to higher dimension without substantial modifications both of statements and proofs."

Indeed, whereas $I * J = IJ$ in dimension two (i.e. the product of complete ideals is still complete), counterexamples to such a statement in three-dimensional regular local rings have recently been given by Huneke[Hu, §3]. And the main theorem in dimension two, on *unique factorization* of complete ideals into simple complete ideals, also breaks down in higher dimension. The first counterexample is due to Cutkosky [C]. Huneke and I subsequently found the following counterexample, in a power series ring $k[[x,y,z]]$, k a field:

$$(0.1) \qquad (x,y,z)(x^3,y^3,z^3,xy,yz,xz) = (x^2,y,z)(x,y^2,z)(x,y,z^2).$$

The ideals appearing here are all complete and $*$-simple (cf. beginning of §2), and the product ideal on each side is also complete.

In spite of these discouraging developments, the main results in §2 below may offer a scintilla of hope that some substantial generalizations are not entirely out of reach. What we do in §2, after setting up the foundations in §1, is to associate special $*$-simple complete ideals to certain "infinitely near points"; and then show that any "finitely supported" complete ideal admits a "unique factorization" into special $*$-simple complete ideals, with possibly *negative* exponents. (Precise statements are given in (2.1) and (2.5).) Thus, in (0.1), the simple complete ideals $(x^2,y,z),(x,y^2,z),(x,y,z^2)$ are all special (associated to the infinitely near points (= quadratic transforms) in the directons $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ respectively), and they all appear in the "factorization" of (x^3,y^3,z^3,xy,yx,xz) with exponent $+1$, while the maximal ideal (x,y,z), which is also special (associated to $k[[x,y,z]]$ itself), may be said to appear with exponent -1.

Unique factorization in dimension two is derived from Theorem (2.5) in §4, via some results in §3. The point is that in dimension two there are no negative exponents.

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§1. Point bases and completions of ideals in regular local rings.

This section is based on ideas going back to [Z]. The main results are Proposition (1.10) and its elaborations Propositions (1.16) and (1.23).

If R is a noetherian local domain, with maximal ideal $\mathfrak{m}$ and fraction field K, then a *prime divisor* of R is a valuation v of K whose valuation ring R_v dominates R (i.e. $R \subset R_v$ and $\mathfrak{m} \subset \mathfrak{m}_v$, the maximal ideal of R_v), and such that the transcendence degree of the field $R_v/\mathfrak{m}_v$ over R/m is as large as possible, viz. $\dim R - 1$; such a v must be a discrete rank one valuation [A, p.330, Thm.1].

Let I be an ideal in any commutative ring R. An element $x \in R$ is *integral over I* if x satisfies a condition of the form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \qquad (a_j \in I^j, 1 \le j \le n),$$

i.e. if for some $n > 0$, $x^n \in I(I + xR)^{n-1}$. The set of all such x, denoted $\overline{I}$, is called the *integral closure*, or *completion*, of I. The completion $\overline{I}$ is itself an ideal, and we have $I \subset \overline{I} = \overline{\overline{I}}$. [1] I is *integrally closed*, or *complete*, if $I = \overline{I}$.

Proposition (1.1). *Let R be an integral domain with fraction field K, and let I be an ideal in R, with completion $\overline{I}$. Then for any valuation v of K whose valuation ring R_v contains R, we have $\overline{I}R_v = IR_v$. Conversely, if R is noetherian, local, and universally catenary* [EGA IV, (5.6.2)] *(for example R regular* [*ibid.* (5.6.4)]) *then for every $x \notin \overline{I}$ there exists a prime divisor v of R such that $x \notin IR_v$.*

Proof: For the first assertion, cf. [ZS_2, p.350, proof of Thm.1]. For the second, $\mathfrak{m}$ being the maximal ideal of R, it will suffice to produce a finitely generated R-subalgebra S of K and a height one prime ideal p in S such that $\mathfrak{m}+x^{-1}I \subset p$; for then, since R is universally catenary, the field S_p/pS_p will have transcendence degree $\dim R - 1$ over its subfield $R/\mathfrak{m}$, and so any valuation ring R_v in K dominating S_p (i.e. any localization at a maximal ideal of the integral closure of S_p in K, cf. [B, §2, no.5, Cor.2]) will give a prime divisor v with $x^{-1}I \subset \mathfrak{m}_v$, i.e. $x \notin IR_v$.

Actually we need not assume that the prime ideal p has height one, since this can always be arranged by blowing up (argue as in [ZS_2, p.96]). So consider the ring

$$S = R[x^{-1}I] = \bigcup_{n\ge 0} (I + xR)^n/x^n$$

and note that $p = \mathfrak{m} + x^{-1}IS$ is an ideal in S such that S/p is a homomorphic image of $R/\mathfrak{m}$. Moreover $1 \notin p$, since otherwise there would be a relation of the form

$$1 + a \in I(I + xR)^{n-1}/x^n \qquad a \in \mathfrak{m}, \quad n > 0,$$

and multiplying this by $x^n(1 + a)^{-1}$ would yield $x \in \overline{I}$, a contradiction. Thus p is a maximal ideal in S, and we are done.

* * *

Next, after a preparatory lemma, we describe the *transform* of an ideal (Definition (1.4), Proposition (1.5)).

[1]These assertions follow e.g. from the fact that $x \in \overline{I}$ if and only if, in the polynomial ring $R[T]$, the element xT is integral over the graded subring $R[IT]$.

Lemma (1.2). *Let R be an integral domain with fraction field K, and let p be a prime ideal in R such that the localization R_p is a discrete valuation ring (d.v.r.). Let S be a ring with $R \subset S \subset K$, and set $S_p = S \otimes_R R_p \subset K$ (i.e. S_p is the ring of fractions $S[M^{-1}]$, $M = R - p$). Let*

$$p^S = pS_p \cap S.$$

Then the following are equivalent:

(i) $p^S \neq S$.

(ii) $q = p^S$ *is the unique prime ideal in* S *whose intersection with* R *is* p; *and* $S_q = R_p$.

(iii) *There exists an ideal* J *in* S *such that* $J \cap R = p$.

(iii)′ $pS \cap R = p$.

(iv) $S_p \neq K$.

(v) $S_p = R_p$.

(v)′ $S \subset R_p$.

Proof: The implications (ii) ⇒ (i) ⇒ (iv), (ii) ⇒ (iii) ⇔ (iii)′, and (ii) ⇒ (v)′ ⇔ (v) are all trivial. (iii) ⇒ (iv) is immediate (since $J \cap R = p$ implies $0 \neq JS_p \neq S_p$), as is (iv) ⇒ (v) (since $R_p \subset S_p \subset K$ and R_p is a d.v.r).

It remains to prove that (v) ⇒ (ii). If $S_p = R_p$, then $q = p^S = pR_p \cap S$ is a prime ideal in S, and $q \cap R = pR_p \cap R = p$; and furthermore $R_p \subset S_q \neq K$, so that R_p being a d.v.r. we must have $R_p = S_q$. Similarly for *any* prime ideal q' in S such that $q' \cap R = p$, we have $S_{q'} = R_p (= S_q)$; and therefore $q' = q$. q.e.d.

Suppose now that R is a *unique factorization domain* (UFD) with fraction field K. For any non-zero ideal I in R, let x be a greatest common divisor of the elements in I (i.e., among the principal ideals containing I, xR is the smallest) and set

$$I^{-1} = x^{-1}R = \{z \in K \mid zI \subset R\}.$$

Then II^{-1} is the unique ideal J in R such that:

(i) $J^{-1} = R$ (i.e., R itself is the only principal ideal containing J), and

(ii) $I = yJ$ for some y in R.

Thus: *every non-zero ideal I in R is uniquely of the form*

$$I = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} J \qquad (a_i > 0) \tag{1.3}$$

where the p_i are principal prime ideals, the a_i are (strictly) positive integers, and J is an ideal with $J^{-1} = R$.

Definition (1.4). *Let R be a UFD with fraction field K, and let S be a UFD with $R \subset S \subset K$. Let $I \neq (0)$ be an ideal in R, factor I as in (1.3), and* *for each* $i = 1, 2, \ldots, n$, *set* $q_i = p_i^S$ *(cf.* (1.2)*). We define the transform of* I *in* S *to be the ideal*

$$I^S = q_1^{a_1} \cdots q_n^{a_n} (JS)(JS)^{-1}.$$

In particular, if I *is a principal prime ideal then* I^S *is the same here as in* (1.2).

Some basic properties of the "transform" operation are given in the next Proposition.

Proposition (1.5). *Let* $R \subset S \subset K$ *be as in* (1.4), *let* I, I_1, I_2 *be non-zero ideals in* R *and let* T *be a UFD with* $S \subset T \subset K$.

(i) *If* I *is a principal prime ideal, then either* $IS \cap R = I$, *in which case* I^S *is a principal prime ideal with* $I^S \cap R = I$; *or* $IS \cap R \neq I$, *in which case* $I^S = S$.

(ii) *If* $I^{-1} = R$, *then* $I^S = (IS)(IS)^{-1}$ *(so that* $(I^S)^{-1} = S$*).*

(iii) *(Compatibility of transform with products.)* $(I_1 I_2)^S = I_1^S I_2^S$.

(iv) *(Transitivity of transform.)* $(I^S)^T = I^T$.

(v) *(Localization.) If* S *is a ring of fractions of* R, *then* $I^S = IS$.

(vi) *(Compatibility with integral dependence.) If* $I_2 \subset I_1 \subset \overline{I_2}$ *(the completion of* I_2*) then* $I_2^S \subset I_1^S \subset \overline{(I_2^S)}$.

Proof: (i)–(v) are left as an exercise. As for (vi), we first check, by applying the first assertion in (1.1) to the discrete valuation rings obtained by localizing R at height one primes p, that $I_1^{-1} = I_2^{-1}$ (note that $I_1^{-1} = J_1^{-1}$, where $J_1 = \bigcap_p I_1 R_p$). Hence for suitable $x \in R$ and with $L_i = I_i I_i^{-1}$ $(i = 1, 2)$ we have

$$I_1 = xL_1, \quad I_2 = xL_2, \quad L_1^{-1} = L_2^{-1} = R, \quad L_2 \subset L_1 \subset \overline{L_2} = x^{-1}\overline{I_2};$$

and if $y \in S$ is such that $(xR)^S = yS$, (cf. (i)) then (by (iii))

$$I_1^S = yL_1^S, \qquad I_2^S = yL_2^S. \tag{1.5.1}$$

Now since $L_1 \subset \overline{L_2}$, therefore every $z \in L_1$ is integral over $L_2 S$, and hence, since $\overline{L_2^S}$ is an S-ideal, we have

$$L_2 S \subset L_1 S \subset \overline{L_2 S}.$$

As above, then, $(L_1 S)^{-1} = (L_2 S)^{-1}$, and (by (ii))

$$L_2^S = (L_2 S)(L_2 S)^{-1} \subset (L_1 S)(L_1 S)^{-1} = L_1^S \subset \overline{L_2^S}. \tag{1.5.2}$$

The conclusion follows from (1.5.1) and (1.5.2).

* * *

We come now to "infinitely near points" (Definition (1.6 below) and their relation to prime divisors (Proposition (1.7)).

Let K be a field. We denote by Greek letters $\alpha, \beta, \gamma, \ldots$ regular local rings of Krull dimension ≥ 2, with fraction field K; and refer to such objects as "points". For any point α, let $\mathfrak{m}_\alpha$ be its maximal ideal, and $\operatorname{ord}_\alpha$ the corresponding order valuation, i.e., the unique (discrete, rank one) valuation of K such that for $0 \neq x \in \alpha$,

$$\operatorname{ord}_\alpha(x) = \max\{\, n \mid x \in \mathfrak{m}_\alpha^n \,\}.$$

Recall that a *quadratic transform* of a point α is a local ring of the form $Q = (\alpha[x^{-1}\mathfrak{m}_\alpha])_p$ where $x \in \mathfrak{m}_\alpha, x \notin \mathfrak{m}_\alpha^2$, and p is a prime ideal in the ring $\alpha[x^{-1}\mathfrak{m}_\alpha]$ such that $\mathfrak{m}_\alpha \subset p$. Such a Q is necessarily regular (hence a point if its dimension $\dim Q$ is ≥ 2), and the residue field $Q/\mathfrak{m}_Q$ has transcendence degree $\dim \alpha - \dim Q$ over $\alpha/\mathfrak{m}_\alpha$ (cf. [A, p.334, Lemma 10], or [EGA IV, (5.6.4)]). There is a unique one-dimensional quadratic transform of α, namely the valuation ring of $\operatorname{ord}_\alpha$.

Definition (1.6). *A point β is* infinitely near *to α, $\beta \succ \alpha$ (or $\alpha \prec \beta$) in symbol, if there exists a sequence*

$$\alpha = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n = \beta \quad (n \geq 0)$$

such that for each $i = 0, 1, \ldots, n-1$, α_{i+1} is a quadratic transform of α_i. Such a sequence, if it exists, is unique; we call it the quadratic sequence from α to β.

In case $\dim \alpha = 2$, the factorization theorem of Zariski and Abhyankar states that *any* β containing α is infinitely near to α (cf. [A, p.343, Thm.3]. [2] No such statement holds when $\dim \alpha > 2$ (e.g. [Sa]).

If $\alpha \prec \beta$, then ord_β is a prime divisor of α (by the above remarks on quadratic transforms). In fact, *every* prime divisor v of α is of the form ord_β, where β is found as follows: let $\alpha_0 = \alpha$, and having defined α_i for some $i \geq 0$, let α_{i+1} be the unique quadratic transform of α_i dominated by (the valuation ring of) v, unless $v = \operatorname{ord}_{\alpha_i}$ in which case set $\beta = \alpha_i$ and stop; then this process must terminate after a finite number of steps (cf. [A, p.336, Prop.3]); thus β is the largest point infinitely near to α and dominated by v. In summary (with remaining details left to the reader):

Proposition (1.7). *The map of sets*

$$\{\textit{points infinitely near to } \alpha\} \longrightarrow \{\textit{prime divisors of } \alpha\}$$

which takes β to ord_β is a bijection.

* * *

[2] A similar result holds if we assume only that α and β are two-dimensional local rings with fraction field K, such that α is rational and β is factorial (cf. [L, p.203, Prop.(3.1)] and [He, Thm.1]).

We are now almost ready to state the first main result (1.10) of this section.

Let $I \neq (0)$ be an ideal in a point α. [3] I is finitely generated, so for any valuation v of K whose valuation ring contains α, we can set

$$v(I) = \min\{v(z) | z \in I\}.$$

If $\beta \succ \alpha$ is an infinitely near point, then, since α and β are both regular, hence UFD's, the transform I^β can be defined as in (1.4). In particular, when $\beta = \alpha$, then $I^\beta = I$.

Definition (1.8). *The* point basis *of a non-zero ideal $I \subset \alpha$ is the family of non-negative integers*

$$\mathbf{B}(I) = \{\mathrm{ord}_\beta(I^\beta)\}_{\beta \succ \alpha}.$$

A base point *of I is a point $\beta \succ \alpha$ such that* $\mathrm{ord}_\beta(I^\beta) \neq 0$ *(i.e., $I^\beta \neq \beta$).*

Remarks (1.9). (a): For any $\beta \succ \alpha$, the point basis $\mathbf{B}(I^\beta)$ is obtained by restricting $\mathbf{B}(I)$ to the set of $\gamma \succ \beta$ (because $(I^\beta)^\gamma = I^\gamma$, by (1.5)(iv)).

(b): For two non-zero ideals I, J in α, (1.5)(iii) gives:

$$\mathbf{B}(IJ) = \mathbf{B}(I) + \mathbf{B}(J).$$

Proposition (1.10). *Two non-zero ideals I, J in α have the same point basis if and only if their integral closures are equal:*

$$\mathbf{B}(I) = \mathbf{B}(J) \Leftrightarrow \overline{I} = \overline{J}.$$

Proof: [4] For any $\beta \succ \alpha$, let R_β be the valuation ring of ord_β. It follows from (1.1) and (1.7) that

$$\begin{aligned} \{\overline{I} = \overline{J}\} \quad &\Leftrightarrow \quad \{IR_\beta = JR_\beta \text{ for all } \beta \succ \alpha\} \\ &\Leftrightarrow \quad \{\mathrm{ord}_\beta(I) = \mathrm{ord}_\beta(J) \text{ for all } \beta \succ \alpha\}. \end{aligned}$$

The question is whether this last condition is equivalent to:

$$\mathrm{ord}_\beta(I^\beta) = \mathrm{ord}_\beta(J^\beta) \text{ for all } \beta \succ \alpha.$$

An affirmative answer can be deduced from the following useful fact:

Lemma (1.11). *Let $\beta \succ \alpha$, and let*

$$\alpha = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n = \beta$$

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[3] Note that α, being integrally closed in K, is determined by I: $\alpha = \{z \in K | zI \subset I\}$.

[4] The implication $\overline{I} = \overline{J} \Rightarrow \mathbf{B}(I) = \mathbf{B}(J)$ also follows from (1.5)(vi) and (1.1).

be the quadratic sequence from α *to* β *(cf.* (1.6)). *Let* L *be a non-zero ideal in* α. *For* $0 \le j \le n$ *set*

$$\begin{aligned} \mathfrak{m}_j &= \text{maximal ideal of } \alpha_j \\ \operatorname{ord}_j &= \operatorname{ord}_{\alpha_j} \\ L_j &= L^{\alpha_j}. \end{aligned}$$

Then for any valuation v *whose valuation ring contains* β, *we have*

$$v(L) = v(L_n) + \sum_{j=0}^{n-1} \operatorname{ord}_j(L_j) v(\mathfrak{m}_j).$$

Indeed, taking $v = \operatorname{ord}_\beta$, we see from (1.11) that if $\mathbf{B}(I) = \mathbf{B}(J)$ then $\operatorname{ord}_\beta(I) = \operatorname{ord}_\beta(J)$, whence, as above, $\overline{I} = \overline{J}$.

Conversely, if $\overline{I} = \overline{J}$, so that $\operatorname{ord}_j(I) = \operatorname{ord}_j(J)$ for $0 \le j \le n$, then we find by induction on n that $\operatorname{ord}_n(I_n) = \operatorname{ord}_n(J_n)$, i.e., $\operatorname{ord}_\beta(I^\beta) = \operatorname{ord}_\beta(J^\beta)$; and thus $\mathbf{B}(I) = \mathbf{B}(J)$.

Proof of (1.11): Proceeding by induction on n, we need only show that

$$v(L_{n-1}) = v(L_n) + \operatorname{ord}_{n-1}(L_{n-1}) v(\mathfrak{m}_{n-1})$$

i.e., (since $L_n = (L_{n-1})^{\alpha_n}$, (1.5)(iv)), we need only treat the case $n = 1$. So set $\mathfrak{m} = \mathfrak{m}_0$, and assume that $\beta = \alpha_1$, say β is a localization of $\alpha[x^{-1}\mathfrak{m}], x \in \mathfrak{m}$, so that $\mathfrak{m}\beta = x\beta$ and $v(\mathfrak{m}) = v(x)$. We want to show, with $l = \operatorname{ord}_\alpha(L)$, that

$$v(L) = v(L_1) + l v(\mathfrak{m}) = v(L_1) + l v(x).$$

So it will be enough to check that

$$L_1 = x^{-l} L\beta. \tag{1.11.1}$$

Since "transform" respects products (1.5)(iii), it is in fact enough to check (1.11.1) when

(a): $L^{-1} = \alpha$

and when

(b): $L = p$, a principal prime ideal in α.

For this purpose, note that if q is a prime ideal in β with $\mathfrak{m} \not\subset q$, and if $q' = q \cap \alpha$, then $\alpha_{q'} = \beta_q$ (for, $x \notin q'$, so $\alpha_{q'}$ and β_q are both localizations of $\alpha[x^{-1}\mathfrak{m}]$, and since β_q dominates $\alpha_{q'}$, therefore $\beta_q = \alpha_{q'}$). In particular, every principal prime ideal in β *other than* $x\beta$ intersects α in a principal prime. Since $l = \operatorname{ord}_\alpha(L)$, and since the valuation ring of $\operatorname{ord}_\alpha$ is the localization of β at the prime ideal $x\beta$, therefore

$$x^{-l} L\beta \not\subset x\beta,$$

and in case (a) it follows that no principal prime contains $x^{-l}L\beta$, i.e., (cf. (1.5)(ii)):

$$x^{-l}L\beta = (L\beta)^{-1}(L\beta) = L^{\beta} = L_1.$$

As for case (b), it follows that except for $x\beta$ every associated prime ideal of the principal ideal $p\beta$ intersects α in p, whence, (e.g., by[ZS_1, p.225, Thm.17]) with notation as in (1.2):

$$x^{-l}p\beta = p\beta_p \cap \beta = p^{\beta} = L_1.$$

q.e.d.

Exercise (1.12). Generalize (1.11.1) by showing, for *any* $\beta \gneqq \alpha$, that an ideal L' in β equals L^{β} if and only if:

(i) $L' = y^{-1}L\beta$ for some $y \in \beta$ such that $\mathfrak{m}_\alpha\beta \subset \sqrt{y\beta}$, and
(ii) $(\mathfrak{m}_\alpha\beta) : L' = \mathfrak{m}_\alpha\beta$.
(In other words, $y\beta$ is the smallest principal ideal containing $L\beta$ and all of whose associated primes contain $\mathfrak{m}_\alpha\beta$)

* * *

We continue with some simple—but necessary— supplements to the foregoing material.

Definition (1.13). *For any two ideals I, J in a commutative ring R, we set*

$$I * J = \overline{IJ} \quad (\textit{the completion of } IJ).$$

Lemma (1.14). *Assume that R is a commutative integral domain.*
(i) *For any two ideals I, J in R we have $I * J = \overline{I} * \overline{J}$.*
(ii) *If I is complete and $J \neq (0)$ is finitely generated then*

$$(I * J) : J = I.$$

(iii) *The non-zero complete ideals in R, with the $*$-product, form a commutative monoid $\mathcal{M}_R$ with cancellation (i.e., $I * J = I' * J \Rightarrow I = I'$).*

Proof: (i) is a consequence of the fact that if $x \in \overline{I}$ and $y \in \overline{J}$ then $xy \in \overline{IJ}$, a fact which follows easily from [ZS_2, bottom of p.349] (where $N \neq (0)$). Similarly one shows that if J is finitely generated then $xJ \subset I * J$ implies that $x \in \overline{I}$, and (ii) results. As for (iii), associativity of the $*$-product can be shown as follows:

$$\begin{aligned}(I_1 * I_2) * I_3 = (I_1I_2) * I_3 \text{ (cf. (i))} = \overline{I_1I_2I_3} &= I_1 * (I_2I_3) \\ &= I_1 * (I_2 * I_3);\end{aligned}$$

while commutativity and the existence of an identity (viz. R) are obvious. Cancellation follows from (ii).
q.e.d.

From (1.10) we have that for complete ideals I, J,

$$\mathbf{B}(I * J) = \mathbf{B}(IJ) = \mathbf{B}(I) + \mathbf{B}(J) \tag{1.15}$$

and furthermore

$$\{\mathbf{B}(I) = \mathbf{B}(J)\} \Rightarrow \{I = J\}.$$

Thus:

Proposition (1.16). *By associating to each non-zero complete ideal I in a point α the point basis $\mathbf{B}(I)$, we obtain an injective homomorphism of monoids*

$$\mathcal{M}_\alpha \hookrightarrow \prod_{\beta \succ \alpha} \mathbf{N}_\beta$$

where, for each $\beta \succ \alpha$, $\mathbf{N}_\beta$ is the monoid of non-negative integers (under addition).

* * *

Definition (1.17). *With $R \subset S \subset K$ as in (1.4), and I a non-zero ideal in R, we define the* complete transform $I^{\overline{S}}$ *to be $\overline{I^S}$, the completion of I^S.*

Proposition (1.18). *With notation as in (1.5), we have:*
(i) $(I_1 * I_2)^{\overline{S}} = (I_1 I_2)^{\overline{S}} = I_1^{\overline{S}} * I_2^{\overline{S}}$.
(ii) $(I^{\overline{S}})^{\overline{T}} = (I^S)^{\overline{T}} = I^{\overline{T}}$.

Proof: By (vi) of (1.5) we have

$$(I_1 I_2)^S \subset (I_1 * I_2)^S \subset \overline{(I_1 I_2)^S}$$

and hence

$$(I_1 * I_2)^{\overline{S}} = \overline{(I_1 I_2)^S} = (I_1 I_2)^{\overline{S}}.$$

Furthermore, by (iii) of (1.5) we have

$$(I_1 I_2)^{\overline{S}} = \overline{I_1^S I_2^S} = I_1^S * I_2^S = I_1^{\overline{S}} * I_1^{\overline{S}}$$

(the last equality by (1.14)(i)), proving (i).

Again by (vi) of (1.5),

$$(I^S)^T \subset (I^{\overline{S}})^T \subset (I^S)^{\overline{T}}$$

and hence

$$(I^{\overline{S}})^{\overline{T}} = (I^S)^{\overline{T}}.$$

And by (iv) of (1.5),

$$(I^S)^{\overline{T}} = \overline{(I^S)^T} = \overline{I^T} = I^{\overline{T}}.$$

q.e.d.

Remark (1.19): If S is an integrally closed domain, then every principal ideal in S is complete (for, if $0 \neq y \in S$, then an equation of integral dependence of x over yS yields an equation of integral dependence of x/y over S ...). Moreover, if I is an ideal in S such that $\overline{I}$ is principal, say $\overline{I} = xS$, then $I = \overline{I}$. (This is clear if $x = 0$; and otherwise $x^{-1}I \subset S$ and an equation of integral dependence of x over I yields

$$1 \in x^{-1}I + x^{-2}I^2 + \cdots = x^{-1}I$$

whence $x \in I$.). In particular, with $I \subset R \subset S$ as in (1.17):

$$\{I^S \text{ is principal}\} \Leftrightarrow \{I^{\overline{S}} \text{ is principal}\} \Rightarrow \{I^S = I^{\overline{S}}\};$$

and if $I^{-1} = R$, then (cf. (1.5)(ii)):

$$\{I^S = S\} \Leftrightarrow \{I^{\overline{S}} = S\}$$

* * *

Definition (1.20). *An ideal I in α is* finitely supported *if $I \neq (0)$ and I has at most finitely many base points (cf. (1.8)).*

Proposition (1.21). *If I is a finitely supported ideal in α, then for all $\beta \succ \alpha$, the transform I^β is finitely supported, and the ring β/I^β is artinian.*

Proof: The first assertion follows from (1.9) (a). For the second, we may then assume that $\beta = \alpha$. Suppose that I is contained in a non-maximal prime ideal p. Then any $\gamma \succ \alpha$ such that $\gamma \subset \alpha_p$ is a base point of I, because by (1.5)(v) and (1.5)(iv),

$$\alpha_p \neq I\alpha_p = I^{\alpha_p} = (I^\gamma)^{\alpha_p}$$

so that $I^\gamma \neq \gamma$. Thus (1.21) results from the following elementary fact:

Lemma (1.21.1). *Let p be a non-maximal prime ideal in α. Then there exists a quadratic transform α_1 of α with $\alpha_1 \subset \alpha_p$; and hence there is an infinite sequence*

$$\alpha = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_p$$

where each α_i $(i > 0)$ is a quadratic transform of α_{i-1}.

Proof: The second assertion follows from the first, since $p_1 = p\alpha_p \cap \alpha_1$ is a non-maximal prime (because $p_1 \cap \alpha = p$ whereas α_1 dominates α), and $(\alpha_1)_{p_1} = \alpha_p$, so that we can apply the first assertion to find a quadratic transform α_2 of α_1 with $\alpha_2 \subset \alpha_p$, etc. etc. The first assertion follows from the fact that the map

$X \to \mathrm{Spec}(\alpha)$ obtained by blowing up the maximal ideal $\mathfrak{m}$ of α is surjective and proper. Or, we can argue directly as follows.

The graded ring

$$gr_{\mathfrak{m}}(\alpha/p) = \bigoplus_{n\geq 0} (\mathfrak{m}^n + p)/(\mathfrak{m}^{n+1} + p)$$

is not artinian, hence has non-nilpotent elements of degree one, i.e., there is an $x \in \mathfrak{m}$ such that for all $n > 0$,

(1.21.2) $$x^n \notin \mathfrak{m}^{n+1} + p.$$

In particular, $x \notin \mathfrak{m}^2 + p$, and so

$$A = \alpha[x^{-1}\mathfrak{m}] = \bigcup_{n>0} \mathfrak{m}^n/x^n \subset \alpha_p.$$

Set

$$p' = p\alpha_p \cap A = \bigcup_{n>0} (p \cap \mathfrak{m}^n)/x^n.$$

Then

$$1 \notin \mathfrak{m}A + p' = xA + p'$$

since otherwise, for some $n > 0$, we would have

$$1 \in \mathfrak{m}^{n+1}/x^n + (p \cap \mathfrak{m}^n)/x^n$$

contradicting (1.21.2). So there exists a prime ideal q in A containing $\mathfrak{m}A + p'$, and $\alpha_1 = A_q \subset \alpha_p$ is a quadratic transform as desired.

Corollary (1.22). *If $\beta \succ \alpha$ is a base point of a finitely supported ideal $I \subset \alpha$, then $\dim \beta = \dim \alpha$.*

Proof: Let

$$\alpha = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n = \beta$$

be the quadratic sequence from α to β (1.6), and argue by induction on n. There being nothing to prove when $n = 0$, assume that $n > 0$ and that $\dim \alpha_{n-1} = \dim \alpha$. For some $x \in \mathfrak{m}$ (the maximal ideal of α_{n-1}), β is of the form $\beta = A_q$, where $A = \alpha_{n-1}[x^{-1}\mathfrak{m}]$ and $q \supset \mathfrak{m}$ is a prime ideal in A. Let $Q \subset A$ be a maximal ideal in A containing q. Then $Q/\mathfrak{m}A$ is a maximal ideal in $A/\mathfrak{m}A$, which is a finitely generated algebra over the field $\alpha_{n-1}/\mathfrak{m}$. It follows that A/Q is a finite field extension of $\alpha_{n-1}/\mathfrak{m}$, and hence that

(1.22.1) $$\dim A_Q = \dim \alpha_{n-1} = \dim \alpha$$

(cf. remarks preceding (1.6)).

Now if $q \neq Q$, then β is a localization of $\gamma = A_Q$ at a non-maximal prime ideal; but by (1.21) γ/I^γ is artinian, whence, by (1.5)(v) and (1.5)(iv)

$$\beta = (I^\gamma)\beta = (I^\gamma)^\beta = I^\beta,$$

i.e., β is not a base point of I. Thus $q = Q$, $\beta = A_Q$, and by (1.22.1), $\dim \beta = \dim \alpha$. *q.e.d.*

Remark (*not used elsewhere*): If $f : X \to \mathrm{Spec}(\alpha)$ is the map obtained by blowing up I, then I is finitely supported if and only if there exists a sequence

$$\sigma : X_n \underset{f_n}{\longrightarrow} X_{n-1} \to \cdots \to X_1 \underset{f_1}{\longrightarrow} X_0 = \mathrm{Spec}(\alpha)$$

where each f_i ($i > 0$) is obtained by blowing up a closed point of X_{i-1} (for example a base point of I) and such that X_n dominates X, i.e., there is a map $g : X_n \to X$ such that

$$f \circ g = f_1 \circ f_2 \circ \cdots \circ f_n;$$

in other words, "the indeterminacies of f^{-1} can be eliminated by a finite number of point blow ups". (Indeed, if σ exists, the base points of I must be among those which are blown up in σ.) It follows that *whenever a suitable local version of resolution of singularities is available* (for example if α is excellent and equicharacteristic, with $\alpha/\mathfrak{m}_\alpha$ a perfect field, of characteristic zero if $\dim \alpha > 3$, [Hi, p.142. Thm.II], [A′, p.149, (5.2.1)]) *then I is finitely supported if* (and by (1.22) only if) *every base point β of I satisfies* $\dim \beta = \dim \alpha$. (For, the indeterminacies of f^{-1} can then be eliminated by blowing up finitely many base points of I, and I will have no other base points.)

(1.23). For non-zero ideals I, J in α, the complete ideal $I * J$ is finitely supported if and only if both I and J are. (This is because $\mathbf{B}(I * J) = \mathbf{B}(I) + \mathbf{B}(J)$, cf. (1.15).) Thus (and by (1.22)):

Proposition (1.23). *The finitely supported complete ideals in α, together with the $*$-product, form a commutative monoid $\mathcal{M}^f_\alpha$, isomorphic under the injective monoid map* (1.16) *to a submonoid of the free commutative monoid $\mathcal{F}$ generated by all $\beta \succ \alpha$ with* $\dim \beta = \dim \alpha$. *($\mathcal{F}$ is the submonoid $\{\sum n_\beta \beta \mid n_\beta \geq 0$ for all $\beta\}$ of the free abelian group $\mathcal{G}^f_\alpha$ generated by such β).*

(1.24). A basic question now is to understand the structure of the monoid $\mathcal{M}^f_\alpha$. A partial result along these lines is given in Theorem (2.5) below; but it raises more questions than it answers. When $\dim \alpha = 2$, there is a satisfactory result, due to Zariski (Theorem (4.2)): $\mathcal{M}^f_\alpha$ is itself a free commutative monoid.

§2. Simple complete ideals corresponding to infinitely near points.

We say that a complete ideal I in a commutative ring R is **$*$-simple** if $I \neq R$ and if whenever $I = J * L$ with ideals J, L in R (cf. (1.13)) then either $J = R$ or $L = R$.

As in §1, we consider "points" $\alpha, \beta, \gamma, \ldots$ all having the same fraction field K. Recall the definitions of "infinitely near points" (1.6) and of "complete transform" (1.17). The main results in this section are contained in (2.1) and (2.5).

Proposition (2.1). *For each pair of points α, β with $\dim \alpha = \dim \beta$ there exists a unique complete ideal $\mathfrak{p}_{\alpha\beta}$ in α such that for every $\gamma \succ \alpha$: if $\gamma \prec \beta$ then the complete transform $(\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}}$ is $*$-simple, and otherwise $(\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}} = \gamma$.*

Corollary (2.2). (i). *γ is a base point of $\mathfrak{p}_{\alpha\beta}$ if and only if $\alpha \prec \gamma \prec \beta$; and hence $\mathfrak{p}_{\alpha\beta}$ is finitely supported* (1.20).
(ii). *The ring $\alpha/\mathfrak{p}_{\alpha\beta}$ is artinian.*
(iii). *$\mathfrak{p}_{\alpha\alpha}$ is the maximal ideal $\mathfrak{m}_\alpha$ of α.*
(iv). *For all $\gamma \succ \alpha$ with $\dim \gamma = \dim \alpha$ we have*

$$(\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}} = \mathfrak{p}_{\gamma\beta}.$$

Proof of (2.2): (i). γ is a base point of $\mathfrak{p}_{\alpha\beta}$ (1.8) iff $(\mathfrak{p}_{\alpha\beta})^{\gamma} \neq \gamma$, i.e. (clearly, or by (1.19)) iff $(\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}} \neq \gamma$, i.e. (by (2.1)) iff $\alpha \prec \gamma \prec \beta$. If $\alpha \prec \beta$, there are only finitely many such γ, viz. the members of the quadratic sequence from α to β (1.6); and otherwise there are no such γ. In any case, $\mathfrak{p}_{\alpha\beta}$ has at most finitely many base points.

(ii). This follows from (i) together with (1.21).

(iii). The ideal $(\mathfrak{m}_\alpha)^{\overline{\alpha}} = \mathfrak{m}_\alpha$ is $*$-simple; and for every $\gamma \succ \alpha$ with $\gamma \neq \alpha$ we have $(\mathfrak{m}_\alpha)^{\overline{\gamma}} = \gamma$.

(iv). For any $\delta \succ \gamma$, we have that $((\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}})^{\overline{\delta}} = (\mathfrak{p}_{\alpha\beta})^{\overline{\delta}}$ (cf. (1.18)(ii)), which is $*$-simple if $\delta \prec \beta$ and equal to δ otherwise.

Proof of (2.1): If β is not infinitely near to α, then $\mathfrak{p}_{\alpha\beta} = \alpha$ is an ideal in α having the required properties; and by taking $\gamma = \alpha$ in (2.1) we see that there is no other such ideal. So suppose that $\alpha \prec \beta$, and let

$$\alpha = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n = \beta$$

be the corresponding quadratic sequence (cf. (1.6)). We proceed by induction on n.

For $n = 0$, i.e. $\beta = \alpha$, we have already noted that $\mathfrak{p}_{\alpha\alpha} = \mathfrak{m}_\alpha$ has the required behavior with respect to $\gamma \succ \alpha$ (proof of (2.2)(iii)). That no other ideal in α has this behavior follows easily from (2.3) below (with $\mathcal{I} = \mathcal{O}_X$).

For $n > 0$, we already have $\mathfrak{p}_{\alpha_1\beta}$ (by the inductive hypothesis). Any $\gamma \succ \alpha$, other than $\gamma = \alpha$, satisfies $\gamma \succ \delta$ for a unique quadratic transform δ of α. As in the proof of (2.2)(iv), we find then that a $*$-simple complete ideal I in α satisfies the defining properties of $\mathfrak{p}_{\alpha\beta}$ if and only if: (i) $I^{\overline{\alpha_1}} = \mathfrak{p}_{\alpha_1\beta}$, and (ii) $I^\delta = \delta$ for every quadratic transform δ of α except $\delta = \alpha_1$.

We consider the map $f : X \to \operatorname{Spec}(\alpha)$ obtained by blowing up $\mathfrak{m}_\alpha$.[5] The quadratic transforms of α are just the local rings $\mathcal{O}_{X,x}$ of points $x \in f^{-1}\{\mathfrak{m}_\alpha\}$ (the closed fibre). Let $y \in f^{-1}\{\mathfrak{m}_\alpha\}$ be such that $\mathcal{O}_{X,y} = \alpha_1$. Since $\alpha_1/\mathfrak{p}_{\alpha_1\beta}$ is artinian (2.2)(ii), there exists a unique coherent $\mathcal{O}_X$-ideal $\mathcal{I}(\alpha,\beta)$ whose stalk at y is $\mathfrak{p}_{\alpha_1\beta}$ and whose stalk at any $x \neq y$ is $\mathcal{O}_{X,x}$. Thus, to complete the proof of (2.1), it suffices to show:

Lemma (2.3). *With $f : X \to \operatorname{Spec}(\alpha)$ as above, and $\mathfrak{m} = \mathfrak{m}_\alpha$, let $\mathcal{I}$ be a coherent $\mathcal{O}_X$-ideal whose stalk $\mathcal{I}_x$ is a complete $\mathcal{O}_{X,x}$-ideal for all $x \in X$, with $\mathcal{I}_x = \mathcal{O}_{X,x}$ if $x \notin f^{-1}\{\mathfrak{m}\}$. Assume also that $\mathcal{I} \not\subset \mathfrak{m}\mathcal{O}_X$. Then there exists a unique $\mathfrak{m}$-primary complete ideal I in α such that*

(i): *for every quadratic transform $\gamma = \mathcal{O}_{X,x}$ of α, we have*

$$I^{\overline{\gamma}} = \mathcal{I}_x;$$

and (ii): *any complete ideal $J \subset \mathfrak{m}$ such that $J^{\overline{\gamma}} = I^{\overline{\gamma}} = \mathcal{I}_x$ for all $\gamma = \mathcal{O}_{X,x}$ as in* (i) *must be of the form*

$$J = \mathfrak{m} * \mathfrak{m} * \cdots * \mathfrak{m} * I.$$

Furthermore, if $\mathcal{I}$ is $*$-simple (in the sense that $\mathcal{I} \neq \mathcal{O}_X$ and whenever $\mathcal{J} \neq \mathcal{O}_X$ and $\mathcal{L}$ are $\mathcal{O}_X$-ideals such that for all $x \in X$ we have $\mathcal{I}_x = \mathcal{J}_x * \mathcal{L}_x$, then $\mathcal{L} = \mathcal{O}_X$), then I is $*$-simple.

Proof: The $\mathcal{O}_X$-ideal $\mathfrak{m}\mathcal{O}_X$ is invertible, and hence for every $n \geq 0$ and every $x \in X$, $\mathfrak{m}^n\mathcal{I}_x$ is complete. Also, $\mathfrak{m}\mathcal{O}_X$ is very ample, so for some integer $N > 0$ the $\mathcal{O}_X$-ideal $\mathfrak{m}^N\mathcal{I}$ is generated by its global sections [Ha, p.121, Thm. 5.17]; in other words, if for any $n \geq 0$, I_n is the α-ideal

$$I_n = H^0(X, \mathfrak{m}^n\mathcal{I}) = \bigcap_{x\in X} \mathfrak{m}^n\mathcal{I}_x \subseteq \bigcap_{x\in X} \mathcal{O}_{X,x} = \alpha \tag{2.3.1}$$

then, for each $x \in X$,

$$\mathfrak{m}^N\mathcal{I}_x = I_N\mathcal{O}_{X,x}.$$

So we can define r to be the *least* among all integers $n > 0$ such that for all $x \in X$, $\mathfrak{m}^n\mathcal{I}_x$ is the *completion* $(I_n\mathcal{O}_{X,x})^-$; and then we set

$$I = I_r.$$

Let us check that this I is as asserted in (2.3).

---

[5] $X = \operatorname{Proj} S$, where S is the graded α-algebra $\bigoplus_{n\geq 0} \mathfrak{m}_\alpha^n$. In the language of models, [ZS$_2$, p.116 ff.], [EGA I, §8], X is the projective model determined by any basis of $\mathfrak{m}_\alpha$, and f is the domination mapping to $V(\alpha) = \operatorname{Spec}(\alpha)$.

First of all, for any $n \geq 0$, any $\xi \in \alpha$ which is integral over I_n is integral over $I_n\mathcal{O}_{X,x} \subset \mathfrak{m}^n\mathcal{I}_x$ for every $x \in X$, so that

$$\xi \in \bigcap_{x\in X} \mathfrak{m}^n\mathcal{I}_x = I_n,$$

and thus I_n is *complete*. Also, since $\mathcal{O}_X/\mathcal{I}$ is supported in $f^{-1}\{\mathfrak{m}\}$, there is an integer $N' > 0$ such that $\mathfrak{m}^{N'}\mathcal{O}_X \subset \mathcal{I}$ (this can be checked in each member of a finite affine open covering of X); hence

$$\mathfrak{m}^{n+N'} \subset \bigcap_{x\in X} \mathfrak{m}^{n+N'}\mathcal{O}_{X,x} \subset \bigcap_{x\in X} \mathfrak{m}^n\mathcal{I}_x = I_n,$$

and so I_n is $\mathfrak{m}$-primary, unless $n = 0$ and $\mathcal{I} = \mathcal{O}_X$ in which case $1 \in I_n$, i.e. $I_n = \alpha$. In particular, $I = I_r$ is complete and (since $r > 0$) $\mathfrak{m}$-primary.

Now note that $r = \text{ord}_\alpha(I)$: for if x is the generic point of the closed fibre $f^{-1}(\mathfrak{m})$, so that $\mathcal{O}_{X,x}$ is just the valuation ring of ord_α, then

$$I \subset \alpha \cap \mathfrak{m}^r\mathcal{O}_{X,x} = \{\xi \in \alpha \mid \text{ord}_\alpha(\xi) \geq r\} = \mathfrak{m}^r;$$

and if $I \subset \mathfrak{m}^{r+1}$ then for all $x \in X$:

$$\mathfrak{m}^r\mathcal{I}_x = (I\mathcal{O}_{X,x})^- \subset (\mathfrak{m}^{r+1}\mathcal{O}_{X,x})^- = \mathfrak{m}^{r+1}\mathcal{O}_{X,x},$$

whence $\mathcal{I} \subset \mathfrak{m}\mathcal{O}_X$, contrary to assumption. As in (1.11.1) we see then that for any quadratic transform $\gamma = \mathcal{O}_{X,x}$ ($x \in f^{-1}\{\mathfrak{m}\}$), we have

$$I^\gamma = (\mathfrak{m}\gamma)^{-r}(I\gamma)$$

and so

$$\overline{I^\gamma} = (\mathfrak{m}\gamma)^{-r}(I\gamma)^- = (\mathfrak{m}\mathcal{O}_{X,x})^{-r}(\mathfrak{m}^r\mathcal{I}_x) = \mathcal{I}_x,$$

i.e. (i) in (2.3) is satisfied.

Next we prove (2.3) (ii) (from which, in particular, the uniqueness of I follows). Let $s = \text{ord}_\alpha(J)$. Then for any quadratic transform $\gamma = \mathcal{O}_{X,y}$, we have, as above,

$$\mathcal{I}_y = \overline{J^\gamma} = (\mathfrak{m}\gamma)^{-s}(J\gamma)^-.$$

Hence

(2.3.2) $$\mathfrak{m}^s\mathcal{I}_y = (J\mathcal{O}_{X,y})^-$$

so that (since $\mathfrak{m}^s\mathcal{I}_x = \mathcal{O}_{X,x}$ for all $x \in X$ such that $x \notin f^{-1}\{\mathfrak{m}\}$)

$$J \subset \bigcap_{x\in X} \mathfrak{m}^s\mathcal{I}_x = I_s$$

(cf. (2.3.1)); and then (since, as above, $\mathfrak{m}^{s+N'} \subset I_s$ so that $I_s\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for $x \notin f^{-1}\{\mathfrak{m}\}$):

$$\mathfrak{m}^s\mathcal{I}_x = (I_s\mathcal{O}_{X,x})^-$$

for *all* $x \in X$. By the definition of r, therefore, $s \geq r$, and for all $x \in X$:

$$\mathfrak{m}^s \mathcal{I}_x = (\mathfrak{m}^{s-r} I \mathcal{O}_{X,x})^{-}. \tag{2.3.3}$$

Now, I claim, *every element* $\xi \in I_s$ *is integral over* J (whence $I_s = J$ since $J \subset I_s$ and J is complete). Indeed, if v is any valuation of the fraction field K of α whose valuation ring R_v dominates α (i.e. $R_v \supset \alpha$ and $v(\eta) > 0$ for all $\eta \in \mathfrak{m}$) then R_v dominates $\mathcal{O}_{X,y}$ for some $y \in f^{-1}\{\mathfrak{m}\}$ (since f is a proper map, cf. e.g. [ZS$_2$, p.120, (b)]), so by (2.3.2) and (1.1)

$$\xi \in I_s \subset (\mathfrak{m}^s \mathcal{I}_y) R_v = J R_v;$$

and thus (cf. (1.1)) $\xi \in \overline{J}$. Similarly, we see from (2.3.3) that I_s $(= J)$ is integral over $\mathfrak{m}^{s-r} I$, i.e. that

$$J = \underbrace{\mathfrak{m} * \mathfrak{m} * \cdots * \mathfrak{m}}_{s-r \text{ times}} * I$$

proving (ii).

Finally, suppose that $\mathcal{I}$ is $*$-simple and that $I = J * L$ where J, L are complete ideals in α; and let us deduce that either $J = \alpha$ or $L = \alpha$ (i.e. I is $*$-simple). Set

$$p = \operatorname{ord}_\alpha(J), \quad q = \operatorname{ord}_\alpha(L).$$

Then, since $I = (JL)^{-}$, (1.1) gives

$$\begin{aligned} r = \operatorname{ord}_\alpha(I) &= \operatorname{ord}_\alpha(J) + \operatorname{ord}_\alpha(L) \\ &= p + q, \end{aligned} \tag{2.3.4}$$

and for all $x \in X$, since

$$\begin{aligned} \mathfrak{m}^{p+q} \mathcal{I}_x = \mathfrak{m}^r \mathcal{I}_x &= (I \mathcal{O}_{X,x})^{-} \\ &= (JL\mathcal{O}_{X,x})^{-} \end{aligned} \tag{2.3.5}$$

we see that

$$\mathcal{I}_x = [(\mathfrak{m}\mathcal{O}_{X,x})^{-p} J \mathcal{O}_{X,x}] * [(\mathfrak{m}\mathcal{O}_{X,x})^{-q} L \mathcal{O}_{X,x}].$$

So if $\mathcal{J}, \mathcal{L}$ are the $\mathcal{O}_X$-ideals

$$\mathcal{J} = \mathfrak{m}^{-p} J \mathcal{O}_X, \qquad \mathcal{L} = \mathfrak{m}^{-q} L \mathcal{O}_X,$$

then either $\mathcal{J} = \mathcal{O}_X$ or $\mathcal{L} = \mathcal{O}_X$.

Suppose for example that $\mathcal{L} = \mathcal{O}_X$. Then for all $x \in X$ we have

$$L\mathcal{O}_{X,x} = \mathfrak{m}^q \mathcal{O}_{X,x},$$

and consequently (cf. (2.3.5))

$$I^{\tilde{\gamma}} = J^{\tilde{\gamma}}, \quad \gamma = \mathcal{O}_{X,x},$$

so that by (ii) above:

$$J = \mathfrak{m} * \mathfrak{m} * \cdots * \mathfrak{m} * I \subset I.$$

It follows at once (from (2.3.4)) that $q = 0$, i.e. $L = \alpha$. q.e.d.

Remarks (2.4) (*not used elsewhere*): For *any* $n \geq 0$, the α-ideal I_n defined by (2.3.1) is *contracted*, i.e.

(2.4.1) $$I_n = H^0(X, I_n \mathcal{O}_X) = \bigcap_{x \in X} I_n \mathcal{O}_{X,x}.$$

Moreover, if $n \geq r$ (r as in the proof of (2.3)), then, for all $x \in X$,

$$\begin{aligned} \mathfrak{m}^n \mathcal{I}_x = \mathfrak{m}^{n-r}(I_r \mathcal{O}_{X,x})^- &= (\mathfrak{m}^{n-r} I_r \mathcal{O}_{X,x})^- \quad (\text{since } \mathfrak{m}^{n-r}\mathcal{O}_{X,x} \text{ is principal}) \\ &\subset (I_n \mathcal{O}_{X,x})^- \subset \mathfrak{m}^n \mathcal{I}_x{}^{6} \end{aligned}$$

so that $\mathfrak{m}^n \mathcal{I}_x = (I_n \mathcal{O}_{X,x})^-$;[7] and if $\mathcal{I}_x = \mathcal{O}_{X,x}$ then

$$\mathfrak{m}^n \mathcal{O}_{X,x} = \mathfrak{m}^n \mathcal{I}_x = (I_n \mathcal{O}_{X,x})^- = I_n \mathcal{O}_{X,x}$$

(the last equality e.g. by (1.19), since $\mathfrak{m}^n \mathcal{O}_{X,x}$ is a principal $\mathcal{O}_{X,x}$-ideal). Thus if

(2.4.2) $$J_n = \Big(\bigcap_{\substack{x \in X \\ \mathcal{I}_x \neq \mathcal{O}_{X,x}}} I_n \mathcal{O}_{X,x} \Big) \cap \alpha$$

then

$$I_n = J_n \cap \Big(\bigcap_{x \in X} \mathfrak{m}^n \mathcal{O}_{X,x} \Big) = J_n \cap \mathfrak{m}^n.$$

Moreover, if $\mathcal{I}_x \neq \mathcal{O}_{X,x}$ for some $x \in f^{-1}\{\mathfrak{m}\}$ (i.e. $\mathcal{I} \neq \mathcal{O}_X$) then

$$J_n \subset (I_n \mathcal{O}_{X,x}) \cap \alpha \subset \mathfrak{m}^n \mathcal{O}_{X,x} \cap \alpha = \mathfrak{m}^n$$

(for the last equality, just note that $\mathrm{ord}_\alpha(\xi) \geq n$ for any $\xi \in \mathfrak{m}^n \mathcal{O}_{X,x}$). We conclude that

(2.4.3) $$I_n = J_n \qquad (n \geq r).$$

In particular:

(2.4.4)(cf. [ZS₂, top of p.373]). *Suppose that* $S = \{$ support of $\mathcal{O}_X/\mathcal{I} \}$ *is finite and non-empty,*[8] *and let* $\alpha_\mathcal{I}$ *be the semi-local ring*

$$\alpha_\mathcal{I} = \bigcap_{x \in S} \mathcal{O}_{X,x}.$$

---

[6] For any non-negative integers $p \geq q$, it follows easily from (2.3.1) that $I_q = I_p : \mathfrak{m}^{p-q}$. In particular, $\mathfrak{m}^{n-r} I_r \subset I_n$.

[7] As in the proof of (2.3), it follows that $I_n^{\overline{\gamma}} = \mathcal{I}_x$ for any quadratic transform $\gamma = \mathcal{O}_{X,x}$ of α, whence, by (2.3) (ii), $I_n = \mathfrak{m}^{n-r} * I$.

[8] Equivalently: if $k = \alpha/\mathfrak{m}$ and $L_r(I)$ is the k-vector space of leading forms of elements $\xi \in I$ such that $\mathrm{ord}_\alpha(\xi) = r$, then the set of zeros of $L_r(I)$ in the projective space $\mathrm{Proj}(\bigoplus_{t \geq 0} \mathfrak{m}^t/\mathfrak{m}^{t+1})$ is finite and non-empty.

Then, for $n \geq r$

$$I_n = (I_n \alpha_{\mathcal{I}}) \cap \alpha.$$

* * *

The next result is the central one in this paper. It expresses a kind of "unique factorization with possibly negative exponents". In the two-dimensional case, the exponents all turn out to be non-negative (Theorem (4.2)).

Theorem (2.5). *For each finitely supported* (cf. (1.20)) *complete ideal I in a point α there exists a unique family of integers*

$$(n_\beta) = (n_\beta(I))_{\beta \succ \alpha, \dim \beta = \dim \alpha}$$

such that $n_\beta = 0$ for almost all (i.e. *all but finitely many*) *β and such that*

$$\left(\prod_{n_\beta < 0}^{*} \mathfrak{p}_{\alpha\beta}^{-n_\beta} \right) * I = \prod_{n_\gamma > 0}^{*} \mathfrak{p}_{\alpha\gamma}^{n_\gamma}$$

where $\mathfrak{p}_{\alpha\beta}$ is as in (2.1)*, $\prod^{*}_{n_\beta<0}$ denotes $*$-product over all $\beta \succ \alpha$ such that $n_\beta < 0$, and similarly for $\prod^{*}_{n_\gamma>0}$.*

It is straightforward to see that (2.5) can be restated as follows:

Theorem (2.5)′. *For fixed α and variable $\beta \succ \alpha$ with $\dim \beta = \dim \alpha$, the images of the ideals $\mathfrak{p}_{\alpha\beta}$ under the canonical injection described in* (1.23) *form a basis of the free abelian group $\mathcal{G}_\alpha^f$ generated by all the β.*

Proof of (2.5)′: Fixing α, set

$$\Gamma = \{\gamma \mid \gamma \succ \alpha, \quad \dim \gamma = \dim \alpha\}.$$

According to (1.23), the canonical image of $\mathfrak{p}_{\alpha\beta}$ is the family of non-negative integers $(p_{\alpha\beta,\gamma})_{\gamma \in \Gamma}$ where

$$\begin{aligned} p_{\alpha\beta,\gamma} &= \operatorname{ord}_\gamma((\mathfrak{p}_{\alpha\beta})^\gamma) = \operatorname{ord}_\gamma((\mathfrak{p}_{\alpha\beta})^{\overline{\gamma}}) && \text{(cf. (1.1))} \\ &= \operatorname{ord}_\gamma(\mathfrak{p}_{\gamma\beta}) && \text{(cf. (2.2) (iv)).} \end{aligned}$$

By (2.2) (i), then, $p_{\alpha\beta,\gamma} = 0$ unless $\gamma \prec \beta$; and by (2.2) (iii), $p_{\alpha\beta,\beta} = 1$ for all β.

Now (2.5)′ asserts that for any family of integers $\mathbf{g} = (g_\gamma)_{\gamma \in \Gamma}$, with $g_\gamma = 0$ for almost all γ, there is a unique family of integers $\mathbf{h} = (h_\beta)_{\beta \in \Gamma}$ with $h_\beta = 0$ for almost all β and such that for all $\gamma \in \Gamma$

$$g_\gamma = \sum_{\beta \in \Gamma} h_\beta p_{\alpha\beta,\gamma}.$$

For the existence of h, argue by induction on the number $\nu_{\mathbf{g}}$ of points γ such that $g_\delta \neq 0$ for some $\delta \succ \gamma$: if $\nu_{\mathbf{g}} > 0$ then choose β such that $g_\beta \neq 0$ and $g_\delta = 0$ for all $\delta \succ \beta$, and set

$$g'_\gamma = g_\gamma - g_\beta p_{\alpha\beta,\gamma} \qquad (\gamma \in \Gamma);$$

then $g'_\gamma = g_\gamma$ unless $\gamma \prec \beta$, and moreover $g'_\beta = 0$, so $\nu_{\mathbf{g}'} < \nu_{\mathbf{g}} \cdots$.

For the uniqueness of h, assume that $g_\gamma = 0$ for all γ, but that $h_\gamma \neq 0$ for some γ. Then for some γ, $h_\gamma \neq 0$ and $h_\beta = 0$ for all $\beta \succ \gamma$, $\beta \neq \gamma$, so $g_\gamma = h_\gamma \neq 0$, contradiction. *q.e.d.*

§3. The length of a complete ideal (dimension 2).

In this section we derive a formula of Hoskin and Deligne for the length of an $\mathfrak{m}_\alpha$-primary complete ideal I in a two-dimensional regular local ring α, in terms of the point basis of I,[9] and deduce a number of consequences, some of which will be needed in §4.

We denote the length of an α-module M by $\lambda_\alpha(M)$. If β is a point infinitely near to α (i.e. $\alpha \subset \beta \subset$ fraction field of α, cf. note following (1.6)) then $[\beta : \alpha]$ denotes the (finite) degree of the residue field extension $\beta/\mathfrak{m}_\beta \supset \alpha/\mathfrak{m}_\alpha$.

Theorem (3.1). [Ho, p.85, Thm.(5.2)], [D, p.22, Thm. (2.13)]. *Let α be a two-dimensional regular local ring, with maximal ideal $\mathfrak{m}$, and let I be a complete $\mathfrak{m}$-primary ideal with point basis*

$$\mathbf{B}(I) = \{r_\beta\}_{\beta \succ \alpha} \qquad \text{cf. (1.8).}$$

Then $r_\beta = 0$ for all but finitely many β (i.e. *I is finitely supported,* cf. (1.20)), *and*

$$\lambda_\alpha(\alpha/I) = \sum_\beta [\beta : \alpha] r_\beta(r_\beta + 1)/2.$$

Proof: From (1.5) (ii) it follows that at most finitely many quadratic transforms of α, say $\alpha_1, \alpha_2, \ldots, \alpha_n$, are base points of I. Let

$$I_i = I^{\alpha_i} = (I\alpha_i)(I\alpha_i)^{-1} \qquad (1 \leq i \leq n),$$

[9] Geometrically speaking, the length of I is the number of conditions imposed on curves of sufficiently high degree by requiring their local equations to lie in I. In other words, if α is the local ring of a point x on a non-singular projectively embedded surface X over an algebraically closed field k, and $\mathcal{I}$ is the $\mathcal{O}_X$-ideal whose stalk at x is I and which coincides with $\mathcal{O}_X$ at all points other than x, then for sufficiently large n there is an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{I}(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)/\mathcal{I}(n)) \longrightarrow 0,$$

and, since the support of $\mathcal{O}_X/\mathcal{I}$ is the single point x, we have, for all n,

$$\dim_k H^0(X, \mathcal{O}_X(n)/\mathcal{I}(n)) = \dim_k(\alpha/I)$$

and set $f_i = [\alpha_i : \alpha]$. By a theorem of Zariski, [ZS_2, p.381, Prop.5], [L, p.209, (6.5)], the ideal $I\alpha_i$ is complete, whence so is I_i.

It is clear (cf. (1.9) (a)) that Theorem (3.1) implies:

$$\lambda_\alpha(\alpha/I) = \frac{1}{2}r_\alpha(r_\alpha + 1) + \sum_{i=1}^{n} f_i\lambda_{\alpha_i}(\alpha_i/I_i). \tag{3.1.1}$$

On the other hand, the validity of (3.1.1) for all α and I implies (by a straightforward induction) the validity of (3.1).

To prove (3.1.1), consider the map $X \to \operatorname{Spec}(\alpha)$ obtained by blowing up $\mathfrak{m}$. Since I is complete, we deduce from (1.1) that

$$H^0(X, I\mathcal{O}_X) = \bigcap_{x\in X} I\mathcal{O}_{X,x} = I. \tag{3.1.2}$$

(For another proof, cf. [L, p.208, Prop.(6.2)]). The argument which follows applies to any I satisfying (3.1.2), i.e. to any I which is "contracted from X".[10]

Using the affine open covering

$$X = \operatorname{Spec}(\alpha[b/c]) \cup \operatorname{Spec}(\alpha[c/b])$$

where $b, c \in \alpha$ generate $\mathfrak{m}$, one checks that $H^1(X, \mathcal{O}_X) = 0$ (cf. e.g. [L, p.200]), and that for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, $H^2(X, \mathcal{F}) = 0$ (since the Čech complex corresponding to the covering vanishes in dimension $\neq 0, 1$). There exists an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^N \longrightarrow I\mathcal{O}_X \longrightarrow 0,$$

whence an exact sequence

$$0 = H^1(X, \mathcal{O}_X^N) \longrightarrow H^1(X, I\mathcal{O}_X) \longrightarrow H^2(X, \mathcal{F}) = 0$$

so that

$$H^1(X, I\mathcal{O}_X) = 0. \tag{3.1.3}$$

Now with

$$r = r_\alpha = \operatorname{ord}_\alpha(I)$$

we have an exact sequence

$$0 \to I\mathcal{O}_X \to \mathfrak{m}^r\mathcal{O}_X \to \mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_X \to 0$$

whence an exact sequence

$$0 \to H^0(X, I\mathcal{O}_X) \to H^0(X, \mathfrak{m}^r\mathcal{O}_X) \to H^0(X, \mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_X) \to 0 \tag{3.1.4}$$

[10] It also applies, with slight modifications, when α is replaced by any local ring of a two-dimensional (pseudo-) rational singularity.

(where the 0 on the right comes from (3.1.3)). By (3.1.2)

$$H^0(X, I\mathcal{O}_X) = I,$$

and also (as in easily seen)

$$H^0(X, \mathfrak{m}^r\mathcal{O}_X) = \mathfrak{m}^r.$$

Thus

(3.1.5) $\lambda_\alpha(\alpha/I) - (1/2)r(r+1) = \lambda_\alpha(\mathfrak{m}^r/I) = \lambda_\alpha(H^0(X, \mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_X))$.

Moreover, for each x in the closed fibre $X \otimes_\alpha (\alpha/\mathfrak{m})$, the local ring $\beta = \mathcal{O}_{X,x}$ is a quadratic transform of α, $\mathfrak{m}\beta$ is invertible, and we have, as in (1.11.1),

(3.1.6) $$I^\beta = (\mathfrak{m}\beta)^{-r}(I\beta).$$

Hence $\mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_X$ is supported in the finite set of closed points $x_1, \ldots x_n \in X$ whose local rings are

$$\mathcal{O}_{X,x_i} = \alpha_i \qquad (1 \le i \le n),$$

and for each i, the stalk $(\mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_{X,x})_{x_i}$ is isomorphic to α_i/I_i, so that

(3.1.7) $$\lambda_\alpha(H^0(X, \mathfrak{m}^r\mathcal{O}_X/I\mathcal{O}_X)) = \sum_{i=1}^{n} \lambda_\alpha(\alpha_i/I_i) = \sum_{i=1}^{n} f_i\lambda_{\alpha_i}(\alpha_i/I_i).$$

Together, (3.1.5) and (3.1.7) give (3.1.1). *q.e.d.*

Henceforth we write "λ" for "λ_α".

Corollary (3.2). *Let I be an $\mathfrak{m}$-primary ideal satisfying* (3.1.2) (*e.g. I complete*), *and set $r = \mathrm{ord}_\alpha(I)$. Then any minimal generating set of I contains $r+1$ elements; in other words:*

$$\lambda(I/\mathfrak{m}I) = (say)\ \mu(I) = r + 1.$$

Proof: The ideal $\mathfrak{m}I$ is also contracted from X (cf. [ZS$_2$, p.376, Cor.1], or [L, p.209, Thm.(7.2)]), so we can replace I by $\mathfrak{m}I$ in (3.1.1) (cf. remarks following (3.1.2)). Since

$$\mathrm{ord}_\alpha(\mathfrak{m}I) = \mathrm{ord}_\alpha(I) + 1 = r + 1,$$

and since for all quadratic transforms β of α, we have

$$(\mathfrak{m}I)^\beta = (\mathfrak{m}\beta)^{-r-1}(\mathfrak{m}I\beta) = (\mathfrak{m}\beta)^{-r}(I\beta) = I^\beta$$

(cf. (3.1.6)), we deduce that

$$\begin{aligned}\lambda(I/\mathfrak{m}I) = \lambda(\alpha/\mathfrak{m}I) - \lambda(\alpha/I) &= \frac{1}{2}(r+1)(r+2) - \frac{1}{2}r(r+1) \\ &= r+1,\end{aligned}$$

proving (3.2).

Remark (3.3): In [Hy,Thm. 2.1], Huneke and Sally prove a converse to (3.2), at least when $\alpha/\mathfrak{m}$ is infinite: *if I is an $\mathfrak{m}$-primary ideal with $\mu(I) = \mathrm{ord}_\alpha(I)+1$ then I satisfies* (3.1.2). In their proof, they point out that for any $\mathfrak{m}$-primary I,

$$\mathrm{Tor}_2^\alpha(\alpha/I, \alpha/\mathfrak{m}) \cong (I:\mathfrak{m})/I$$

(as can be seen from the Koszul resolution of $\alpha/\mathfrak{m}$), and then, calculating Tor_2 via an exact sequence

$$0 \to \alpha^{\mu-1} \to \alpha^\mu \to \alpha \to \alpha/I \to 0 \quad (\mu = \mu(I)),$$

they conclude that

$$\lambda((I:\mathfrak{m})/I) = \mu(I) - 1.$$

Hence:

Corollary (3.4). *For any I as in* (3.2),

$$\lambda((I:\mathfrak{m})/I) = \mathrm{ord}_\alpha(I).$$

Corollaries (3.2) and (3.4) yield a proof, suggested to me by Craig Huneke, of the following result of Zariski:

Corollary (3.5). (cf. [ZS$_2$, p.368, Prop.3]). *Let I and $r = \mathrm{ord}_\alpha(I)$ be as in* (3.2), *and assume that $I \neq \mathfrak{m}(I:\mathfrak{m})$. Then*

$$\lambda(I/\mathfrak{m}^{r+1} \cap I) = 1.$$

Proof: If I satisfies (3.1.2), then

(3.5.1) $$\begin{aligned} x \in I:\mathfrak{m} &\iff x\mathfrak{m} \subset I \iff x\mathfrak{m}\mathcal{O}_X \subset I\mathcal{O}_X \\ &\iff x \in H^0(X, (\mathfrak{m}\mathcal{O}_X)^{-1} I\mathcal{O}_X)\end{aligned}$$

and consequently $I:\mathfrak{m}$ also satisfies (3.1.2), so that by (3.4) and (3.2)

$$\begin{aligned} r = \lambda((I:\mathfrak{m})/I) &< \lambda((I:\mathfrak{m})/\mathfrak{m}(I:\mathfrak{m}) \\ &= \mathrm{ord}_\alpha(I:\mathfrak{m}) + 1 \leq r+1.\end{aligned}$$

Hence

(3.5.2) $$\mathrm{ord}_\alpha(I:\mathfrak{m}) = r$$

and

$$(3.5.3)\qquad \lambda(I/\mathfrak{m}(I:\mathfrak{m})) = \lambda((I:\mathfrak{m})/\mathfrak{m}(I:\mathfrak{m})) - \lambda((I:\mathfrak{m})/I) = 1.$$

From (3.5.2) we get

$$\mathfrak{m}(I:\mathfrak{m}) \subset \mathfrak{m}^{r+1} \cap I \underset{\neq}{\subseteq} I,$$

and so (3.5.3) gives the conclusion.

Remark(3.6): Since $I:\mathfrak{m}$ satisfies (3.1.2) whenever I does (proof of (3.5)), (3.4) can be restated as:

$$\lambda(\alpha/I) = \sum_{n=0}^{\infty} \mathrm{ord}_{\alpha}(I:\mathfrak{m}^n)$$

* * *

We conclude with some remarks on "intersection numbers" and Hilbert-Samuel functions of complete finitely supported ideals I, J in α. (cf. (4.1)(C) below).

If I, J have respective point bases

$$\mathbf{B}(I) = \{r_\beta\}_{\beta \succ \alpha} \qquad \mathbf{B}(J) = \{s_\beta\}_{\beta \succ \alpha}$$

then we set

$$(I \cdot J) = \sum_{\beta} [\beta : \alpha] r_\beta s_\beta.$$

This integer can be interpreted as the "intersection multiplicity at α of generic member of I and a generic member of J" (cf. e.g. [N, p.189, Thm.8]). Its negative is the total intersection number of the curves defined by the ideals $I\mathcal{O}_Y, J\mathcal{O}_Y$, where $g: Y \to \mathrm{Spec}(\alpha)$ is any proper birational map such that $I\mathcal{O}_Y$ and $J\mathcal{O}_Y$ are both invertible (cf. e.g. [D, p.17, Thm.(2.9)]).

The following corollaries of (3.1) may be compared with [L, p.223, (13.1)(c)] and [L, p.253, (23.2)] respectively.

Corollary (3.7). *If I,J are complete finitely supported ideals in α, then*

$$\lambda(\alpha/IJ) = \lambda(\alpha/I) + \lambda(\alpha/J) + (I \cdot J).$$

Proof: The ideal IJ is complete (cf. (4.1)(A) below) and

$$\mathbf{B}(IJ) = \mathbf{B}(I) + \mathbf{B}(J)$$

(cf. (1.9)(b)). So we need only note that

$$(r_\beta + s_\beta)(r_\beta + s_\beta + 1) = r_\beta(r_\beta + 1) + s_\beta(s_{\beta+1}) + 2r_\beta s_\beta,$$

and apply (3.1).

Corollary (3.8). *If I is a complete finitely supported ideal in α, then for every $n \geq 0$ we have*

$$\lambda(I^n/I^{n+1}) = n(I \cdot I) + \lambda(\alpha/I).$$

Proof: The ideal I^n is complete for all $n \geq 0$ (cf. (4.1)(A) below), so (3.7) with $J = I^n$ gives

$$\begin{aligned} \lambda(I^n/I^{n+1}) = \lambda(\alpha/I^{n+1}) - \lambda(\alpha/I^n) &= \lambda(\alpha/I) + (I \cdot I^n) \\ &= \lambda(\alpha/I) + n(I \cdot I) \end{aligned}$$

(where the last equality follows from $\mathbf{B}(I^n) = n\mathbf{B}(I)$). *q.e.d.*

§4. Unique factorization for complete ideals (dimension 2).

We assume that all points $\alpha, \beta, \ldots$ are two-dimensional regular local rings with the same fraction field K. As noted following (1.6), β is then infinitely near to α if and only if $\beta \supseteq \alpha$.

The theory of complete ideals in the two-dimensional case is due to Zariski [Z],[ZS$_2$, Appendix 5.]. (Generalizations to rational singularities can be found in [L, Chapters II, V].) Here we review some of the main results in light of the preceding material in this paper.

First of all, concerning notions introduced in §1 above we have the following simplifications:

(4.1) **(A).** *Any product of complete ideals in α is again complete* (in other words, the $*$-product of (1.13) is just the usual product), [ZS$_2$, p.385, Thm.2′], [L, (7.1)].

(B). *If I is a complete ideal in α and $\beta \supseteq \alpha$, then $I\beta$ is a complete ideal in β* (and hence $I^\beta = \overline{I^\beta}$,cf. (1.5),(1.17)), [ZS$_2$, p.381, Prop.5], [L, (6.5)].

(C). *An ideal I in α is finitely supported if (cf.* (3.1) and (1.10)*) and only if (cf.* (1.21)*) α/I is artinian* (i.e. α contains some power $\mathfrak{m}_\alpha^n$ of the maximal ideal $\mathfrak{m}_\alpha$).

We will say that an ideal L in α is **simple** if $L \neq \alpha$ and if whenever $L = IJ$ with ideals I, J in α then $I = \alpha$ or $J = \alpha$.

Since α is a two-dimensional local unique factorization domain, every non-zero ideal in α is uniquely a product IJ, with I principal and J containing some power $\mathfrak{m}_\alpha^n$. Hence a simple ideal must be either principal (and prime) or (by (4.1)(C)) finitely supported.

The main results to be proved here are:

Theorem (4.2). [ZS$_2$, p.386, Thm.3],[L, p.244, Thm.(20.1)].
When dim $\alpha = 2$, *all the integers $n_\beta(I)$ in Theorem* (2.5) *are* ≥ 0, *and consequently every $\mathfrak{m}_\alpha$-primary complete ideal in α is in a unique way a product of simple complete ideals of the form $\mathfrak{p}_{\alpha\beta}$ ($\beta \supseteq \alpha$).*

Corollary (4.3). [ZS_2, p.389,(B)]. *Every $\mathfrak{m}_\alpha$-primary simple complete ideal is of the form $\mathfrak{p}_{\alpha\beta}$ for some $\beta \supseteq \alpha$.*

Corollary (4.4). [ZS_2, p.386, Lemma 6],[L, p.247, Prop.(21.5)]. *If I is a simple complete ideal in α, and $\beta \supseteq \alpha$, then the transform I^β is a simple complete ideal in β, unless $I^\beta = \beta$.*

Proof of (4.4): If I is principal use (1.5)(i). If I is $\mathfrak{m}_\alpha$-primary, use (4.3), (4.1)(B), and (2.2)(iv).

Proof of (4.2): In view of (4.1)(A), it is clear from (2.5) that we can characterize the integers $n_\beta(I)$ as follows:

Let $\mathcal{M}^f_\alpha$ be the monoid of complete ideals in α containing some power $\mathfrak{m}^n_\alpha$ (*cf.* (1.23) *and* (4.1)(A),(C)) *and let*

$$\nu_\beta : \mathcal{M}^f_\alpha \longrightarrow \mathbf{Z} \tag{4.2.1}$$

be a monoid homomorphism such that for $\gamma \supseteq \alpha$

$$\begin{aligned} \nu_\beta(\mathfrak{p}_{\alpha\gamma}) &= 1 \quad \textit{if} \quad \gamma = \beta \\ &= 0 \quad \textit{if} \quad \gamma \neq \beta. \end{aligned} \tag{4.2.2}$$

Then for all $I \in \mathcal{M}^f_\alpha$, we have

$$\nu_\beta(I) = n_\beta(I).$$

Thus, to prove (4.2), it will suffice to exhibit (for each β) such a map ν_β which satisfies in addition the property:

$$\nu_\beta(I) \geq 0 \qquad \text{for all} \qquad I \in \mathcal{M}^f_\alpha. \tag{4.2.3}$$

For this purpose we use the "characteristic form" $c(I)$ of a non-zero ideal $I \subset \alpha$, defined as follows (cf. [ZS_2, p.363]): we fix a basis (x,y) of $\mathfrak{m} = \mathfrak{m}_\alpha$, and correspondingly identify the graded ring $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ with the polynomial ring $k[X,Y]$ ($k = \alpha/\mathfrak{m}$); then, with $r = \operatorname{ord}_\alpha(I)$, we have an identification of the k-vector space

$$(I + \mathfrak{m}^{r+1})/\mathfrak{m}^{r+1} \cong I/\mathfrak{m}^{r+1} \cap I$$

with a k-vector space $L(I)$ consisting of forms of degree r in $k[X,Y]$, and we let $c(I)$ be a greatest common divisor of all the members of $L(I)$. Thus $c(I)$ is a form, uniquely determined up to multiplication by a non-zero element in k; and the *degree* $s(I)$ of $c(I)$ satisfies

$$s(I) \leq \operatorname{ord}_\alpha(I). \tag{4.2.4}$$

We define:

$$\nu_\beta(I) = \operatorname{ord}_\beta(I^\beta) - s(I^\beta).$$

Then (4.2.3) is immediate (by (4.2.4)), and the fact that ν_β is a monoid homomorphism (i.e. $\nu_\beta(IJ) = \nu_\beta(I) + \nu_\beta(J)$) follows from the easily proved identities

$$\begin{aligned} L(IJ) &= L(I)L(J) \\ c(IJ) &= a.c(I)c(J) \qquad 0 \neq a \in k \end{aligned}$$

together with the fact that $(IJ)^\beta = I^\beta J^\beta$ (cf. (1.5)(iii)).

It remains then to prove (4.2.2). Since (by (4.1)(B) and (2.2))

$$(\mathfrak{p}_{\alpha\beta})^\beta = \mathfrak{p}_{\beta\beta} = \mathfrak{m}_\beta,$$

it is clear that $\nu_\beta(\mathfrak{p}_{\alpha\beta}) = 1$. If $\alpha \subseteq \gamma$ and $\beta \not\subseteq \gamma$, then, by definition (cf. (2.1)) $(\mathfrak{p}_{\alpha\gamma})^\beta = \beta$, and so $\nu_\beta(\mathfrak{p}_{\alpha\gamma}) = 0$. Suppose then that $\alpha \subseteq \beta \subsetneq \gamma$. Then (cf. (2.2), (4.1)(B)):

$$(\mathfrak{p}_{\alpha\gamma})^\beta = \mathfrak{p}_{\beta\gamma} \neq \mathfrak{m}_\beta,$$

and furthermore $\mathfrak{p}_{\beta\gamma}$ is complete and $*$-simple (cf. (2.1)), hence not divisible by $\mathfrak{m}_\beta$ (i.e. not of the form $\mathfrak{m}_\beta J$ for some β-ideal J). What we have to show then is that

$$\operatorname{ord}_\beta(\mathfrak{p}_{\beta\gamma}) - s(\mathfrak{p}_{\beta\gamma}) = 0$$

or, equivalently, that *the* $(\beta/\mathfrak{m}_\beta)$*-vector space* $L(\mathfrak{p}_{\beta\gamma})$ *has dimension one.*

But this is a special case of Corollary (3.5). *q.e.d.*

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