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Topological invariants of quasi-ordinary singularities

ABSTRACT

A complex-analytic d-dimensional hypersurface germ (X, x) is *quasi-ordinary* if there exists a finite projection $\pi\colon (X, x) \to (\mathbf{C}^d, 0)$ which is unramified outside the union of the coordinate hyperplanes in $\mathbf{C}^d$. The branching number $m_i = m_i(\pi)$ over a coordinate hyperplane H_i $(1 \leq i \leq d)$ is then the local covering degree of $\pi^{-1}(H_i) \to H_i$ at a generic point of the irreducible variety $\pi^{-1}(H_i)$. The main result is that *except for its largest member, the family* $\{m_i\}_{1 \leq i \leq d}$ *is (modulo permutations) a topological invariant of* (X, x). This result is a major ingredient in the embedded topological classification of quasi-ordinary singularities via "characteristic monomials", given by Y.-N. Gau in the accompanying paper.

Key Words: Topology of analytic hypersurface germs; characteristic monomials of quasi-ordinary singularities; rational equivalence and local homology (for codimension one cycles).

Embedded topological classification of quasi-ordinary singularities

ABSTRACT

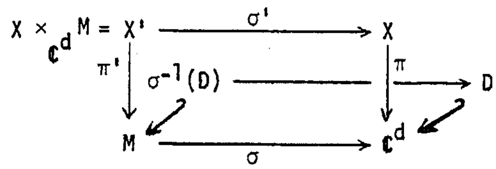
We give the embedded topological classification of the quasi-ordinary hypersurface singularities. The proof involves the topology of plane curve germs (Puiseux expansion, characteristic pairs, intersection multiplicity) and the result of Lipman on the local homology of the quasi-ordinary singularities. The classification is in terms of "distinguished tuples" which is a generalization of characteristic pairs.

Key Words: Analytic hypersurface germ, quasi-ordinary singularity, link, singular locus, topological type.

INTRODUCTION

This paper deals with topological invariants of quasi-ordinary singularities on complex analytic hypersurfaces. Before describing the main results, we give some motivating background.

(0.1) The Jungian strategy for resolving the singularities of a hypersurface $X \subset \mathbb{C}^{d+1}$ begins with diagrams like



where π is a finite projection, with (reduced) discriminant locus D, M is smooth, and σ is a bimeromorphic map such that $\sigma^{-1}(D)$ has only normal crossings as singularities. (The existence of σ would be given by induction on d.) Then σ' is bimeromorphic, and X' admits a finite projection π' to M whose discriminant locus has only normal crossings. This leads one to consider d-dimensional hypersurface singularities admitting locally a finite projection to $\mathbb{C}^d$ with normal-crossing discriminant. Such singularities are called *quasi-ordinary* [Z2 Def. 1.1]. Every plane curve singularity is quasi-ordinary $(d = 1)$. The case $d = 2$ is discussed in [L].

An irreducible quasi-ordinary singularity $x \in X$ can be represented explicitly by means of certain *fractional power series parametrizations*, which we now describe briefly. (More details are given at the beginning of §5). The local ring $\mathcal{O}_{X,x}$ of germs of holomorphic functions is isomorphic to a $\mathbb{C}$-algebra

$$\mathbb{C}\langle T_1,\dots,T_d\rangle[\zeta]$$

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where $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ is the convergent power series ring in d variables, and, for some integer n and some convergent power series H:

$$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n}).$$

Moreover the fractional power series ζ satisfies the following condition. Let $\zeta_1,\ldots,\zeta_m$ be the distinct $\mathbb{C}\langle T_1,\ldots,T_d\rangle$-conjugates of ζ (obtained from ζ by various substitutions of the form $T_i^{1/n} \to \omega_i T_i^{1/n}$ with $\omega_i^n = 1$ $(1 \le i \le d)$); then for all $i \ne j$,

$$\zeta_i - \zeta_j = M_{ij}\varepsilon_{ij}(T_1^{1/n},\ldots,T_d^{1/n}) \qquad \varepsilon_{ij}(0,\ldots,0) \ne 0$$

where

$$M_{ij} = T_1^{a_1/n} \cdots T_d^{a_d/n}$$

with integers $a_1,\ldots,a_d$ depending on i,j. Such a ζ is called a *quasi-ordinary branch*, and the monomials M_{ij} are called the *characteristic monomials* of ζ. We say then that ζ is a *quasi-ordinary (Puiseux-)parametrization* of the analytic germ (X,x); and (X,x) can be realized geometrically as the image of the map $\psi_\zeta : U \to \mathbb{C}^{d+1}$ (U some neighborhood of 0 in $\mathbb{C}^d$) defined by

$$\psi_\zeta(s_1,\ldots,s_d) = (s_1^n,\ldots,s_d^n, H(s_1,\ldots,s_d)). \qquad (0.1.1)$$

The plane curve case $(d = 1)$ is of course classical.

A given germ (X,x) may be parametrized by many different quasi-ordinary branches ζ. We will naturally be interested in those features of ζ which reflect intrinsic geometric information about (X,x). It turns out that much of the geometry of (X,x) is determined just by the characteristic monomials of a parametrizing branch ζ (cf. [L, p. 163]). This is illustrated below in the Appendix, §7, where we describe the singular locus of (X,x).

For another example, in the plane curve case the characteristic monomials of any Puiseux parametrization ζ determine and are determined by the "characteristic pairs" of ζ; and via knot theory, the characteristic pairs determine the *local topology* of the pair $(X,x) \subset (\mathbb{C}^2,0)$, and vice versa (cf. [R]).[1] So for $d > 1$, characteristic monomials are a natural higher-dimensional generalization of characteristic pairs; and it is natural to ask about the *relation between the topology of* $(X,x) \subset (\mathbb{C}^{d+1},0)$ *and the characteristic monomials of a (normalized) quasi-ordinary parametrization.*

It is this question which motivates the present paper.

(0.2). Part of the question is not hard to answer. As indicated in [L, §2], *the characteristic monomials of a quasi-ordinary parametrization* ζ *of* (X,x) *determine the* $\mathbb{C}\langle T_1,\ldots,T_d\rangle$*-saturation of* $\mathcal{O}_{X,x} = \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta]$, *and hence, by a theorem of Zariski, determine the topology of* $(X,x) \subset (\mathbb{C}^{d+1},0)$.

While this is nice to know, it is not very illuminating, in the sense that it gives no insight into the actual nature of the pair $(X,x) \subset (\mathbb{C}^{d+1},0)$. A better approach is suggested by the case $d = 1$, where one can realize the pair explicitly as the cone over a compound toroidal knot specified by the characteristic pairs. But for $d > 1$, little seems to have been done along such lines.

The converse is harder:

Conjecture (0.2.1).* *Let* $(X,x) \subset (\mathbb{C}^{d+1},0)$, $(X',x') \subset (\mathbb{C}^{d+1},0)$ *be* d-*dimensional germs parametrized by normalized quasi-ordinary branches* ζ, ζ' *respectively. If there exists a germ-homeomorphism*

[1]Here ζ needs to be "normalized", in the sense that no characteristic monomial has the form T_1^λ with $\lambda < 1$. There is a well-known "inversion" procedure which transforms any non-normalized parametrization ζ into a normalized one; and this procedure works in the higher-dimensional case as well (where "normalized" means "having no characteristic monomial of the form T_i^λ with $\lambda < 1$"), cf. Appendix to accompanying paper of Gau.

*This has now been confirmed by Y.-N. Gau (see accompanying paper).

$$\varphi:(\mathbb{C}^{d+1},0) \to (\mathbb{C}^{d+1},0)$$

such that

$$\varphi(X,x) = (X',x')$$

then ζ *and* ζ' *have the same characteristic monomials.*

Note that the truth (or falsehood) of this statement is not at all obvious even when $(X,x) = (X',x')$ and φ is the identity map.

For $d = 2$, the conjecture was affirmed by Gau in [G]. Gau recovers the characteristic monomials of a parametrization essentially from the following data (A), (B) at a generic point z_i of each codimension-one component Z_i of the singular locus Sing(X) (there are at most two such components):

(A) *The homotopy type of a pair*

$$(X \cap P, z_i) \subset (P, z_i)$$

where $P \subset \mathbb{C}^3$ *is a plane transversal to* Z_i *at* z_i, (so that $X \cap P$ is a possibly reducible plane curve).

Even though "transversal planes" are not preserved by homeomorphisms φ as above, the "transversal homotopy types" *are* topological invariants of $(X,x) \subset (\mathbb{C}^3,0)$, roughly because φ preserves components of Sing(X), and because X is equisingular along Z_i at z_i, so that there is a local homeomorphism of triples

$$(P \times \mathbb{C}^1, (X \cap P) \times \mathbb{C}^1, \{z_i\} \times \mathbb{C}^1) \xrightarrow{\sim} (\mathbb{C}^3, X, Z_i).$$

(B). *The branching order* m_i *at* z_i *of the projection* $\pi:(X,x) \to (\mathbb{C}^2,0)$ *taking* $(s_1^n, s_2^n, H(s_1,s_2))$ *to* (s_1^n, s_2^n) *cf. (0.1.1)* [2]

[2]This information is needed to carry out the inversion procedure mentioned in the footnote[1] above, since the restriction of π to $X \cap P$ (cf. (A)) might not be sufficiently generic - consider for example the germ parametrized by $\zeta = T_1^{2/3}T_2$, and see what happens at a generic point of Z_1 = zero-set of T_1.

The problem here is that there is no apparent reason why the m_i, which depend *a priori* on the chosen parametrization ζ, should be topological invariants of (X,x). A key result of Gau is that nevertheless they are! In fact, Gau proves that:

(0.2.2) m_i *is the order of the local singular homology group* $H_2(X,X-x)$.[3]

The main result of this paper (Theorem (6.1)) is the *topological invariance of the branching numbers* m_i (with one exception - cf. (0.3) below) *for arbitrary dimension* d.

With this generalization of step (B) above, and a suitable elaboration of step (A), Conjecture (0.2.1) has been shown by Gau to hold in all dimensions.

(0.3). We will now describe the main result, Theorem (6.1), more closely.

So let $(X,x) \subset (\mathbb{C}^{d+1},0)$ be parametrized by a quasi-ordinary branch ζ, and let $\pi:(X,x) \to (\mathbb{C}^d,0)$ be projection on the first d coordinates, the map germ corresponding to the inclusion of local rings

$$\mathcal{O}_{\mathbb{C}^d,0} = \mathbb{C}\langle T_1,\ldots,T_d\rangle \hookrightarrow \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta] \xrightarrow{\sim} \mathcal{O}_{X,x} .$$

The discriminant locus D of π is defined by

$$\Delta(\zeta) = \prod_{i\neq j} (\zeta_i - \zeta_j) = T_1^{b_1}\ldots T_d^{b_d}\varepsilon(T_1,\ldots,T_d) = 0$$

(where $\zeta_1,\ldots,\zeta_m$ are the conjugates of ζ, the b_i are non-negative integers, and $\varepsilon(0,\ldots,0) \neq 0$. Suppose that D has $c \leq d$ components, so that after

[3]This requires ζ to be "reduced", a technical condition which can be arranged by replacing (X,x) by a homeomorphic germ, cf. Proposition (6.3) below.

relabelling the variables T_i, we may assume that D is given by $T_1T_2\dots T_c = 0$. Let $Z_i \subset X$ be the zero set of T_i $(1 \le i \le d)$. One shows that Z_i is irreducible (beginning of §4). Let m_i be the branching order of π at a generic point z_i of Z_i (m_i = number of points in $\pi^{-1}(y)$ approaching z_i as $y \in \mathbb{C}^d$ approaches $\pi(z_i)$ in a generic direction). After further relabelling of the variables, we have (cf. (5.10.1)):

(0.3.1) $\quad 1 = m_d = m_{d-1} = \dots = m_{c+1} < m_c | m_{c-1} | \dots | m_1$ = (branching order of π at x)

(where "|" denotes "divides"). Furthermore we have the following generalization of (0.2.2) (cf. (5.10.3)):

(0.3.2) *The group* $H_{2d-2}(X, X-x)$ *has order* $m_2m_3\dots m_c$.

Thus the product $m_2m_3\dots m_c$ is a topological invariant of (X,x).

Better yet, Theorem (6.1) says that:

(0.3.3) *The entire sequence* $m_c | m_{c-1} | \dots | m_2$ *is a topological invariant of* (X,x), *in the following sense: if* (X',x') *is parametrized by a quasi-ordinary branch* ζ', *and if there exists a germ homeomorphism* $\psi: (X',x') \to (X,x)$ *then, with self-explanatory notation,* $c' = c$ *and the two sequences* $m'_c | m'_{c-1} | \dots | m'_2$, $m_c | m_{c-1} | \dots | m_2$ *coincide.*

Remarks (i). We could not have expected m_1, the branching order of π at x, to be topologically invariant, since when $d = 1$ all irreducible plane curve germs are homeomorphic to $(\mathbb{C}^1, 0)$). However, when ζ is *reduced*, (cf. footnote[3]) then $m_1 = m_2$.

(ii) Consider the map $\psi: \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\psi(t_1,t_2,t_3) = (t_1,t_2,t_1t_3).$$

If $X' \subset \mathbb{C}^3$ (resp. $X \subset \mathbb{C}^3$) is the surface parametrized by $\zeta' = T_1^{1/2}T_2^{1/2}$ (resp. $\zeta = T_1^{3/2}T_2^{1/2}$) then ψ induces a *homeomorphism* $(X',0) \to (X,0)$, which is identical with the blow-up of the line $T_1 = T_3 = 0$ on the surface X (whose equation is $T_3^2 = T_1^3T_2$). Because of (0.2.1), which holds when $d = 2$, this homeomorphism *cannot extend* to a homeomorphism of $(\mathbb{C}^3,0)$ to itself. Many such examples can be found by blowing up codimension-one smooth subgerms of quasi-ordinary singularities. Another kind of bimeromorphic homeomorphism appears in (6.3).

(iii) At present I don't know much beyond what is given by (6.1) concerning the *classification* of quasi-ordinary germs under homeomorphism.[4] (Of course (0.2.1) gives a complete classification under "embedded" homeomorphism.)

(0.4). Now here is a brief summary of the contents of the paper, leading up to the proof of Theorem (6.1).

In §1, we define, for any analytic germ (X,x) and any integer $k \geq 0$, a *local fundamental class map*

$$c_k: A_k(X)_x \to H_{2k}(X,X-x)$$

where $A_k(X)_x$ is the group of *rational equivalence classes* of germs at x of k-dimensional cycles on X.

For d-dimensional quotient singularities ($= (\mathbb{C}^d,0)$/finite group) the map c_{d-1} is an *isomorphism*. This holds, more generally, for rational singularities, as shown by Flenner. We will need only the quotient-singularity case, where a relatively simple proof can be given (and is, in §2).

[4]Progress has been made by Gau [G'].

In §3 we show that c_{d-1} *is an isomorphism for any* d-*dimensional quasi-ordinary singularity* (X,x) (i.e. a - not necessarily hypersurface - germ admitting a finite projection $\pi:(X,x) \to (\mathbb{C}^d,0)$ with normal-crossing discriminant). The idea is to compare c_{d-1} with the corresponding map for the normalization, which is a quotient singularity. Since the rational equivalence group $A_{d-1}(X)_x$ can be calculated by algebraic methods (applied to the local ring $\mathcal{O}_{X,x}$), this gives us an explicit presentation of the local homology $H_{2d-2}(X,X-x)$ (cf. §4). In particular (cf. (4.1.3)) the order of $H_{2d-2}(X,X-x)$ is

(0.3.2)' $$m_1m_2\ldots m_d/(\text{branching order of } \pi \text{ at } x)$$

where the branching numbers m_i are defined as above.

In §5, we consider hypersurface germs (X,x) parametrized by quasi-ordinary branches. The presentation of H_{2d-2} given in §4 is then described in quite computable terms, via the characteristic monomials of ζ, as are the integers m_i (cf. (5.9)). From this, (0.3.1) results, and hence (0.3.2) follows from (0.3.2)' (cf. (5.10.3)).

Finally, in §6, we prove Theorem (6.1) by applying (0.3.2)' to generic points of certain topologically distinguished subvarieties $\hat{Z}_i$ of the singular locus of X (and to the inverse images of such points on the normalization of X). One problem here is that the singular locus Sing(X) itself is not always topologically distinguishable: if X is irreducible at a generic point z of a codimension one component Z of Sing(X), then X is a topological manifold at z (because X is locally the topological product of Z - which is smooth at z - with a transversal plane section, which, being an irreducible plane curve germ, is homeomorphic to $(\mathbb{C}^1,0)$). Now if Z is topologically invisible, how can we capture (topologically) m_z, the branching order of π at z? It turns out, as indicated above, that the varieties $\hat{Z}_i \subset X$ $(1 \leq i \leq c)$, which are given by $T_1 = \ldots = T_{i-1} = T_{i+1} = \ldots = T_c = 0$ (cf. (0.3)), carry enough information to determine the m_i; but it takes some time to show that the collection $(\hat{Z}_i)_{1 \leq i \leq c}$ is actually a topological invariant of (X,x), cf. (6.7). Secondly one has to make some technical adjustments to ζ, as alluded to in footnote [3] above, and this also adds to the length of §6.

(0.5) In closing, I want to thank Y.-N. Gau for some stimulating conversations and suggestions, in particular the idea to look at the varieties $\hat{Z}_i$ in §6. Furthermore his result (0.2.2) is the spark which set this paper going.

I. RATIONAL EQUIVALENCE AND LOCAL HOMOLOGY IN CODIMENSION ONE

§1 Local fundamental class map

(1.1) Let U be a non-empty open subset of $\mathbb{C}^n$, let $X \subset U$ be the set of common zeros of a finite number of holomorphic functions on U, and let $x \in X$.

By excision, the singular homology groups (with $\mathbb{Z}$ coefficients)

$$H_i(X)_x = H_i(X, X-x) \qquad (i \geq 0)$$

do not change if we replace X by an open neighborhood of x in X. Consequently these groups are topological invariants of the analytic germ (X,x).

Here are two other interpretations of these local invariants.

First, fix a Riemannian metric on U, and let d be the induced distance function. For $\varepsilon > 0$ set

$$X_\varepsilon = \{z \in X \mid d(z,x) < \varepsilon\}$$

$$\overline{X}_\varepsilon = \{z \in X \mid d(z,x) \leq \varepsilon\}$$

$$L_\varepsilon = \overline{X}_\varepsilon - X_\varepsilon .$$

The *conic structure lemma* [BV, p.58, Lemma 3.2] says that for sufficiently small ε, $\overline{X}_\varepsilon$ is homeomorphic to the cone over L_ε, with x corresponding to the vertex. Hence, for such ε, and all $i > 0$, the natural maps (cf. [Sp, p.184])

$$\tilde{H}_{i-1}(L_\varepsilon) \longleftarrow H_i(\overline{X}_\varepsilon, L_\varepsilon) \longrightarrow H_i(\overline{X}_\varepsilon, \overline{X}_\varepsilon - x) \longleftarrow H_i(X_\varepsilon, X_\varepsilon - x) = H_i(X)_x$$

are *isomorphisms*; and so we have an interpretation of $H_i(X)_x$ in terms of the (reduced) homology of the "link" L_ε.(1)

(1) For $i = 0$, $H_0(X)_x = 0$ *unless* x is an isolated point of X.

Next, if we take the usual metric on U then $\overline{X}_\varepsilon$ is a finite simplicial complex and L_ε is a subcomplex [Lo, p.464, Thm. 3], so that there is a natural identification

$$H_i(\overline{X}_\varepsilon, L_\varepsilon) = H_i^{BM}(X_\varepsilon) \tag{1.1.1}$$

where "BM" signifies Borel-Moore homology with coefficients in the integers $\mathbb{Z}$ [BH, p.463, bottom]. As above, for ε small we have $H_i(\overline{X}_\varepsilon, L_\varepsilon) = H^i(X)_x$. Thus

$$H_i(X)_x = \varinjlim_{\varepsilon} H_i^{BM}(X_\varepsilon)$$

is the *local Borel-Moore homology at* x [BH, p.464].

(1.2) Let R be the local ring of germs at x of holomorphic functions on X. Imitating [F, §1.3; or p.396, 20.1.3] we define the local k-cycle group

$$Z_k(X)_x = Z_k(\mathrm{Spec}(R)) \qquad (k \geq 0)$$

to be the free abelian group generated by prime ideals $p \subset R$ such that the local ring R/p has dimension k (or, as we shall say, p is a "k-dimensional prime").

For any $(k+1)$-dimensional prime $q \subset R$ and any non-zero $r \in \overline{R} = R/q$, the k-cycle $[\mathrm{div}(r)]$ is defined to be

$$[\mathrm{div}(r)] = \sum_p \mathrm{ord}_p(r).p$$

where p runs through all k-dimensional primes containing q, and $\mathrm{ord}_p(r)$ is the *length* of the artin local ring $(\overline{R}/r\overline{R}) \otimes_R R_p$. The set of all cycles $[\mathrm{div}(r)]$ (as both q and r vary) generates a subgroup $\mathrm{Rat}_k(X)_x$ of $Z_k(X)_x$; and the quotient group

$$A_k(X)_x = Z_k(X)_x / \mathrm{Rat}_k(X)_x$$

is, by definition, the group of *local* k-*cycles modulo rational equivalence.*

(1.3) There is a natural map

$$c : Z_k(X)_x \longrightarrow H_{2k}(X)_x \qquad (k \geq 0)$$

defined as follows. The Nullstellensatz [GR, p.79][(1)] implies that the zeros of a k-dimensional prime ideal p form an irreducible k-dimensional analytic subgerm $(S,x) \subset (X,x)$. The *fundamental class* of S gives, for sufficiently small ε, an element of $H^{BM}_{2k}(S_\varepsilon)$, which maps canonically to $H^{BM}_{2k}(X_\varepsilon)$ and thence to $H_{2k}(X)_x$ (cf. [BH, p.481, §4.2]). Thus we have an element

$$c(p) = c^X_S \in H_{2k}(X)_x$$

(not depending on ε), *the local fundamental class of* S *in* X, *at* x. Extending by linearity, we get the asserted map c.

Proposition (1.4). *For any* [div(r)] *as in* (1.2), *we have*

$$c[\mathrm{div}(r)] = 0.$$

Thus c *induces a homomorphism* (*still to be denoted by* "c")

$$A_k(X)_x \longrightarrow H_{2k}(X)_x \qquad (k \geq 0).$$

Proof.[(2)] For any subgerm (V,x) of (X,x), there is a canonical map $H_{2k}(V)_x \longrightarrow H_{2k}(X)_x$. Hence with q as in (1.2), we may replace (X,x) by the subgerm corresponding to q. In other words, we may assume that $q = (0)$, i.e. that X is irreducible, of dimension $k + 1$.

(1) which reduces to the case when R is a one-dimensional domain, by virtue of the fact that {Spec(R)- maximal ideal} is a Jacobson space [GrD., p.67]; and is easily proved in this case by Puiseux-parametrizing the normalization of R.

(2) A different proof is given in (1.6) below.

Now let ρ be a holomorphic function defined in a neighborhood of x in U, and inducing the function germ r. Shrinking U if necessary, we may assume that ρ is defined everywhere on U, and we may assume that $\rho(x) = 0$ (otherwise $[\mathrm{div}(r)] = 0$). Let

$$Y = \{y \in U \mid \rho(y) = 0\}.$$

Since $r \not\equiv 0$, therefore ρ does not vanish identically on X, and the intersection $X \cap Y$ is a pure k-dimensional analytic set. I claim that *if* $X\cdot Y$ *is the analytic intersection cycle defined in* [BH, p.483, §4.6], *and* $(X\cdot Y)_x \in Z_k(X)_x$ *is its germ at* x, *then*

$$c[(X\cdot Y)_x] = 0.$$

Indeed, we have a commutative diagram (compatibility of intersections and "enlargement of families of supports" [BH, p.468, 1.12]):

$$\begin{array}{ccc} H^X_{2k+2}(U) \times H^Y_{2n-2}(U) & \xrightarrow{\text{intersection}} & H^{X\cap Y}_{2k}(U) \\ \text{natural}\downarrow & & \downarrow \\ H^X_{2k+2}(U) \times H^U_{2n-2}(U) & \xrightarrow[\text{intersection}]{} & H^X_{2k}(U) = H_{2k}(X) \end{array}$$

and moreover (supposing, as we may, that U is an open ball) we have $H^U_{2n-2}(U)$ $[= H^{BM}_{2n-2}(U)] = 0$, cf. (1.1.1).

Thus it will be enough to show that

$$[\mathrm{div}(r)] = (X\cdot Y)_x.$$

This follows at once from the "Fact" on p.184 of [GL] (equality of algebraic and topological intersection numbers) *provided that* each component of $X \cap Y$ contains a point where Y is smooth, i.e. where the partial derivatives of ρ

do not all vanish. But this can always be arranged: for, to prove (1.4), which is intrinsic to the germ (X,x), we may choose any embedding whatsoever of X in any $\mathbb{C}^n$, and so we may assume that some coordinate function ξ in $\mathbb{C}^n$ vanishes on X; then we can replace ρ by $\rho + t\xi$ for any $t \in \mathbb{C}$ (without changing $X \cap Y$), and almost any choice of t will ensure that the partial derivative $(\partial/\partial\xi)(\rho+t\xi)$ does not vanish identically on any component of $X \cap Y$. Q.E.D.

Example (1.5) (Codimension-one cycles). Assume that every component of X at x has dimension d. Let $Z \ni x$ be a closed analytic subset of X such that $\dim Z < d$ and $X^0 = X - Z$ is smooth. Let $k = d-1$, and let p, S be as in (1.3). After shrinking X we may assume that

$$H_i(X)_x = H_i^{BM}(X) \qquad (i \geq 0)$$

(cf (1.1)) and that S is a closed codimension-one subvariety of X.

Then $S^0 = S \cap X^0$ is a divisor on the smooth variety X^0, to which corresponds a line bundle $\mathcal{O}(S^0)$, and hence an element $\xi \in H^1(X^0, \mathcal{O}^*_{X^0})$. The *Chern class*

$$c_1(S^0) \in H^2(X^0, \mathbb{Z})$$

of $\mathcal{O}(S^0)$ is the image of ξ under the connecting homomorphism

$$\delta : H^1(X^0, \mathcal{O}^*_{X^0}) \longrightarrow H^2(X^0, \mathbb{Z})$$

induced by the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X^0} \xrightarrow{\exp} \mathcal{O}^*_{X^0} \longrightarrow 0$$

where

$$\exp(f) = e^{2\pi\sqrt{-1}f}$$

for any holomorphic f defined on an open subset of X^0.

We also have an isomorphism (Poincaré duality)

$$H^2(X^0, \mathbb{Z}) \longrightarrow H^{BM}_{2d-2}(X^0)$$

given by cap product with the fundamental class $[X^0]$. According to [BH, p.487, 4.13], the image of $c_1(S^0)$ under this isomorphism is identical with the natural image of the fundamental class $[S^0] \in H_{2d-2}(S^0)$. Thus, if (abusing notation) we let

$$c(p) \in H^{BM}_{2d-2}(X)$$

be the element corresponding to the fundamental class of S, and if

$$j: H^{BM}_{2d-2}(X) \longrightarrow H^{BM}_{2d-2}(X^0)$$

is the restriction map, then we have:

$$jc(p) = [X^0] \cap c_1(S^0). \tag{1.5.1}$$

In case $\dim Z \leq d-2$ (for example if X is normal and Z is the singular locus of X), then j is an *isomorphism*: there is an exact sequence

$$H^{BM}_{2d-2}(Z) \longrightarrow H^{BM}_{2d-2}(X) \xrightarrow{j} H^{BM}_{2d-2}(X^0) \longrightarrow H^{BM}_{2d-3}(Z)$$

[BH, p.465, 1.6], and the two outside terms vanish [*ibid*, **1.10**, 3.1]. So in this case, *the map* c *of (1.3) with* $k=d-1$, *is uniquely determined by (1.5.1).*

Remark (1.6). Example (1.5) gives another proof of (1.4). For, we may assume as in (1.4) that X is irreducible, of dimension $d=k+1$. Then c_1 extends to a homomorphism from the group of divisors of $X^0 = \{X - \text{singular locus}\}$ into the group $H^2(X^0, \mathbb{Z})$, and the kernel contains the divisor of any function. So if X is normal, then (1.4) follows easily from (1.5.1). And in the general case, if $\nu\colon \overline{X} \longrightarrow X$ is the normalization of X, and $\bar{x}$ is the unique point in $\nu^{-1}\{x\}$, then an argument similar to the one in the proof of Lemma 19.1.2 in [F, p.372] gives the commutativity of

(1.6.1)
$$\begin{array}{ccc} Z_k(\overline{X})_{\bar{x}} & \xrightarrow{c} & H_{2k}(\overline{X})_{\bar{x}} \\ {\scriptstyle \nu_*}\downarrow & & \downarrow{\scriptstyle \nu_*} \\ Z_k(X)_x & \xrightarrow[c]{} & H_{2k}(X)_x ; \end{array}$$

and since

$$[\operatorname{div}(r)] = \nu_*[\operatorname{div}(r\circ\nu)]$$

[F, p.12, Prop.1.4(b)], therefore (1.4) for X follows from (1.4) for $\overline{X}$.

§2. Codimension one cycles at quotient singularities

In this section, the hurried reader need only look at Theorem (2.2) and the first two sentences in its proof.

Let X be as in (1.1), and let $x \in X$ be a *quotient singularity*. This means that the local ring R of X at x (cf.(1.2)) is the ring of invariants of a finite group of $\mathbb{C}$-automorphisms of the convergent power series ring $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ ($d = \dim_x X$), or, equivalently, that R is normal and that the fraction field K of R has a finite Galois extension L in which the integral closure R_L of R is a *regular* local ring. Such an L, if one exists, is not unique, but can be chosen so that the extension R_L/R is *unramified in codimension one*; the corresponding Galois group $G = G(L/K)$ is then uniquely determined by the germ (X,x): it is the *local fundamental group* (cf.[P1, p.381, Thm. 1], or [DR]).

And the *divisor class group*

$$C\ell(R) = A_{d-1}(X)_x \qquad (d = \dim . R = \dim_x X)$$

(cf.(1.2)) is isomorphic to the abelianization of G:

$$(2.1) \qquad A_{d-1}(X)_x = \mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{C}^*) \cong G/[G,G]$$

(where "=" means "canonically isomorphic to") cf. [P2, p.78, Prop. III.4][(1)].

Theorem (2.2). *If $x \in X$ is, as above, a quotient singularity, then the class map*

$$c: A_{d-1}(X)_x \longrightarrow H_{2d-2}(X)_x \qquad (d = \dim_x X > 0)$$

(cf. (1.4)) is an isomorphism.

(1) or the footnote to Remark (2.3) at the end of this section.

Proof. The quotient singularity x is *rational* (cf. e.g. [BS, Korollar 5.8]) and so (in view of (1.5)) the Theorem is a special case of a result of Flenner [Fr, Satz (6.1)]. However, since Flenner's proof involves some deep theorems (resolution of singularities, extension of coherent sheaves,...), we give here a more elementary proof, adapted to quotient singularities.

We will need a basis of "conical" neighborhoods of x (as in (1.1)), well-chosen with respect to G. For this purpose, we recall some more results of Prill, concerning *standard models* for quotient singularities [Pl, IV].

For a given d-dimensional quotient singularity (X,x), the local fundamental group G is isomorphic to a "small" subgroup of $GL(d,\mathbb{C})$, unique up to conjugation; and the germ (X,x) is isomorphic to the germ $(\mathbb{C}^d/G, p(0))$, where p is the natural map of $\mathbb{C}^d$ onto the orbit space $\mathbb{C}^d/G$ (G acting on $\mathbb{C}^d$ in the obvious way). By [C, §3], the analytic space $\mathbb{C}^d/G$ (in which holomorphic functions are holomorphic functions in $\mathbb{C}^d$ constant on the orbits of G) admits an *isomorphism*

$$\phi:\mathbb{C}^d/G \xrightarrow{\sim} X'$$

with X' a normal closed algebraic subvariety of some $\mathbb{C}^n$, $\pi = \phi\circ p$ being given by

$$\pi(y) = (\phi_1(y),\ldots,\phi_n(y)) \qquad (y \in \mathbb{C}^d)$$

where $\{\phi_1,\ldots,\phi_n\}$ is a finite set of homogeneous polynomials generating the $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[T_1,\ldots,T_d]$ consisting of all elements invariant under the obvious action of G. For studying local properties of (X,x), we may therefore identify X with X' and x with $0 \in \mathbb{C}^n$. The surjective map $\pi:\mathbb{C}^d \to X$ is proper, with finite fibres; and is étale (= locally biholomorphic) outside the singular locus of X [Pl, p.379].

For any $\varepsilon > 0$, with the usual norm $\|\cdot\|$ on $\mathbb{C}^n$ we set

$$X_\varepsilon = \{x \in X \mid \|x\|<\varepsilon\} \qquad Y_\varepsilon = \pi^{-1}(X_\varepsilon)$$

$$\overline{X}_\varepsilon = \{x \in X \mid \|x\|\leq\varepsilon\} \qquad \overline{Y}_\varepsilon = \pi^{-1}(\overline{X}_\varepsilon)$$

$$X_\varepsilon^0 = X_\varepsilon - \{\text{singular locus}\} \qquad Y_\varepsilon^0 = \pi^{-1}(X_\varepsilon^0).$$

Note that G operates on Y_ε, with orbit space X_ε; and that G operates on Y_ε^0 *without fixed points*, the orbit space being X_ε^0.

For every $\varepsilon > 0$, $\overline{X}_\varepsilon$ (respectively $\overline{Y}_\varepsilon$) is homeomorphic to the cone over the real algebraic variety $\overline{X}_\varepsilon - X_\varepsilon$ (respectively $\overline{Y}_\varepsilon - Y_\varepsilon$), with the origin corresponding to the vertex: indeed $\overline{X}_\varepsilon - X_\varepsilon$ is compact (as is $\overline{Y}_\varepsilon - Y_\varepsilon$, π being proper); and the map

$$\xi:(\overline{X}_\varepsilon - X_\varepsilon) \times [0,1] \longrightarrow \overline{X}_\varepsilon$$

given by

$$\xi(x_1,\ldots,x_n,t) = (t^{d_1}x_1,\ldots,t^{d_n}x_n) \qquad d_i = \text{degree of } \phi_i$$

(respectively

$$\eta:(\overline{Y}_\varepsilon - Y_\varepsilon) \times [0,1] \longrightarrow \overline{Y}_\varepsilon$$

$$\eta(y,t) = ty\ )$$

induces the asserted homeomorphism. As in (1.1), we have then, for all $\varepsilon > 0$, natural isomorphisms

$$H_i^{BM}(X_\varepsilon) = H_i(X)_x \qquad (i\geq 0)$$

$$H_i^{BM}(Y_\varepsilon) = H_i(\mathbb{C}^d)_0 = \mathbb{Z} \qquad \text{if } i = 2d$$

$$= 0 \qquad \text{otherwise}$$

* * *

It is now easy to see that

$$H_{2d-2}(X)_x \cong G/[G,G] \qquad (\cong A_{d-1}(X)_x,\ \text{cf.}(2.1)).$$

Indeed, as in (1.5), we have isomorphisms

$$H_{2d-2}(X)_x = H^{BM}_{2d-2}(X_\varepsilon) \cong H^{BM}_{2d-2}(X^0_\varepsilon) \cong H^2(X^0_\varepsilon,\mathbb{Z}).$$

Furthermore, there is a spectral sequence

(2.2.1) $$H^p(G,H^q(Y^0_\varepsilon,\mathbb{Z})) \Rightarrow H^{p+q}(X^0_\varepsilon,\mathbb{Z})$$

(cf. [Gr, p.205, Cor.3] or [CE, p.355]). But, for $q = 1,2$, we have, as above, isomorphisms

(2.2.2) $$H^q(Y^0_\varepsilon,\mathbb{Z}) \cong H^{BM}_{2d-q}(Y^0_\varepsilon) \cong H^{BM}_{2d-q}(Y_\varepsilon) = 0,$$

whence, from (2.2.1), an (edge-)isomorphism

(2.2.3) $$H^2(G,H^0(Y^0_\varepsilon,\mathbb{Z})) \cong H^2(X^0_\varepsilon,\mathbb{Z}).$$

Finally, Y^0_ε is connected, so that $H^0(Y^0_\varepsilon,\mathbb{Z}) = \mathbb{Z}$, with trivial G-operations (since complex-linear maps are orientation-preserving); and so

$$H_{2d-2}(X)_x \cong H^2(X^0_\varepsilon,\mathbb{Z}) \cong H^2(G,\mathbb{Z}) \cong G/[G,G]$$

(cf [CE, p.250, (7)]).

Thus, to prove Theorem (2.2), it will suffice to show that c is *injective*. In other words, if $p_1,\dots,p_e$ are height one primes in R, and $D = \sum_{i=1}^{e} n_i p_i$ is such that $c(D) = 0$, then $D = [\mathrm{div}(r_1)] - [\mathrm{div}(r_2)]$ for some $r_1, r_2 \in R$. For this, after adding to D a suitable $[\mathrm{div}(r)]$, we may assume that the integers n_i are all > 0.

For sufficiently small ε, there are irreducible codimension-one subvarieties P_i of X_ε with germs at 0 corresponding to p_i. Let $\mathcal{D}$ be the $\mathcal{O}_{X_\varepsilon}$-ideal consisting of functions vanishing on each P_i to order at least n_i. The restriction of $\mathcal{D}$ to X_ε^0 is invertible (= locally free of rank one), and, as in (1.5), there is a corresponding element

$$\xi_D \in H^1(X_\varepsilon^0, \mathcal{O}_X^*)$$

such that, modulo Poincaré duality, we have

$$c(D) = \delta(\xi_D).$$

Since the class of D in $A_{d-1}(X)_x = G/[G,G]$ has finite order (i.e. for some integer m, mD is the divisor of zeros of some function germ at x) it follows (after shrinking ε) that also ξ_D *has finite order.*

Arguing as above, we have an isomorphism

$$H^1(X_\varepsilon^0, \mathbb{Z}) \cong H^1(G, \mathbb{Z}) = 0$$

(for the last equality cf. [CE, p.237,(4)].). The exponential sequence in (1.5) gives then an exact sequence

$$0 = H^1(X_\varepsilon^0, \mathbb{Z}) \longrightarrow H^1(X_\varepsilon^0, \mathcal{O}_X) \longrightarrow H^1(X_\varepsilon^0, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X_\varepsilon^0, \mathbb{Z}),$$

and the kernel of δ (viz. the $\mathbb{C}$-vector space $H^1(X_\varepsilon^0, \mathcal{O}_X)$) contains no non-zero element of finite order. Thus if $0 = c(D) = \delta(\xi_D)$, then $\xi_D = 0$, i.e. the restriction of $\mathcal{D}$ to X_ε^0 is generated by a single holomorphic function ρ. Since X_ε is normal, ρ extends uniquely to a holomorphic function on X_ε. Then the ideals $\mathcal{D}$ and $\rho\mathcal{O}_{X_\varepsilon}$ agree outside a set of codimension two (viz. $X_\varepsilon - X_\varepsilon^0$), so they are equal, and hence $D = [\mathrm{div}(r_1)]$ with r_1 the germ of ρ at 0.

Q.E.D.

Remark (2.3) One can also prove (2.2)[1] by showing that c factors (modulo Poincaré duality, cf. (1.5)) into the following sequence of three maps, *each of which is an isomorphism:*

$$C\ell(R) \xrightarrow{d} H^1(G,R_L^*) \xrightarrow{e} H^2(G,\mathbb{Z}) \xrightarrow{f} H^2(X_\varepsilon^0,\mathbb{Z}).$$

Here:

- d is Samuel's descent isomorphism [Fo, p.82] (note that R_L is a unique factorization domain, unramified over R in codimension 1).

- e is induced by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow R_L \xrightarrow{\exp} R_L^* \longrightarrow 0$$

($\exp(r) = e^{2\pi ir}$), which is a sequence of G-modules (G acting trivially on $\mathbb{Z}$); and e is an isomorphism because multiplication by the order $|G|$ is bijective in R_L, so that $H^i(G,R_L) = 0$ for all $i > 0$ [CE, p.236, Prop.2.5].

- f is the edge-isomorphism (2.2.3).

The idea is that since $H^1(Y_\varepsilon^0,\mathbb{Z}) = 0$ (cf. 2.2.2)), we have an exact "exponential" sequence of G-modules:

$$0 \longrightarrow H^0(Y_\varepsilon^0,\mathbb{Z}) \longrightarrow H^0(Y_\varepsilon^0,\mathcal{O}_{Y_\varepsilon}) \longrightarrow H^0(Y_\varepsilon^0,\mathcal{O}_{Y_\varepsilon}^*) \longrightarrow 0$$

whence a commutative diagram

$$\begin{array}{ccc} H^1(G,H^0(Y_\varepsilon^0,\mathcal{O}_{Y_\varepsilon}^*)) & \longrightarrow & H^2(G,H^0(Y_\varepsilon^0,\mathbb{Z})) = H^2(G,\mathbb{Z}) \\ \downarrow & & \downarrow f \\ H^1(X_\varepsilon^0,\mathcal{O}_{X_\varepsilon}^*) & \xrightarrow{\delta} & H^2(X_\varepsilon^0,\mathbb{Z}) \end{array}$$

where the vertical arrows are edge homomorphisms for the spectral sequence (2.2.1) and its analogue for $\mathcal{O}^*$, and the horizontal arrows are connecting homomorphisms

(1) and incidentally (2.1), since $H^2(G, \mathbb{Z}) \cong \mathrm{Hom}(G,\mathbb{C}^*)$, cf. [CE, p.250, (7)].

induced by the exponential. Then one checks that for a divisor D in R, a one-cocycle representing d(D) "spreads out" to a one-cocycle with values in $H^0(Y^0_\varepsilon, \mathcal{O}^*_{Y_\varepsilon})$ (ε sufficiently small), giving an element in $H^1(G, H^0(Y^0_\varepsilon, \mathcal{O}^*_{Y_\varepsilon}))$ whose image in $H^1(X^0_\varepsilon, \mathcal{O}^*_{X_\varepsilon})$ is the element ξ_D of (2.2.4)...

§3. Quasi-ordinary singularities

This section is devoted entirely to Theorem (3.2). It is not necessary to read the proof of (3.2) to understand the rest of the paper.

Let (X,x) be an analytic germ, with local ring R (germs of holomorphic functions). Assume that X is *irreducible* at x, i.e. that R is an integral domain, of dimension, say, d. Corresponding to any set of elements $r_1,\dots,r_d$ in R such that $R/(r_1,\dots,r_d)R$ is a finite-dimensional $\mathbb{C}$-vector space, there is an injective finite homomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}\langle T_1,\dots,T_d\rangle \hookrightarrow R$$

(where $\mathbb{C}\langle T_1,\dots,T_d\rangle$ is the ring of convergent power series in d variables) taking T_i to r_i $(1\le i\le d)$; and hence (by a standard equivalence of categories) a *finite map germ*

$$\pi:(X,x)\to(\mathbb{C}^d,0).$$

In fact we can choose a neighborhood X_0 of x in X such that for each $i = 1, 2,\dots,d$, r_i is the germ of a holomorphic function $\rho_i\colon X_0\to\mathbb{C}$; and then π is the germ of the map - still to be denoted by π-from X_0 to $\mathbb{C}^d$ given by

$$\pi(y) = (\rho_1(y),\dots,\rho_d(y)) \qquad\qquad (y\in X_0).$$

Moreover if X_0 is sufficiently small, then the map π is proper, with finite fibres, and open; in particular $N=\pi(X_0)$ is a neighborhood of 0 in $\mathbb{C}^d$. We assume henceforth that all these conditions hold.

For all z in some open dense subset of N, the fibre cardinality $\#\{\pi^{-1}(z)\}$ has the same value, say m. The set

$$D = \{y\in N \mid \#\{\pi^{-1}(y)\} < m\}$$

is an analytic subvariety of $\mathbb{C}^d$, the *discriminant locus* of π. Outside of D, π is an m-sheeted étale covering map (= local isomorphism of d-dimensional manifolds). Of course the *germ* (D,0) depends only on the *germ* π, so we can think of the discriminant locus of π either in the category of analytic spaces or in the category of germs. It is much easier to use the same notation etc. in both cases; and the proper interpretation should always be clear from the context.

Definition (3.1). *Let* $\pi:(X,x) \to (\mathbb{C}^d,0)$ *be a finite map germ, as above. We say that* π *is* quasi-ordinary *if for some choice of analytic coordinates* $(t_1,\ldots,t_d)$ *at* 0, *the discriminant locus* D *of* π *is contained (germwise) in the subvariety of* $\mathbb{C}^d$ *given by the equation*

$$t_1 t_2 \ldots t_d = 0.$$

We say that (X,x) *(or its local ring* R*) is quasi-ordinary if there exists such a quasi-ordinary* π.

Remark (3.1.1). After changing coordinates in $\mathbb{C}\langle T_1,\ldots,T_d\rangle$, we may assume in (3.1) that $t_i = T_i$. In other words, corresponding to a quasi-ordinary π there is a finite injective $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}\langle T_1,\ldots,T_d\rangle \hookrightarrow R$$

which is étale at any prime p in $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ not containing $T = T_1T_2 \ldots T_d$. It is then well-known that *for some integer* $n > 0$, R *is a* $\mathbb{C}\langle T_1,\ldots,T_d\rangle$*-subalgebra of*

$$\mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle = \mathbb{C}\langle T_1,\ldots,T_d\rangle[T_1^{1/n},\ldots,T_d^{1/n}].$$

(This was deduced classically by Jung from the fact that the fundamental group of a product of d punctured discs is $\mathbb{Z}^d$. A purely algebraic proof is given in [A, p.585, Thm.3].)

Theorem (3.2). *If* (X,x) *is quasi-ordinary, then the class map*

$$c: A_{d-1}(X)_x \longrightarrow H_{2d-2}(X)_x$$

(cf. (1.4)) is an isomorphism.

Remark (3.2.1). The abelian group $A_{d-1}(X)_x$ is *finite* (cf. Corollary (3.5.3) below). More information about this group will be given in §4.

Proof of (3.2). From (3.1.1) it follows that the normalization $(\bar{X},\bar{x})$ of the germ (X,x) has a quotient singularity, with local fundamental group, say,

$$G \subset ((1/n)\mathbb{Z})^d.$$

As in (2.1), we have then

(3.2.1) $$A_{d-1}(\bar{X})_{\bar{x}} \cong G;$$

and by Theorem (2.2), the class map

$$\bar{c}: A_{d-1}(\bar{X})_{\bar{x}} \xrightarrow{\sim} H_{2d-2}(\bar{X})_{\bar{x}}$$

is an *isomorphism.*

The goal of the rest of this section is to prove (3.2) by showing:

(3.2)': *the kernel and cokernel of the class map* c *in (3.2) are respectively isomorphic to the kernel and cokernel of the preceding map* $\bar{c}$.

(3.3) We first look at the algebraic aspects of the situation. With notation as in (3.1.1), let

$$F \subseteq \operatorname{Spec}(\mathbb{C}\langle T_1,\dots,T_d\rangle)$$

be the closed subscheme where $T = T_1T_2 \dots T_d$ vanishes, and let

$$E = \operatorname{Spec}(R/TR) \subset \operatorname{Spec}(R).$$

Then $\mathrm{Spec}(R) - E$ is étale over $\mathrm{Spec}(\mathbb{C}\langle T_1,\ldots,T_d\rangle) - F$, and so is normal. Let $\overline{R}$ be the normalization of R,

$$\mu\colon \mathrm{Spec}(\overline{R}) \longrightarrow \mathrm{Spec}(R)$$

the normalization map,

$$\overline{E} = \mu^{-1}(E)$$

and

$$U = \mathrm{Spec}(\overline{R}) - \overline{E} = \mathrm{Spec}(R) - E.$$

In $\mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle$ there is, for each $i = 1,2,\ldots,d$, a unique height one prime containing T_i (namely the ideal generated by $T_i^{1/n}$); so by the "going-up" theorem the same is true both in R and in $\overline{R}$. Thus E (resp. $\overline{E}$) is a pure $(d-1)$-dimensional subscheme of $\mathrm{Spec}(R)$ (resp. $\mathrm{Spec}(\overline{R})$) whose set of irreducible components is mapped bijectively by π (resp. $\pi\circ\mu$) onto the set of components - d in number - of F. Let

$$\iota\colon E \longrightarrow \mathrm{Spec}(R) \qquad \bar{\iota}\colon \overline{E} \longrightarrow \mathrm{Spec}(\overline{R})$$

be the inclusion maps. As in [F, §1.8], we get a commutative diagram, with exact rows:

$$\begin{array}{ccccccc}
Z_{d-1}(\overline{E}) = A_{d-1}(\overline{E}) & \xrightarrow{\bar{\iota}_*} & A_{d-1}(\mathrm{Spec}(\overline{R})) & \longrightarrow & A_{d-1}(U) & \longrightarrow & 0 \\
\downarrow \mu_*^E & & \downarrow \mu_*^R & & \| & & \\
Z_{d-1}(E) = A_{d-1}(E) & \xrightarrow[\iota_*]{} & A_{d-1}(\mathrm{Spec}(R)) & \longrightarrow & A_{d-1}(U) & \longrightarrow & 0.
\end{array}$$

(Here, as in (1.2), Z_{d-1} denotes $(d-1)$-dimensional cycles, and A_{d-1} denotes rational equivalence classes of such cycles.)

Lemma (3.3.1) (cf. also [F, p.22, Ex. 1.8.1]). *The kernels (resp. cokernels) of* μ_*^E *and* μ_*^R *are naturally isomorphic.*

Proof. As noted above, μ maps the set of components of $\overline{E}$ *bijectively* onto the set of components of E; and so from the definition of μ_*^E [F, §1.4] it follows that μ_*^E is *injective.*

Moreover, for any $r \neq 0$ in the fraction field of R (= fraction field of $\overline{R}$) [F, p.12, 1.4(b)] shows that the map μ_*^E takes the divisor

$$\overline{C} = [\mathrm{div}(r)] \quad \text{on} \quad \mathrm{Spec}(\overline{R})$$

to the divisor

$$C = [\mathrm{div}(r)] \quad \text{on} \quad \mathrm{Spec}(R).$$

The kernel of ι_* (resp. $\bar{\iota}_*$) consists of all such C (resp. $\overline{C}$) supported on E (resp. $\overline{E}$). Hence μ_*^E maps the kernel of $\bar{\iota}_*$ *isomorphically* onto the kernel of ι_*.

From this, (3.3.1) follows formally (i.e. by diagram chasing, or by two applications of the "snake lemma"). Q.E.D.

As mentioned in the preceding proof, μ_*^E is injective. Thus:

Corollary (3.3.2). *The map* μ_*^R *is injective.*

Corollary (3.3.3). *The group* $A_{d-1}(\mathrm{Spec}(R)) = A_{d-1}(X)_x$ *is finite.*

Proof. The cokernel of μ_*^E is easily seen to be finite, whence so is the cokernel of μ_*^R. By (3.2.1)

$$A_{d-1}(\mathrm{Spec}(\overline{R})) = A_{d-1}(\overline{X})_{\overline{x}}$$

is finite, and the conclusion follows.

(3.4) Now we pass from algebra to geometry.

As before, we may think of π as mapping X onto an open neighborhood N of 0 in $\mathbb{C}^d$, π being étale outside the hypersurface $D^* \subset \mathbb{C}^d$ given by $T_1 T_2 \cdots T_d = 0$, where $\{T_1, \ldots, T_d\}$ is a coordinate system in $\mathbb{C}^d$. (This D^* contains the discriminant locus D of π.) For any $\varepsilon > 0$ such that the polydisc

$$P_\varepsilon = \{(y_1, \ldots, y_d) \in \mathbb{C}^d \mid |y_i| < \varepsilon \ \text{ for all } \ i\}$$

is contained in N, we set:

$$X_\varepsilon = \pi^{-1}(P_\varepsilon), \qquad E_\varepsilon = \pi^{-1}(D^* \cap P_\varepsilon)$$

and, with $\nu: \overline{X} \to X$ the normalization map:

$$\overline{X}_\varepsilon = \nu^{-1}(X_\varepsilon), \qquad \overline{E}_\varepsilon = \nu^{-1}(E_\varepsilon).$$

Then ν induces an *isomorphism*

$$\overline{X}_\varepsilon - \overline{E}_\varepsilon \xrightarrow{\ \sim\ } X_\varepsilon - E_\varepsilon = \text{(say) } U_\varepsilon;$$

and

$$\pi: U_\varepsilon \longrightarrow P_\varepsilon - D^*$$

is étale.

For convenience, we drop the subscript ε, and denote Borel-Moore homology H_i^{BM} simply by H_i. Using [BH, p.465, 1.5 and 1.6], we get a commutative diagram, with exact rows:

$$\begin{array}{ccccccccc} H_{2d-1}(U) & \longrightarrow & H_{2d-2}(\overline{E}) & \xrightarrow{\bar{j}} & H_{2d-2}(\overline{X}) & \longrightarrow & H_{2d-2}(U) & \longrightarrow & H_{2d-3}(\overline{E}) \\ \| & & \downarrow{\scriptstyle \nu_*^E} & & \downarrow{\scriptstyle \nu_*^X} & & \| & & \downarrow \\ H_{2d-1}(U) & \longrightarrow & H_{2d-2}(E) & \xrightarrow[j]{} & H_{2d-2}(X) & \longrightarrow & H_{2d-2}(U) & \longrightarrow & H_{2d-3}(E) \end{array}$$

By diagram-chasing, we verify:

Lemma (3.4.1) (a) *The natural map* $\ker(\bar{j}) \to \ker(j)$ *is surjective.*

(b) *The natural map* $\ker(\nu_*^E) \to \ker(\nu_*^X)$ *is surjective.*

(c) *The natural map* $\operatorname{coker}(\nu_*^E) \to \operatorname{coker}(\nu_*^X)$ *is injective.*

Moreover:

Lemma (3.4.2). *The above map* ν_*^X *is injective.*

Proof. By [BH, p.482, 4.4], $H_{2d-2}(\bar{E})$ (resp. $H_{2d-2}(E)$) is freely generated by the components of $\bar{E}$ (resp. E). As in (3.3), the set of components of $\bar{E}$ is mapped bijectively by ν to the set of components of E. (The algebraic argument in (3.3) yields the corresponding geometric statement if ε is sufficiently small.) It follows that ν_*^E is injective; and hence, by (3.4.1)(b), so is ν_*^X. Q.E.D.

The following Lemma is crucial.

Lemma (3.4.3). *The map*

$$H_{2d-3}(\bar{E}) \longrightarrow H_{2d-3}(E)$$

in the above diagram is injective.

The *proof* will be given in (3.6) below.

Corollary (3.4.4). *The above maps* ν_*^E, ν_*^X *have naturally isomorphic cokernels.*

Proof. By diagram chasing, we deduce from (3.4.3) that the map in (3.4.1)(c) is surjective. Q.E.D.

(3.5) Assuming (3.4.4), let us now combine the algebra in (3.3) and the local geometry in (3.4) to complete the proof of (3.2)'.

Since π is proper and (as we may assume, after shrinking X) $\pi^{-1}\{\pi(x)\} = \{x\}$, the varieties X_ε in (3.4) form a basis of neighborhoods of x. Similarly the $\overline{X}_\varepsilon$ form a basis of neighborhoods of $\bar{x}$. The abelian group $A_{d-1}(X)_x$ is finite (3.3.3), hence finitely presentable as a $\mathbb{Z}$-module. So for all sufficiently small $\varepsilon > 0$, with X_ε as in (3.4) the class map c of (1.4) factors as

$$(3.5.1)\quad A_{d-1}(X)_x \longrightarrow H^{BM}_{2d-2}(X_\varepsilon) \longrightarrow \varinjlim_{\varepsilon} H^{BM}_{2d-2}(X_\varepsilon) = H_{2d-2}(X)_x$$

(cf.(1.1), where X_ε has a different meaning, which however doesn't matter for present purposes because $\varinjlim$ can be taken over *any* neighborhood basis of X). A similar statement holds for $(\overline{X},\bar{x})$.

As in (3.4), we write $H_\bullet(X)$ for $H^{BM}_\bullet(X_\varepsilon)$. For sufficiently small ε, we obtain, just as in (1.6.1), a commutative diagram

$$\begin{array}{ccc} A_{d-1}(\overline{X})_{\bar{x}} & \xrightarrow{c_{\bar{x}}} & H_{2d-2}(\overline{X}) \\ \mu^R_* \downarrow & & \downarrow \nu^X_* \\ A_{d-1}(X)_x & \xrightarrow[c_x]{} & H_{2d-2}(X) \end{array}$$

where the horizontal arrows come from (3.5.1), and the vertical arrows from the diagrams in (3.3), (3.4) respectively. Similarly, we have (for small ε) a commutative diagram

$$\begin{array}{ccc} A_{d-1}(\overline{E}_{\bar{x}}) & \xrightarrow{c_{\overline{E}}} & H_{2d-2}(\overline{E}) \\ \mu^E_* \downarrow & & \downarrow \nu^E_* \\ A_{d-1}(E_x) & \xrightarrow[c_E]{} & H_{2d-2}(E) \end{array}$$

(where we have written E_X (resp. $\overline{E}_{\overline{X}}$) for the scheme denoted by E (resp. $\overline{E}$) in (3.3)) in which the maps $c_{\overline{E}}$ and c_E are *isomorphisms* (c_E just sends components of E_X to the fundamental classes of the corresponding components of E, i.e. free generators of $A_{d-1}(E_X)$ to the corresponding free generators of $H_{2d-2}(E)$; and similarly for $c_{\overline{E}}$). Thus μ_*^E and ν_*^E have isomorphic cokernels; and so by (3.3.1) and (3.4.4), the cokernels of μ_*^R and ν_*^X are naturally isomorphic. The same is true of the kernels of μ_*^R and ν_*^X, both of these maps being injective (3.3.2), (3.4.2). It follows formally that the maps $c_{\overline{X}}$ and c_X in (3.5.2) have isomorphic kernels and cokernels. Applying $\varinjlim_{\varepsilon}$, we get (3.2)'. Q.E.D.

(3.6) It remains to prove (3.4.3).

With the notation of (3.4), let S be the singular locus of the hypersurface

$$D_\varepsilon^* = D^* \cap P_\varepsilon = \{(y_1,\dots,y_d) \in \mathbb{C}^d \mid |y_i| < \varepsilon \text{ for all } i, \text{ and } y_i = 0 \text{ for some } i\}.$$

Set

$$E^0 = \pi^{-1}(D_\varepsilon^* - S) \subset E_\varepsilon = E$$

$$\overline{E}^0 = \nu^{-1}(E^0) \subset \overline{E}_\varepsilon = \overline{E}$$

(cf.(3.4)) and let

$$\nu^0 : \overline{E}^0 \to E^0$$

be the map induced by ν. Since the variety $E - E^0 = \pi^{-1}(S)$ has complex dimension $\leq d-2$, hence cohomological dimension $\leq 2d-4$ [BH, 1.10, 3.1], and similarly for $\overline{E} - \overline{E}^0 = \nu^{-1}\pi^{-1}(S)$, we have, by [*ibid*, 1.5, 1.6] a natural commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 = H_{2d-3}(\overline{E}-\overline{E}^0) & \longrightarrow & H_{2d-3}(\overline{E}) & \longrightarrow & H_{2d-3}(\overline{E}^0) \\ & & \downarrow \nu_* & & \downarrow \nu_*^0 \\ 0 = H_{2d-3}(E-E^0) & \longrightarrow & H_{2d-3}(E) & \longrightarrow & H_{2d-3}(E^0). \end{array} \qquad (3.6.1) \qquad (H_\bullet = H_\bullet^{BM})$$

So it will be enough to show that ν_*^0 *is injective.*

For this purpose we will need:

Lemma (3.6.2). *For each connected component* D_i $(1 \le i \le d)$ *of the smooth analytic space* $D_\varepsilon^* - S$, *the spaces*

$$E_i = \pi^{-1}(D_i) \quad \textit{and} \quad \overline{E}_i = \nu^{-1}\pi^{-1}(D_i)$$

are connected, and their normalizations are étale coverings of D_i.

Proof of (3.6.2). The closure of D_i is an irreducible component of D_ε^*. As observed (algebraically, hence for small ε, geometrically) near the beginning of (3.3), the irreducible components of D_ε^* (a subvariety of $\mathbb{C}^d$ whose algebraic germ at 0 is the scheme F in (3.3)) are in one-one correspondence, under π, with those of E_ε. It follows that E_i is the complement of $\pi^{-1}(S)$ in some irreducible component of E_ε, so that E_i is indeed connected. Similarly $\overline{E}_i$ is connected.

For the last assertion, let $\nu_i\colon E_i^! \longrightarrow E_i$ be the normalization map, let $y_0 \in E_i^!$ and let $x_0 = \nu_i(y_0)$. Then ν_i maps some neighborhood W of y_0 in $E_i^!$ onto an irreducible $(d-1)$-dimensional variety $E_i^* \subset E_i$; and if we can show that $\pi\colon E_i^* \longrightarrow D_i$ is étale, then it will follow that E_i^* is normal, whence $W \longrightarrow E_i^*$ is an isomorphism and $\pi\circ\nu_i\colon E_i^! \longrightarrow D_i$ is étale at y_0, as desired.

Let X_0 be a local component of X at x_0 containing E_i^*. Then (X_0 being taken sufficiently small) π induces a branched covering map π_0 of X_0 onto an open set U in $\mathbb{C}^d$, with discriminant locus D_0 satisfying

$$D_0 \subset D_\varepsilon^* \cap U \subset D_i .$$

For convenience, we may assume that D_i is defined by the equation $T_1 = 0$. As in (3.1.1), the local ring R_0 of (X_0,x_0) is then isomorphic to a ring of the form

$$\mathbb{C}\langle T_1,\dots,T_d\rangle[\varphi_1(T_1^{1/n},T_2,\dots,T_d),\dots,\varphi_e(T_1^{1/n},T_2,\dots,T_d)]$$

with convergent power series φ_i vanishing at 0. It follows that (X_0,x_0) is isomorphic to a germ of the form $(\varphi(V),0)$, where V is a neighborhood of 0 in $\mathbb{C}^d$ on which the φ_i converge, and $\varphi\colon V \to \mathbb{C}^{d+e}$ is the map given by

$$\varphi(s_1,\dots,s_d) = (s_1^n,s_2,\dots,s_d,\varphi_1(s_1,\dots,s_d),\dots,\varphi_e(s_1,\dots,s_d));$$

and furthermore π_0 is then identified germwise with the map

$$(\varphi(V),0) \longrightarrow (\mathbb{C}^d,0)$$

given by projection onto the first d coordinates in $\mathbb{C}^{d+e}$. Now $\pi_0^{-1}(D_i)$ consists of points in $\varphi(V)$ of the form

$$(0,s_2,\dots,s_d,\varphi_1(0,s_2,\dots,s_d),\dots,\varphi_e(0,s_2,\dots,s_d)).$$

These points form an irreducible subvariety of $\varphi(V)$, mapped *isomorphically* onto D_i by π_0. Since E_i^* is a component of E_i at x_0, and

$$E_i^* \subset \pi_0^{-1}(D_i) \subset E_i$$

it follows that $E_i^* = \pi_0^{-1}(D_i)$, and so $\pi\colon E_i^* \to D_i$ is indeed étale around x_0.

A similar proof applies to $\overline{E}_i$. (Actually, since $\overline{X}$ is everywhere irreducible, the preceding proof shows that $\overline{E}_i$ itself is étale over D_i, hence normal.)

Q.E.D.

Now ν_*^0 in (3.6.1) is the direct sum of the induced maps

(3.6.3) $$\nu_*^i\colon H_{2d-3}(\overline{E}_i) \longrightarrow H_{2d-3}(E_i)$$

(cf. [BH, p.464, 1.3 or p.466, 1.8]). So if we can show that each ν_*^i is injective, then we will be done.

Again, let $\nu_i\colon E_i' \to E_i$ be normalization, which is an isomorphism outside some (d-2)-dimensional subvariety C_i of E_i. As in (3.6.1), we have a commutative diagram with exact rows:

$$\begin{array}{ccccc}
0 = H_{2d-3}(\nu_i^{-1}(C_i)) & \longrightarrow & H_{2d-3}(E_i') & \longrightarrow & H_{2d-3}(E_i'-\nu_i^{-1}(C_i)) \\
\downarrow & & \downarrow \nu_{i*} & & \| \\
0 = H_{2d-3}(C_i) & \longrightarrow & H_{2d-3}(E_i) & \longrightarrow & H_{2d-3}(E_i-C_i)
\end{array}$$

which shows that ν_{i*} is injective. As noted at the very end of the proof of (3.6.2), $\overline{E}_i$ is normal, so we have a map $\nu'\colon \overline{E}_i \to E_i'$, and the map ν_*^i in (3.6.3) factors naturally as

$$H_{2d-3}(\overline{E}_i) \xrightarrow[\nu_*']{} H_{2d-3}(E_i') \xrightarrow[\nu_{i*}]{} H_{2d-3}(E_i).$$

Thus, to show that ν_*' is injective, it will be enough to prove:

Lemma (3.6.4). *The abelian groups* $H_{2d-3}(\overline{E}_i)$ *and* $H_{2d-3}(E_i')$ *are both free, of the same finite rank; and the cokernel of* ν_*' *is annihilated by some* $n > 0$.

Proof. Recall that cap product with the fundamental class $[E_i']$ gives the Poincaré-duality isomorphism

$$(3.6.5) \qquad H^1(E_i', \mathbb{Z}) \xrightarrow{\sim} H_{2d-3}(E_i')$$

[BH, p.468, 1.12]. For any $\alpha \in H_{2d-3}(E_i')$, there is then a $\beta \in H^1(E_i', \mathbb{Z})$ such that

$$\alpha = [E_i'] \cap \beta;$$

and the *projection formula* [BH, p.507, Th.7.5] says that

$$\nu_*'([\overline{E}_i] \cap \nu'^*\beta) = n[E_i'] \cap \beta = n\alpha$$

where $n > 0$ is the covering degree of $\overline{E}_i$ over E_i' (cf. [BH, p.486,4.11], or just use a diagram analogous to (1.6.1)). This shows that the cokernel of ν_*' is annihilated by n.

Finally, D_i is a $(d-1)$-dimensional polydisc minus the union of its coordinate hyperplanes, i.e. a product of $d-1$ punctured discs, and by (3.6.2), E_i' is a connected covering space of D_i. Hence the fundamental group $\pi_1(E_i')$ is a free abelian group, of rank $d-1$, and so therefore is

$$H_{2d-3}(E_i') \cong H^1(E_i', \mathbb{Z}) \cong \mathrm{Hom}(H_1(E_i'), \mathbb{Z}) = \mathrm{Hom}(\pi_1(E_i'), \mathbb{Z})$$

(the first isomorphism by (3.6.5), and the second by "universal coefficients"). A similar argument shows that also $H_{2d-3}(\overline{E}_i)$ is free, of rank $d-1$.

Q.E.D.

§4 Presentation of the group $A_{d-1} \cong H_{2d-2}$

Let (X,x) be quasi-ordinary, of dimension d, and let R be the local ring of (X,x). In this section we describe an explicit presentation of $A_{d-1}(X)_x \cong H_{2d-2}(X)_x$ (cf. Theorem (3.2)), in terms of a fixed quasi-ordinary projection $\pi:(X,x) \to (\mathbb{C}^d,0)$.

By (3.1.1), corresponding to π there is an integer n together with inclusions

$$\mathbb{C}\langle T_1,\dots,T_d\rangle \subset R \subset \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle = \text{(say) } B$$

such that $R_T = R[1/T_1T_2\dots T_d]$ is étale over $\mathbb{C}\langle T_1,\dots,T_d\rangle_T$ (whence R_T is locally regular). Let

$$p_i = (T_i^{1/n}B) \cap R \qquad\qquad (1 \le i \le d).$$

Any height one prime p in R containing T_i is the intersection of R with a height one prime in B containing T_i ("going up" theorem), i.e. $p = p_i$; consequently $\sqrt{T_iR} = p_i$ is a height one prime ideal in R, and for some integer m_i $(=\text{ord}_{p_i}(T_i))$ we have

$$[\text{div}(T_i)] = m_i p_i.$$

Let

$$\{p_i\} \in A_{d-1}(X)_x = A_{d-1}(\text{Spec}(R))$$

be the rational equivalence class of p_i.

Theorem (4.1). *With preceding notation, the abelian group*

$$A_{d-1}(X)_x \cong H_{2d-2}(X)_x$$

is generated by $\{p_1\}, \{p_2\}, \dots, \{p_d\}$, *subject to the relations given by*

(4.1.1) $$\sum_{i=1}^{d} \alpha_i m_i \{p_i\} = 0$$

as $\alpha = (\alpha_1, \ldots, \alpha_d)$ *runs through the subgroup* Γ *of* $((1/n)\,\mathbb{Z})^d$ *consisting of all* d-*tuples such that* $T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_d^{\alpha_d} \in K$, *the fraction field of* R.

Remarks (i) If $T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_d^{\alpha_d} \in K$, then $\alpha_i m_i$ is an *integer*; in fact

$$\sum_{i=1}^{d} \alpha_i m_i p_i = [\operatorname{div}(T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_d^{\alpha_d})].$$

(To check this equality, multiply both sides by n...).

(ii) We have

$$\mathbb{Z}^d \subset \Gamma \subset ((1/n)\,\mathbb{Z})^d$$

so that Γ is free, of rank d. According to Theorem (4.1), any free basis of Γ leads, via (4.1.1), to a presentation of $A_{d-1}(X)_x$ by a non-singular d×d matrix.

(iii) Denote by $L_n (n \geq 1)$ the fraction field of $\mathbb{C}\langle T_1^{1/n}, \ldots, T_d^{1/n}\rangle$. A basis of the L_1-vector space L_n is given by the monomials

$$T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_d^{\alpha_d}$$

with $\alpha_i \in (1/n)\,\mathbb{Z}$, $0 \leq \alpha_i < 1$, $(1 \leq i \leq d)$. If, as before, $L_1 \subset K \subset L_n$, then these monomials are clearly eigenvectors for any L_1-automorphism of L_n. It follows easily that K (which is the L_1-vector subspace of L_n consisting of all elements invariant under the Galois group of L_n over K) has as basis all such T^α with $\alpha \in \Gamma$. Hence

(4.1.2) $$[\Gamma : \mathbb{Z}^d] = [K : L_1].$$

(Alternatively one can prove (4.1.2) by checking that, with G the Galois group of K over L_1:

$$\langle g, \alpha + \mathbb{Z}^d \rangle = (gT^\alpha)/T^\alpha \qquad (g \in G,\ \alpha \in \Gamma)$$

defines a non-singular pairing $G \times (\Gamma/\mathbb{Z}^d) \to \mathbb{C}^*$.)

Corollary (4.1.3). *The order of* $A_{d-1}(X)_x$ *is*

$$m_1m_2\dots m_d/[\Gamma:\mathbb{Z}^d] = m_1m_2\dots m_d/[R:\mathbb{C}\langle T_1,\dots,T_d\rangle].$$

Proof. By remark (ii), the order of $A_{d-1}(X)_x$ is

$$m_1m_2\dots m_d\cdot\det(D) \qquad (\text{"det"} = \text{"determinant"})$$

where D is a $d\times d$ matrix whose rows form a basis of Γ. But the matrix nD gives a presentation of the quotient group $((1/n)\mathbb{Z})^d/\Gamma$. Hence

$$n^d\cdot\det(D) = \det(nD) = [((1/n)\mathbb{Z})^d:\Gamma]$$
$$= n^d/[\Gamma:\mathbb{Z}^d].$$

The conclusion follows.

* * *

(4.2) For the *proof of (4.1)*, we first show that the $\{p_i\}$ $(1\le i\le d)$ generate $A_{d-1}(X)_x$.

Since $p_1, p_2,\dots, p_d$ are the only height one primes in R containing the element $T = T_1T_2\dots T_d$, we need only show that the localization $R_T = R[1/T]$ is a unique factorization domain (U.F.D.). (As noted above, every localization of R_T at a prime ideal is regular, so R is at least locally a U.F.D.)

Set

$$t_i = T_i^{1/n} \qquad (1\le i\le d)$$

and

$$B = \mathbb{C}\langle t_1,t_2,\dots,t_d\rangle \supset R.$$

Let G be the Galois group of L/K, where L (resp. K) is the fraction field of B (resp. R). Since R_T, being locally regular, is normal, therefore R_T is the ring of G-invariant elements in

$$B_T = B_t = B[1/t].$$

But B_t is a U.F.D., and so by Samuel's "Galois descent" theorem [Fo, p.82, Theorem 16.1] it will suffice to show that

$$H^1(G,B_t^*) = 0 \qquad (4.2.1)$$

where B_t^* is the group of units in B_t, acted on by G in the obvious way. This will be done by elaborating the standard proof of "Noether's theorem": $H^1(G,L^*) = 0$ (cf. e.g. [S, Ch. X, §1, Prop. 2]).

We may just as well proceed without explicit reference to cohomology: if I is any divisorial ideal in R_T, we need to show that I *is a principal ideal*. Since B_t is a U.F.D., there exists $y \in B_t$ such that

$$\text{ord}_p(I) = \text{ord}_p(y)$$

for all height one primes p in B_t; and hence for any $g \in G$ we have

$$\text{ord}_p(g(y)) = \text{ord}_p(g(I)) = \text{ord}_p(I) = \text{ord}_p(y)$$

whence:

$$g(y) = y\beta_g \qquad (\beta_g \in B_t^*); \qquad (4.2.2)$$

and since, for $h \in G$,

$$y\beta_{gh} = gh(y) = g(y\beta_h) = g(y)g(\beta_h) = y\beta_g \cdot g(\beta_h)$$

therefore

$$\beta_{gh} = \beta_g \cdot g(\beta_h) \qquad (g,h \in G). \qquad (4.2.3)$$

If we can deduce from (4.2.3) the existence of $\beta \in B_t^*$ such that

$$(4.2.4) \qquad g(\beta) = \beta_g^{-1}\beta \qquad \text{(for all } g \in G)$$

[i.e. if we can prove (4.2.1)], then

$$g(y\beta) = g(y)g(\beta) = y\beta_g\beta_g^{-1}\beta = y\beta,$$

whence $y\beta \in R_T$; and since for all p as above

$$\mathrm{ord}_p(y\beta) = \mathrm{ord}_p(y) = \mathrm{ord}_p(I)$$

it follows easily that $I = y\beta R_T$, as desired.

So let us find $\beta \in B_t^*$ satisfying (4.2.4).

It is easily verified that $b \in B_t^*$ if and only if b has the form

$$b = t_1^{n_1}t_2^{n_2}\ldots t_d^{n_d}u \qquad (n_i \in \mathbb{Z};\ u \in B^*)$$

(B^* = {units in B}). So, in (4.2.2), we can set

$$\beta_g = \underset{\sim}{t}^{N_g}u_g \qquad (u_g \in B^*)$$

with $N_g = (n_1,\ldots,n_d)$ (the n_i depending on g) and

$$\underset{\sim}{t}^{N_g} = t_1^{n_1}\ldots t_d^{n_d}.$$

Now for any $g \in G$, $g(t_i^n) = t_i^n$, so $g(t_i) = t_i\varepsilon_{ig}$, where ε_{ig} is some n-th root of unity in $\mathbb{C}$, and

$$g(\beta_h) = g(\underset{\sim}{t}^{N_h})g(u_h) \in \beta_h B^*,$$

whence, by (4.2.3):

$$\beta_{gh} \in \beta_g\beta_h B^*$$

from which it follows that

$$N_{gh} = N_g + N_h;$$

and since G is finite, there is no non-trivial group homomorphism from G into $\mathbb{Z}^d$, so we conclude that

$$N_g = (0,0,\dots,0) \qquad \text{(for all } g \in G).$$

Thus $\beta_g \in B^*$; and there exists $b_g \neq 0$ in $\mathbb{C}$ such that $\beta_g - b_g$ is a non-unit in B (b_g is just the "constant term" of β_g).

It will be enough to show that for some monomial

$$\mu = t_1^{n_1} t_2^{n_2} \dots t_d^{n_d}, \qquad (n_i \in \mathbb{Z})$$

the element

$$\beta = \sum_{h \in G} \beta_h \cdot h(\mu)$$

lies in B_t^*; for then

$$g(\beta) = \sum_h g(\beta_h) gh(\mu) = \text{(by (4.2.3))} \sum_h \beta_g^{-1} \beta_{gh} gh(\mu) = \beta_g^{-1} \beta$$

which is the desired relation (4.2.4).

But L is generated as a K-vector space by monomials in the t's. Hence by the linear independence of K-automorphisms of L [La, Ch.VIII, §4] there exists such a monomial, say μ, such that

$$\sum_{h \in G} b_h \cdot h(\mu) \neq 0.$$

As above, $h(\mu^n) = \mu^n$ implies that

$$h(\mu) = \mu \varepsilon_h \qquad (\varepsilon_h \in \mathbb{C}),$$

and so

$$0 \neq \sum_{h \in G} b_h \varepsilon_h \in \mathbb{C}.$$

Thus (since $\beta_h - b_h$ is a non-unit in B):

$$\sum_{h\in G} \beta_h \varepsilon_h \in B^*$$

and

$$\sum_{h\in G} \beta_h \cdot h(\mu) = \sum_{h\in G} \beta_h \varepsilon_h \mu \in B^*\mu \subset B_t^*. \qquad \text{Q.E.D.}$$

(4.3) We prove now that the relations on the generators $\{p_1\},\ldots,\{p_d\}$ are as specified in Theorem (4.1).

By the first remark following (4.1), the relations (4.1.1) do hold. The following Lemma shows that conversely every relation on the $\{p_i\}$ is of the form (4.1.1), thereby completing the proof of (4.1).

Lemma. If $f \in K$ is such that

$$\sum_{i=1}^{d} a_i p_i = [\mathrm{div}(f)] \qquad (a_i \in \mathbb{Z})$$

then

$$f = u\, T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_d^{\alpha_d}$$

where u is a unit in the integral closure $\overline{R}$ of R in K, and $(\alpha_1,\ldots,\alpha_d) \in \Gamma$. Moreover,

(4.3.1) $$\mathrm{ord}_{p_i}(f) = a_i = \alpha_i m_i.$$

Proof. As before, with $T = T_1 T_2 \ldots T_d$, the localization R_T is normal, and

$$p_i R_T = R_T \qquad (1 \le i \le d),$$

so that f is a unit in R_T, hence in B_T, where

$$B = \mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle.$$

Thus

$$f = u\, T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_d^{\alpha_d}$$

where u is a unit in B, and $na_i \in \mathbb{Z}$ $(1 \le i \le d)$. We can write

$$u = c_0 + \sum_\gamma c_\gamma T^\gamma \qquad (c_0, c_\gamma \in \mathbb{C},\ c_0 \neq 0)$$

where $\gamma = (\gamma_1, \ldots, \gamma_d)$ runs through all d-tuples $\neq (0,0,\ldots,0)$ with

$$0 \le \gamma_i \in (1/n)\,\mathbb{Z} \qquad (1 \le i \le d),$$

and

$$T^\gamma = T_1^{\gamma_1} \cdots T_d^{\gamma_d}.$$

If L is the fraction field of B, and G is the Galois group of L/K, then for all $g \in G$, we have (with $\alpha = (\alpha_1, \ldots, \alpha_d)$):

$$c_0 T^\alpha + \sum_\gamma c_\gamma T^{\gamma+\alpha} = f = g(f) = c_0 g(T^\alpha) + \sum_\gamma c_\gamma g(T^{\gamma+\alpha})$$

$$= c_0 \varepsilon_0 T^\alpha + \sum_\gamma c_\gamma \varepsilon_\gamma T^{\gamma+\alpha}$$

where ε_0, ε_γ are suitable n-th roots of unity in $\mathbb{C}$. Hence $\varepsilon_0 = 1$, i.e. $g(T^\alpha) = T^\alpha$, and so $T^\alpha \in K$; and

$$u = f/T^\alpha \in B \cap K = \overline{R}$$

$$1/u = T^\alpha/f \in B \cap K = \overline{R},$$

i.e. u is a unit in $\overline{R}$.

Finally, for (4.3.1), we just need

$$\operatorname{ord}_{p_i}(u) = 0 \qquad 1 \le i \le d.$$

But since u is a unit in $\overline{R}$, this follows at once from [$\underset{\sim}{F}$, p.412, Ex.A.3.1].

Q.E.D.

To apply Theorem (4.1), it is useful to know more about the integers $m_i = \mathrm{ord}_{p_i}(T_i)$. The following examples (4.4) and (4.5) provide such information.

Example (4.4). Suppose that R is *normal*. Then, with Γ as in (4.1), if π_i is the projection of $((1/n)\mathbb{Z})^d$ onto its i-th factor, we have

$$\pi_i(\Gamma) = (1/m_i)\,\mathbb{Z}.$$

Proof. Clearly

$$\mathbb{Z} \subset \pi_i(\Gamma) \subset (1/n)\,\mathbb{Z}$$

so we can write

(4.4.1) $$\pi_i(\Gamma) = (1/n_i)\,\mathbb{Z} \qquad (n_i \in \mathbb{Z},\ n/n_i \in \mathbb{Z})$$

and we can find a d-tuple

$$(\gamma_1,\ldots,\gamma_{i-1},1/n_i,\gamma_{i+1},\ldots,\gamma_d) \in \Gamma$$

with $\gamma_j \geq 0$ for all $j \neq i$. Then

$$\tau = T_1^{\gamma_1}\ldots T_i^{1/n_i}\ldots T_d^{\gamma_d} \in K$$

whence

$$m_i = \mathrm{ord}_{p_i}(T_i) = n_i\ \mathrm{ord}_{p_i}(\tau)$$

and so

(4.4.2) $$n_i \textit{ divides } m_i \qquad (1 \leq i \leq d).$$

Conversely, with notation as in the proof of (4.3), consider an element

$$v = \sum_{\gamma} c_\gamma T^\gamma \in B.$$

Since R is normal, we have $R = B \cap K$, and so $v \in R$ if and only if v is invariant under the Galois group G of L/K. But for any $g \in G$, we have

$$g(v) = \sum_{\gamma} c_{\gamma}\varepsilon_{g\gamma}T^{\gamma}$$

where $\varepsilon_{g\gamma}^{n} = 1$; whence $v \in R$ if and only if $\varepsilon_{g\gamma} = 1$ for all g whenever $c_{\gamma} \neq 0$. Moreover

$$\{\varepsilon_{g\gamma} = 1 \text{ for all } g\} \Leftrightarrow \{T^{\gamma} \text{ is G-invariant}\} \Leftrightarrow \{\gamma \in \Gamma\}.$$

So $v \in R$ if and only if v is of the form

$$v = \sum_{\gamma\in\Gamma} c_{\gamma}T^{\gamma}.$$

Thus we see that

$$(4.4.3) \qquad R \subset \mathbb{C}\langle T_1^{1/n_1}, T_2^{1/n_2}, \ldots, T_d^{1/n_d}\rangle = (\text{say})B_n;$$

and it follows easily (by considering the respective ramification indices of T_i in R and in B_n) that

$$(4.4.4) \qquad m_i \textit{ divides } n_i \qquad (1 \le i \le d).$$

By (4.4.1), (4.4.2), (4.4.4), the proof is complete.

Example (4.5). No longer assuming R to be normal, let $\overline{R}$ be the integral closure of R in its fraction field K. As before, let $p_i \subset R$ (resp. $\bar{p}_i \subset \overline{R}$) be the unique height one prime ideal containing T_i, and let m_i (resp. n_i) be the corresponding order of T_i. By example (4.4), we have

$$\pi_i(\Gamma) = (1/n_i)\mathbb{Z}$$

and (cf.(4.4.2), whose proof doesn't use normality)

$$m_i = e_i n_i \qquad e_i \in \mathbb{Z}.$$

In fact e_i is just the relative degree

$$e_i = [\bar{R}/\bar{p}_i : R/p_i]$$

(i.e. the degree of the corresponding extension of fraction fields) cf. [F, p.412, Ex.A.3.1]].

Now if L_1 is the fraction field of $\mathbb{C}\langle T_1,\dots,T_d\rangle$, then we have the well-known ramification relation

$$(4.5.1)\qquad n_i[\bar{R}/\bar{p}_i : \mathbb{C}\langle T_1,\dots,T_{i-1},T_{i+1},\dots,T_d\rangle] = [K:L_1]$$

(which can also be deduced from (4.1.2) and (4.4)) and it follows that

$$(4.5.2)\qquad m_i[R/p_i : \mathbb{C}\langle T_1,\dots,T_{i-1},T_{i+1},\dots,T_d\rangle] = [K:L_1].$$

The integers m_i, n_i can also be interpreted geometrically as follows. The *branching order (or degree) of* π *at a point* $y \in X$, denoted by $\deg(\pi_y)$, is the integer d such that the germ of π at y is a d-sheeted branched covering. (In other words there is a neighborhood U of $\pi(y) \in \mathbb{C}^d$ and a neighborhood V of $y \in X$ such that π makes $V - \pi^{-1}(D)$ (D = discriminant locus of π) into a d-sheeted étale covering of $U - D$.) For example

$$(4.5.3)\qquad \deg(\pi_x) = [K:L_1].$$

(*Idea of proof*: There are non-zero elements r, $t_0 \in R$ such that

$$rR \subset R_0 = \mathbb{C}\langle T_1,\dots,T_d\rangle[t_0] = \mathbb{C}\langle T_1,\dots,T_d\rangle[T]/F(T)$$

where $F(T)$ is a monic polynomial; by ignoring the zero-set of r, one reduces to the case $R = R_0$ ...)

Proposition (4.5.4). *Let* $\nu:(\bar{X},\bar{x}) \to (X,x)$ *be the normalization map, and set*

$$\bar{\pi} = \pi\circ\nu : (\bar{X},\bar{x}) \longrightarrow (\mathbb{C}^d,0).$$

Let $Z_i \subset X$ *(resp.* $\bar{Z}_i \subset \bar{X}$*) be the zero set of* T_i *(i.e. of* p_i *(resp.* $\bar{p}_i$*)). Then for generic* $z \in Z_i$ *and* $\bar{z} \in \bar{Z}_i$ *we have*

(i) $m_i = \deg(\pi_z)$

(ii) $n_i = \deg(\bar{\pi}_{\bar{z}})$

and

(iii) $e_i = m_i/n_i$ = {*number of irreducible components of the germ* (X,z)}.

Proof. We first prove (iii). For any $z \in Z_i$, the number of components of (X,z) is the cardinality of the fibre $\nu^{-1}(z)$; and for almost all such z, this cardinality is $[\bar{R}/\bar{p}_i : R/p_i] = e_i$ (cf. (4.5.3), as applied to $\nu: \bar{Z}_i \to Z_i$).

Next, for almost all points $\xi \in \mathbb{C}^d$ where $T_i = 0$, the cardinality of the fibre $\bar{\pi}^{-1}(\xi)$ is

$$\begin{aligned}\#(\bar{\pi}^{-1}(\xi)) &= [\bar{R}/\bar{p}_i : \mathbb{C}\langle T_1,\ldots,T_{i-1},T_{i+1},\ldots,T_d\rangle] \\ &= [K:L_1]/n_i \qquad\qquad (\text{cf.}(4.5.1)) \\ &= \#(\bar{\pi}^{-1}(\xi'))/n_i\end{aligned}$$

for any $\xi' \notin D$. Let $\bar{z} \in \bar{\pi}^{-1}(\xi)$. As ξ' approaches ξ, exactly $\deg(\bar{\pi}_{\bar{z}})$ points of the fibre $\bar{\pi}^{-1}(\xi')$ approach $\bar{z}$. So to prove (ii), we need only show that

(*): $\deg(\bar{\pi}_{\bar{z}})$ *has the same value for all* $\bar{z} \in \bar{\pi}^{-1}(\xi)$;

for then that value is necessarily

$$\#(\bar{\pi}^{-1}(\xi'))/\#(\bar{\pi}^{-1}(\xi)) = n_i.$$

But assertion (*) follows at once from the fact that the Galois group G of K/L_1 acts on $\bar{X}$, compatibly with $\bar{\pi}$, the action being transitive on the fibres of $\bar{\pi}$.

[For lack of a convenient reference, we sketch a proof. By the well-known equivalence of the categories of analytic local rings and of analytic germs, each element g of G induces a map, commuting with $\bar{\pi}$, from a neighborhood U_g of $\bar{x}$ in $\bar{X}$ onto some other neighborhood. Setting $U = \cap_{g\in G} U_g$, and $U^* = \cap_{g\in G}(gU)$, we find that every g maps U^* onto U^*, so that, after shrinking $\bar{X}$ to U^*, we can say that G acts on $\bar{X}$, and denote by $\bar{g}:\bar{X}\to\bar{X}$ the map induced by $g \in G$. Now for any $y \in \bar{X}$ we can find a function φ on $\bar{X}$ vanishing at y but not vanishing at any other point of $\bar{\pi}^{-1}\bar{\pi}(y)$ - for example, after embedding $\bar{X}$ into some $\mathbb{C}^N$, we can take φ to be a suitable linear function. But then the function $\Pi_{g\in G}\, \varphi\circ\bar{g}$ is G-invariant, hence is of the form $\psi\circ\bar{\pi}$ for some function (germ) ψ on $\mathbb{C}^d$, and hence has constant value - necessarily zero - on $\bar{\pi}^{-1}\bar{\pi}(y)$. Consequently for each $y' \in \bar{\pi}^{-1}\bar{\pi}(g)$ there must be a g such that $\bar{g}(y') = y$; and so transitivity holds as asserted.]

Finally, for $z \in Z_i$ such that $\pi(z) = \xi$ with ξ as above, and such that $\#(\nu^{-1}(z)) = e_i$, there is a neighborhood U of z such that $\nu^{-1}(U)$ consists of e_i disjoint open sets, each of which ν maps onto a different component of U, bijectively outside $\pi^{-1}(D)$. Since, as just seen, $\deg(\bar{\pi}.)$ is constant $(=n_i)$ along $\nu^{-1}(z)$, it follows easily that

$$\deg(\pi_z) = e_i \deg(\bar{\pi}_\bullet) = e_i n_i = m_i,$$

and (i) is proved. Q.E.D.

Remark (4.6). We can interpret (4.1) via "Kummer theory", as follows.

Let m be the order of $A = A_{d-1}(\mathrm{Spec}(R))$. For each divisor (= codimension-one cycle) D in $\mathrm{Spec}(R)$ we can write

$$mD = [\mathrm{div}(x_D)]$$

for some $x_D \in K^* = K - \{0\}$. Using the fact that units in the integral closure $\overline{R}$ have m-th roots in $\overline{R}$, we find that the element x_D is uniquely determined modulo K^{*m}, and there results a *natural monomorphism* of groups

$$\theta\colon A \hookrightarrow K^*/K^{*m}.$$

The corresponding Kummer extension $K_{\theta(A)}$, obtained by adjoining to K all elements of the form $x_D^{1/m}$, is an abelian extension of K, with Galois group canonically dual to A (cf. [La, Chap.VIII, §8]). It can be shown (using (4.2), (4.4.2) - whose proof doesn't need normality - and (4.4.3)) that *the integral closure of* R *in* $K_{\theta(A)}$ *is* $\mathbb{C}\langle T_1^{1/m_1},\ldots,T_d^{1/m_d}\rangle$. Thus the ring extension

$$R \subset \mathbb{C}\langle T_1^{1/m_1},\ldots,T_d^{1/m_d}\rangle = (\text{say})\tilde{R}$$

depends only on R (not on the chosen monomorphism $\mathbb{C}\langle T_1,\ldots,T_d\rangle \hookrightarrow R$); and it is easily seen that the group given by the generators and relations of Theorem (4.1) is dual to the Galois group of $\tilde{R}$ over R, hence isomorphic to A.

In particular, A *is trivial if and only if* $\tilde{R}$ *is the normalization of* R *in* K.

II. THE HYPERSURFACE CASE

§5. Characteristic monomials of quasi-ordinary parametrizations

From now on we will be mainly concerned with d-dimensional irreducible germs $(X,x) \subset (\mathbb{C}^{d+1},0)$ admitting a quasi-ordinary projection $\pi:(X,x) \to (\mathbb{C}^d,0)$ which induces a surjection of Zariski tangent spaces. Such a hypersurface germ can be parametrized by a special type of fractional convergent power series in d variables, called a *quasi-ordinary branch* (cf.(5.3)). For $d = 1$, this is just the classical Puiseux parametrization of a plane curve singularity. Any quasi-ordinary branch ζ comes equipped with *characteristic monomials*, which are natural higher-dimensional generalizations of the "characteristic pairs" in the plane curve case. These monomials control much of the geometry of the germ (X,x) parametrized by ζ. In particular, we use them in (5.9) and (5.10) to compute $A_{d-1}(X)_x = H_{2d-2}(X)_x$.

So, let (X,x) be quasi-ordinary, with local ring R, an integral domain of dimension d; and let $\mathbb{C}\langle T_1,\dots,T_d\rangle \subset R$ be a finite ring extension with normal crossing discriminant (corresponding to some quasi-ordinary π, cf. (3.1)). We suppose further that *the maximal ideal of* R *is generated by* $(T_1,\dots,T_d,\zeta)$ *for some* $\zeta \in R$. (In other words, π induces a surjective map on Zariski tangent spaces.)

Then there is a surjective $\mathbb{C}\langle T_1,\dots,T_d\rangle$-algebra homomorphism

$$\psi:\mathbb{C}\langle T_1,\dots,T_d,Z\rangle \to R \qquad \psi(Z) = \zeta.$$

The kernel of ψ is a height one prime (since $\dim R = d$), hence generated by a single irreducible element $F(T_1,\dots,T_d,Z)$; and since R is finite over $\mathbb{C}\langle T_1,\dots,T_d\rangle$, therefore

$$\mathbb{C}\langle Z\rangle/F(0,0,\dots,0,Z) \cong R/(T_1,\dots,T_d)R$$

is a finite-dimensional $\mathbb{C}$-vector space, i.e.

$$F(0,0,\dots,0,Z) \neq 0.$$

By the Weierstrass preparation theorem, we may assume (after multiplying F by a unit) that

$$F(T_1,\dots,T_d,Z) = Z^m + f_1(T_1,\dots,T_d)Z^{m-1} + \dots + f_m(T_1,\dots,T_d)$$

is a polynomial in Z, whose coefficients are non-units in $\mathbb{C}\langle T_1,\dots,T_d\rangle$. Then ζ is a root of this polynomial, and we have

$$R \cong \mathbb{C}\langle T_1,\dots,T_d,Z\rangle/(F) \cong \mathbb{C}\langle T_1,\dots,T_d\rangle[\zeta].$$

We may assume, moreover, that the *discriminant* Δ of the polynomial F is of the form

$$\Delta = T_1^{e_1}T_2^{e_2}\dots T_d^{e_d}\,\varepsilon(T_1,T_2,\dots,T_d) \qquad \varepsilon(0,0,\dots,0) \neq 0.$$

As in §3, we have, for some $n > 0$, an injective homomorphism

$$R \hookrightarrow \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle^{(1)}$$

so that for some convergent power series H vanishing at 0, we can make the identification

$$\zeta = H(T_1^{1/n},T_2^{1/n},\dots,T_d^{1/n}).$$

The fraction field of $\mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle$ is a finite Galois extension of the fraction field of $\mathbb{C}\langle T_1,\dots,T_d\rangle$, and so $F = F(Z)$ has m distinct roots

$$\zeta_1 = \zeta,\ \zeta_2,\dots,\zeta_m \in \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle.$$

---

(1) In fact by (4.4.3) — where (by (4.4.2) and (4.4.4) and the definition of m_i at the beginning of §4) $n_i = v_i(T_i)$ is the ramification index of the unique prime divisor v_i of the normalization $\overline{R}$ such that $v_i(T_i) > 0$, so that n_i divides m — we can take $n = m$.

In fact the ζ_j, being the conjugates of ζ over $\mathbb{C}\langle T_1,\ldots,T_d\rangle$, are of the form

$$\zeta_j = H(\omega_{1j}T_1^{1/n},\ldots,\omega_{dj}T_d^{1/n})$$

for suitable complex n-th roots of unity ω_{ij} (cf.(5.5.1) below). Now

$$\prod_{i\neq j}(\zeta_i-\zeta_j) = \Delta = T_1^{e_1}\ldots T_d^{e_d}\varepsilon$$

with ε a unit in the unique factorization domain $\mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle$, and therefore if $i \neq j$, we have

$$\zeta_i - \zeta_j = M_{ij}\varepsilon_{ij}(T_1^{1/n},\ldots,T_d^{1/n}) \qquad \varepsilon_{ij}(0,\ldots,0) \neq 0$$

where

(5.1) $$M_{ij} = T_1^{a_1/n}T_2^{a_2/n}\ldots T_d^{a_d/n},$$

with integers $a_\ell \geq 0$ depending on (i,j). (Note that $\Sigma a_\ell > 0$ since ζ_i and ζ_j are non-units in $\mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle$.)

Definition (5.2) (a) *Given an analytic germ* (X,x) *with local ring* R, *we say that a fractional convergent power series*

(5.2.1) $$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n}) \qquad (H(0,\ldots,0)=0)$$

is a (Puiseux) parametrization *of* (X,x) *if there exists a* $\mathbb{C}$*-isomorphism*

$$R \cong \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta].$$

(b) *We say that* ζ *in (5.2.1) is a* quasi-ordinary branch *if for any two conjugates* $\zeta_i \neq \zeta_j$ *of* ζ *over* $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ *we have that*

(5.2.2) $$\zeta_i - \zeta_j = M_{ij}\varepsilon_{ij}(T_1^{1/n},\ldots,T_d^{1/n}) \qquad \varepsilon_{ij}(0,\ldots,0) \neq 0$$

with M_{ij} *as in* (5.1). *The fractional monomials* M_{ij} *so obtained (as* i,j *vary) are called the* characteristic monomials *of* ζ.

Remarks.(5.3.1) From the preceding discussion (and its converse) we see that (X,x) *is a quasi-ordinary germ satisfying the conditions specified at the beginning of* §5 *if and only if* (X,x) *can be parametrized by a quasi-ordinary branch* ζ *as in* (5.2).

(5.3.2) If (X,x) can be parametrized by a quasi-ordinary branch $\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$, then (X,x) has a representative which is a hypersurface subgerm of $(\mathbb{C}^{d+1},0)$, namely the image of the map $\tilde{\psi}:U \to \mathbb{C}^{d+1}$ (U some neighborhood of 0 in $\mathbb{C}^d$) given by

$$\tilde{\psi}(s_1,\ldots,s_d) = (s_1^n,\ldots,s_d^n, H(s_1,\ldots,s_d));$$

and the map $\pi:(X,x) \to (\mathbb{C}^d,0)$ given by projection to the first d coordinates is a quasi-ordinary projection.

(5.4) Quasi-ordinary parametrizations (when such exist) can obviously be a useful tool for analyzing the geometry of (X,x). A given (X,x) may, however, be parametrized by many different quasi-ordinary branches ζ. We will be interested naturally in geometric information about (X,x) which, though obtained via ζ, is independent of the choice of parametrization. Thus, while (5.5)-(5.8) below are concerned with elementary properties of the characteristic monomials of a given quasi-ordinary branch ζ, our purpose in this section is to use such properties in order to compute the group $A_{d-1}(X)_x = H_{2d-2}(X)_x$ for the germ (X,x) parametrized by ζ, cf.(5.9). Furthermore, in the Appendix (§7) we will use the characteristic monomials of ζ to describe the singular locus of (X,x).

(5.5) Let ζ be a quasi-ordinary branch (cf.(5.2)), and let

$$L \subset K = L(\zeta) \subset L_n$$

be the respective fraction fields of

$$\mathbb{C}\langle T_1,\dots,T_d\rangle \subset \mathbb{C}\langle T_1,\dots,T_d\rangle[\zeta] \subset \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle.$$

L_n - and hence K - is an abelian extension of L. For any L-automorphism θ of L_n, and for $1 \le i \le d$, $\theta(T_i) = T_i$, whence

$$\theta(T_i^{1/n}) = \omega_{i\theta} T_i^{1/n}$$

with $(\omega_{i\theta})^n = 1$; and so for any

$$\xi = G(T_1^{1/n},\dots,T_d^{1/n}) \in \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle$$

we have

$$\theta(\xi) = G(\omega_{1\theta}T_1^{1/n},\dots,\omega_{d\theta}T_d^{1/n}). \tag{5.5.1}$$

Let $\zeta = \zeta_1, \zeta_2,\dots,\zeta_m \in L_n$ be the L-conjugates of ζ. For any j, we can choose an automorphism θ so that $\theta(\zeta_j) = \zeta_1$. Then $\theta(\zeta_i) = \zeta_k$ for some $k \ge 2$, and setting $M_k = M_{k1}$ we find, by applying θ to (5.2.2) (and keeping in mind (5.5.1)), that

$$M_{ij} = M_k.$$

Thus *the set of characteristic monomials of* ζ *(or of any of its conjugates* ζ_i*) is*

$$\{M_k\}_{2\le k\le m}.$$

(We may have $M_k = M_{k'}$ for some $k' \neq k$.)

Lemma (5.6) *The set* $\{M_k\}_{2\le k\le m}$ *of characteristic monomials of a quasi-ordinary branch* ζ *(cf. (5.5)) is* totally ordered *by divisibility (i.e.* $M_i \le M_j$ *if* M_i *divides* M_j *in* $\mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle$).

Proof. Keeping in mind the way in which conjugates are obtained from each other (5.5.1), one easily deduces the assertion from the identity

$$M_i\varepsilon_{i1} - M_j\varepsilon_{j1} = (\zeta_i-\zeta_1) - (\zeta_j-\zeta_1) = M_{ij}\varepsilon_{ij}$$

(which also shows that $M_{ij} = \min(M_i,M_j)$ if $M_i \nmid M_j$).

Lemma (5.7) *If* $\{M_k\}_{2\le k\le m}$ *is the set of characteristic monomials of a quasi-ordinary branch* ζ, *then, with* L *as in* (5.5):

$$L(\zeta) = L(M_2,M_3,\dots,M_m).$$

Proof. Let θ be an L-automorphism of L_n (cf.(5.5)). If $\theta(\zeta) = \zeta_i$ $(i\ge2)$, then, since

$$\theta(\zeta) - \zeta = M_i\varepsilon_{i1}$$

it is clear (cf.(5.5.1)) that

$$\theta(M_i) \nmid M_i.$$

So if $\theta(M_k) = M_k$ for all $k \ge 2$, then $\theta(\zeta) = \zeta$; and we conclude by Galois theory that

$$\zeta \in L(M_2,\dots,M_m).$$

Conversely, it is clear (again by (5.5.1)) that each monomial M_k must appear with non-zero coefficient in the fractional power series ζ (since M_k appears in $\zeta_k-\zeta$), and hence that if $\theta(\zeta) = \zeta$, then $\theta(M_k) = M_k$ for all k. Thus $M_k \in L(\zeta)$, and so

$$L(M_2,\dots,M_m) \subset L(\zeta).$$

Q.E.D.

Remarks (5.8). (i) If e is the number of distinct characteristic monomials of ζ, then according to (5.6) we can reindex these monomials in such a way that

$$M_1 < M_2 < \dots < M_e.$$

For any i with $1 \le i \le e$, there is an automorphism θ_i such that

$$\theta_i(\zeta) - \zeta = M_i \varepsilon_i$$

with ε_i a unit in $\mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle$; and then clearly $\theta_i(M_j) = M_j$ for all $j < i$ (cf.(5.5.1)). Thus, with L as in (5.5), we have

(5.8.1) $$M_i \notin L(M_1,M_2,\dots,M_{i-1}) \qquad 1 \le i \le e.$$

(ii) From (5.6) and (5.8.1), one deduces easy-to-work-with *necessary and sufficient conditions* on the exponents of a set of fractional monomials for this set to be the set of characteristic monomials of a quasi-ordinary branch [L, p.167, Prop.(1.5)]. These conditions provide a convenient algorithm for testing whether or not a given fractional power series is quasi-ordinary, and if it is, for finding its characteristic monomials.

* * *

(5.9). Let (X,x) be a d-dimensional germ parametrized by a quasi-ordinary branch ζ; and let $(\overline{X},\bar{x})$ be the normalization of (X,x). Using the results of §4, we can now describe an algorithm for calculating the groups $A_{d-1}(X)_x$ and $A_{d-1}(\overline{X})_{\bar{x}}$ from the characteristic monomials of ζ.

First of all, from (5.7) we deduce that the group $\Gamma \subset ((1/n)\mathbb{Z})^d$ of Theorem (4.1) is generated by the "characteristic exponent vectors" (i.e. those vectors $(\lambda_1,\dots,\lambda_d)$ such that $T_1^{\lambda_1}\dots T_d^{\lambda_d}$ is a characteristic monomial of ζ - cf. remark 5.8(ii)), together with the standard unit vectors $(1,0,0,\dots)$, $(0,1,0,\dots)$, $(0,0,1,\dots),\dots$. We consider therefore the *characteristic matrix* $M(\zeta)$, whose rows are the characteristic exponent vectors and the standard unit vectors. According to Theorem (4.1):

(5.9.1) *a matrix presenting* $A_{d-1}(X)_x$ (*resp.* $A_{d-1}(\bar{X})_{\bar{x}}$) *is obtained from the characteristic matrix* $\underset{\sim}{M}(\zeta)$ *by multiplying the* i*-th column* $(1 \le i \le d)$ *by the integer* m_i (*resp.* n_i) *specified in Example* (4.5).

The integer n_i is given by

$$\pi_i(\Gamma) = (1/n_i)\mathbb{Z}$$

cf. (4.5)). Thus:

(5.9.2) *the integer* n_i *is the least common denominator of the rational numbers appearing in the* i*-th column of* $\underset{\sim}{M}(\zeta)$.

As for m_i, writing

$$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$$

we set

$$\zeta^{(i)} = H(T_1^{1/n},\ldots,T_{i-1}^{1/n},0,T_{i+1}^{1/n},\ldots,T_d^{1/n})$$

and claim that:

(5.9.3) *if* L *is, as usual, the fraction field of* $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ *then*

$$m_i = [L(\zeta):L(\zeta^{(i)}] = [L(\zeta):L]/[L(\zeta^{(i)}):L].$$

If $T_j^{i/n}$ *divides no characteristic monomial of* ζ, *then* $m_j = n_j = 1$, *and so in* (5.9.1) *we can delete the* j-th *column of* $\underset{\sim}{M}(\zeta)$.

Proof. In view of (4.5.2) (where $L_1 = L$ and $K = L(\zeta)$), it will suffice to demonstrate an isomorphism

$$R/p_i \cong \mathbb{C}\langle T_1,\ldots,T_{i-1},T_{i+1},\ldots,T_d\rangle[\zeta^{(i)}]$$

where

$$R = \mathbb{C}\langle T_1,\dots,T_d\rangle[\zeta]$$

and

$$p_i = \sqrt{T_i R} = T_i^{1/n}\,\mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle \cap R$$

(cf. beginning of §4). To do so, we need only think for a moment about the image of $R \subset \mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle$ under the $\mathbb{C}$-homomorphism

$$\mathbb{C}\langle T_1^{1/n},\dots,T_d^{1/n}\rangle \longrightarrow \mathbb{C}\langle T_1^{1/n},\dots,T_{i-1}^{1/n},T_{i+1}^{1/n},\dots,T_d^{1/n}\rangle$$

taking $T_i^{1/n}$ to 0 and taking $T_j^{1/n}$ to $T_j^{1/n}$ for $j \neq i$.

The last assertion follows from (5.10) below. Q.E.D.

Now note that $\zeta^{(i)}$ is a quasi-ordinary branch whose characteristic monomials coincide with those characteristic monomials of ζ in which T_i does not appear, i.e. which are not divisible by $T_i^{1/n}$. (In view of (5.5.1), this follows easily from the definitions in (5.2)(b).) So in order to use (5.9.3), we just need to know how to compute $[L(\zeta):L]$ for a quasi-ordinary ζ. But by (4.1.3):

(5.9.4) $$[L(\zeta):L] = n_1 n_2 \dots n_d / |A_{d-1}(\overline{X})_{\bar{x}}|$$

where $|A_{d-1}(\overline{X})_{\bar{x}}|$, the order of the finite abelian group $A_{d-1}(\overline{X})_{\bar{x}}$, is the greatest common divisor of the $(d\times d)$-subdeterminants of a matrix presenting the group, as described in (5.9.1); and n_i is given by (5.9.2).

$[L(\zeta):L]$ can also be computed via (5.10.3.1) below.

This completes the description of the algorithm. For a concrete example, cf. (7.4).

* * *

We close this section with some complements concerning the integers m_i and the order of $A_{d-1}(X)_x$.

Lemma (5.10). *With notation as in* (5.9), *for each* $i = 1,2,\dots,d$ *the following are equivalent:*

(i) *The integer* $m_i = [L(\zeta):L(\zeta^{(i)})]$ (*cf.*(5.9.3)) *equals one.*

(ii) $T_i^{1/n}$ *does not divide any characteristic monomial of* ζ.

(ii)' T_i *does not divide the discriminant*

$$\Delta(\zeta) = \prod_{i \neq j} (\zeta_i - \zeta_j)$$

(*where* $\zeta_1,\dots,\zeta_n$ *are the conjugates of* ζ).

Proof. The equivalence of (ii) and (ii)' is clear from the definitions involved ((5.2)(b)). Condition (i) holds if and only if $L(\zeta) = L(\zeta^{(i)})$, i.e. (by (5.7)) every characteristic monomial of ζ lies in $L(\zeta^{(i)})$. But by (5.8.1) and the description of characteristic monomials of $\zeta^{(i)}$ preceding (5.9.4), this means that every characteristic monomial of ζ is a characteristic monomial of $\zeta^{(i)}$, i.e. it is not divisible by $T_i^{1/n}$. Thus (i) and (ii) are equivalent.

Corollary (5.10.1). *Assume that precisely* c *of the variables* T_i *divide the discriminant* $\Delta(\zeta)$. *Then, after relabelling the* T_i, *we have*

$$1 = m_d = m_{d-1} = \dots = m_{c+1} < m_c | m_{c-1} | \dots | m_1 = [L(\zeta):L]$$

(*where* "|" *denotes* "*divides*").

Proof. Let $M_1 < \dots < M_e$ be the characteristic monomials of ζ (5.8). If $T_i^{1/n}$ divides M_j then $T_i^{1/n}$ divides M_k for all $k > j$. Hence for each i there is an integer $e_i \geq 0$ such that $T_i^{1/n}$ divides M_j if and only if $j > e_i$; in other words the characteristic monomials of $\zeta^{(i)}$ are $M_1 < M_2 < \dots < M_{e_i}$. By (5.7), the fields $L(\zeta^{(i)})$ are therefore totally ordered by inclusion; and hence the integers $m_i = [L(\zeta):L(\zeta^{(i)})]$ are totally ordered by divisibility. Because of (5.10), we can arrange that

$$(m_j=1) \Leftrightarrow (j>c).$$

Finally, we can arrange that $T_1^{1/n}$ divides M_1, so that $L(\zeta^{(1)}) = L$, i.e. $m_1 = [L(\zeta):L]$. The conclusion is now obvious. Q.E.D.

Again, let $M_1 < M_2 < \ldots < M_e$ be the characteristic monomials of ζ (5.8). Since $M_i^n \in L$, the field extension

$$L(M_1,\ldots,M_{i-1},M_i) \supset L(M_1,\ldots,M_{i-1})$$

(which is *non-trivial*, by (5.8.1)) is *cyclic*, of order

(5.10.2) $$b_i = \min\{b>0 \mid M_i^b \in L(M_1,\ldots,M_{i-1})\} > 1 \quad (1 \le i \le e).$$

Equivalently, if

$$\underset{\sim}{\lambda}_j = (\lambda_{j1},\ldots,\lambda_{jd}) \in ((1/n)\mathbb{Z})^d$$

is the vector defined by

$$M_j = T_1^{\lambda_{j1}} \ldots T_d^{\lambda_{jd}}$$

then

(5.10.2)' $$b_i = \min\{b>0 \mid b\underset{\sim}{\lambda}_i \in \mathbb{Z}\underset{\sim}{\lambda}_1 + \ldots + \mathbb{Z}\underset{\sim}{\lambda}_{i-1} + \mathbb{Z}^d\}.$$

The integer b_i can be computed from (5.10.2)'; or from relations similar to (5.9.4).

Proposition (5.10.3). *For each* $i = 1,2,\ldots,e$ *let* b_i *be as above and let* ν_i *be the number of variables* T_j *appearing in* M_i *with positive exponent. Assume the variables labelled so that (5.10.1) is satisfied. Then the order* a *of the finite group* $A_{d-1}(X)_x \cong H_{2d-2}(X)_x$ *is given by*

$$a = m_2 m_3 \ldots m_c = b_1^{\nu_1 - 1} \ldots b_e^{\nu_e - 1}.$$

Proof. The equality $a = m_2 m_3 \dots m_c$ is immediate from (4.1.3) and (5.10.1). Also, for each i, we have

$$
\begin{aligned}
m_i &= [L(\zeta):L(\zeta^{(i)})] && (5.9.3)\\
&= [L(M_1,\dots,M_e):L(M_1,\dots,M_{e_i})] && \text{(5.7) and proof of (5.10.1)}\\
&= b_{e_i+1}\, b_{e_i+2} \dots b_e .
\end{aligned}
$$

Noting that $T_i^{1/n}$ divides M_j if and only if $j > e_i$, and that $m_i = 1$ if $i > c$ (5.10.1), we conclude that

$$m_1 m_2 \dots m_c = b_1^{\nu_1} b_2^{\nu_2} \dots b_e^{\nu_e}.$$

Since (by (5.10.1) and (5.7))

(5.10.3.1) $$m_1 = [L(\zeta):L] = b_1 b_2 \dots b_e ,$$

we are done.

Remark (5.10.4). Given (5.10.1) and (5.10.3), one might guess that the sequence $m_c | m_{c-1} | \dots | m_2$ is the sequence of "invariant factors" of the abelian group A_{d-1}. This does not always turn out to be true, cf. Example (7.5).

§6. Topological invariance of the reduced branching sequence

Throughout this section, the notation and assumptions at the beginning of §5 remain in force (*cf. also* (5.3)).

Before stating the main result, Theorem (6.1), we need to define the "reduced branching sequence" of a quasi-ordinary branch ζ.

We will usually label the variables $T_1,\ldots,T_d$ in such a way that (5.10.1) is satisfied:

$$1 = m_d = m_{d-1} = \ldots = m_{c+1} < m_c | m_{c-1} | \ldots | m_1 = [L(\zeta):L],$$

where m_i is the generic branching order of the projection $\pi:(X,x) \longrightarrow (\mathbb{C}^d,0)$ (corresponding to $\mathbb{C}\langle T_1,\ldots,T_d\rangle \subset \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta]$) along the irreducible subvariety $Z_i \subset X$ defined by $T_i = 0$ (cf. (4.5.4)). Also, as in (4.5.4), we let n_i be the generic branching order of $\pi\circ\nu$ (where $\nu:(\overline{X},\bar{x}) \to (X,x)$ is "normalization") along $\overline{Z}_i=\nu^{-1}(Z_i)$. The *branching sequence* of ζ consists of the pairs

$$(m_1,n_1),(m_2,n_2),(m_3,n_3),\ldots,(m_c,n_c)$$

arranged in *decreasing lexicographic order*.

(Note that $m_i \geq n_i$ for all i (4.5); and that $(m_i,n_i) = (1,1)$ for $i > c$.)

The *reduced* branching sequence of ζ is the empty sequence if $c \leq 1$; and otherwise is the sequence obtained from the branching sequence by replacing (m_1,n_1) by (m_1',n_1'), where

$$(m_1',n_1') = (m_2/m_1)(m_1,n_1) = (m_2,m_2n_1/m_1),$$

and then shifting (m_1',n_1') to the right until the resulting sequence is lexicographically decreasing.

The reason for introducing (m_1',n_1') will become apparent momentarily.

Here is the main result, asserting that the reduced branching sequence of any quasi-ordinary parametrization of a germ (X,x) is a topological invariant of (X,x).

Theorem (6.1) *Let* (X,x), (X',x') *be analytic germs parametrized by quasi-ordinary branches* ζ, ζ' *respectively. Suppose that there exists a homeomorphism of topological germs* $\varphi:(X,x) \xrightarrow{\sim} (X',x')$. *Then* ζ *and* ζ' *have the same reduced branching sequence.*

The proof will occupy the rest of §6. The basic argument can be found in §§(6.7), (6.8).

Remark (6.1.1). By considering the case of irreducible plane curve germs - which are always homeomorphic to the germ $(\mathbb{C}^1,0)$ - one sees that the branching sequence itself cannot be a topological invariant. In higher dimensions, a similar situation occurs, as explained in the following Proposition (6.3).

Definition (6.2) *A quasi-ordinary branch*

$$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$$

is reduced *if for each* $i = 1,2,\ldots,d$, *we have*

$$H(0,\ldots,0,T_i^{1/n},0,\ldots,0) \in \mathbb{C}\langle T_i\rangle. \tag{6.2.1}$$

Remarks. (i) With labelling as in (5.10.1), the condition (6.2.1) is automatically satisfied for all $i > 1$. (Indeed any term of the form aT_j^{λ} $(a\neq 0, \lambda\notin \mathbb{Z})$ in $H(T_1^{1/n},\ldots,T_d^{1/n})$ can be "moved" by some automorphism θ as in (5.5), hence appears in $\zeta - \theta\zeta = M\varepsilon$ (5.2.2), hence is divisible by the characteristic monomial M; but $T_1^{1/n}$ divides every such M (cf. proof of (5.10.1)), so $j = 1$.) Thus:

(6.2.2) ζ *is reduced if and only if* $H(T_1^{1/n},0,\ldots,0) \in \mathbb{C}\langle T_1\rangle$.

(ii) From (5.9.3) and the description of the characteristic monomials of $\zeta^{(i)}$ following the proof of (5.9.3), we find that ζ is reduced if and only if $\zeta^{(2)}$ has no characteristic monomials, i.e.

$$m_2 = [L(\zeta):L(\zeta^{(2)})] = [L(\zeta):L] = m_1.$$

Consequently:

(6.2.3) *ζ is reduced if and only if the reduced branching sequence of ζ is the same as the branching sequence of ζ.*

Proposition (6.3). *For any quasi-ordinary branch $\zeta = H(T_1^{1/n},\dots,T_d^{1/n})$ there exists a reduced quasi-ordinary branch ζ' whose branching sequence coincides with the reduced branching sequence of ζ, and such that the germs (X,x), (X',x') parametrized by ζ and ζ' respectively are homeomorphic.*

Proof.[(1)] For some integer b dividing n we have

(6.3.1) $$\mathbb{C}\langle T_1\rangle[H(T_1^{1/n},0,\dots,0)]^{\sim} = \mathbb{C}\langle T_1^{1/b}\rangle$$

where "$\sim$" denotes "normalization". (The integer b is the least common denominator of the exponents λ of terms aT^{λ} $(a\neq 0)$ appearing in $H(T_1^{1/n},0,\dots,0)$.) Set

$$\zeta' = H(T_1^{b/n},T_2^{1/n},\dots,T_d^{1/n}).$$

If $\omega_1,\dots,\omega_d$ are n-th roots of unity, then

(6.3.2) $$H(T_1^{1/n},T_2^{1/n},\dots,T_d^{1/n}) - H(\omega_1^b T_1^{1/n},\omega_2 T_2^{1/n},\dots,\omega_d T_d^{1/n}) = \eta T_1^{a_1/n}\dots T_d^{a_d/n}\,\varepsilon(T_1^{1/n},T_2^{1/n},\dots,T_d^{1/n})$$

(1) May (should) be skipped on first reading.

where η is either 0 or 1, and $\varepsilon(0,0,\ldots,0) \neq 0$ (cf.(5.2.2), (5.5.1)). Replacing $T_1^{1/n}$ by $T_1^{b/n}$, we see that ζ' is *quasi-ordinary* and (by (6.3.1) and the fact that (6.2.1) holds for all $i > 1$) *reduced.*

Now let $(X,x) \subset (\mathbb{C}^{d+1},0)$ [respectively $(X',x') \subseteq (\mathbb{C}^{d+1},0)$] be the germ parametrized by ζ [respectively ζ'], and define $\varphi:\mathbb{C}^{d+1} \longrightarrow \mathbb{C}^{d+1}$ by

$$\varphi(t_1,t_2,\ldots,t_d,z) = (t_1^b,t_2,\ldots,t_d,z).$$

It follows from (5.3.2) that φ maps (X',x') to (X,x), giving a finite bimeromorphic map corresponding to the composition in the middle row of the commutative diagram

$$\begin{array}{ccccc} R & & R[T_1^{1/b}] & & \\ \| & & \| & & \\ \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta] & \subset & \mathbb{C}\langle T_1^{1/b},\ldots,T_d\rangle[\zeta] & \xrightarrow{\sim} & \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta'] \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle & \xrightarrow[\alpha]{\sim} & \mathbb{C}\langle T_1^{b/n},T_2^{1/n},\ldots,T_d^{1/n}\rangle \end{array}$$

where α is the $\mathbb{C}$-isomorphism such that

$$\alpha(T_1^{1/n}) = T_1^{b/n}$$

$$\alpha(T_i^{1/n}) = T_i^{1/n} \qquad (i>1).$$

(Note that $R[T_1^{1/b}]$ is contained in the normalization of R since, as in the proof of (5.7), $H(T_1^{1/n},0,\ldots,0) \subset L(\zeta)$.) This map $(X',x') \longrightarrow (X,x)$ is proper, with image a neighborhood of x; and to show that it is a homeomorphism of germs we need only show that it is *injective* on some neighborhood of x'.

Let

$$y = (s_1^n,\dots,s_d^n,H(s_1,\dots,s_d)) \in X$$

cf. (5.3.2). Then, if ω is a primitive n-th root of unity, we have

$$\varphi^{-1}(y) = \{((\omega^j s_1)^{n/b},s_2^n,\dots,s_d^n,H(s_1,\dots,s_d))\}_{1\le j\le b}.$$

On the other hand, any point $y' \in X'$ is of the form

$$y' = (u_1^n,u_2^n,\dots,u_d^n,H(u_1^b,u_2,\dots,u_d))$$

for suitable complex numbers $u_1,\dots,u_d$. So if $y' \in \varphi^{-1}(y)$, then for some n-th roots of unity $\omega_1,\dots,\omega_d$, and for some j with $1 \le j \le b$, we have:

$$u_1^b = \omega_1\omega^j s_1$$

$$u_k = \omega_k s_k \qquad\qquad k \ge 2$$

and

$$\text{(6.3.3)}\qquad H(s_1,\dots,s_d) = H(\omega_1\omega^j s_1,\omega_2 s_2,\dots,\omega_d s_d).$$

Let us deduce from this that either $s_1 = 0$ or $j = b$, thereby showing that there is just one point in $\varphi^{-1}(y) \cap X'$, as desired.

With these $\omega_1,\dots,\omega_d$ in (6.3.2), note that if $\eta \ne 0$ then $T_1^{a_1/n} \dots T_d^{a_d/n}$ is a characteristic monomial of ζ, and that moreover $a_i > 0$ for some $i \ge 2$, as can be seen, in view of (6.3.1), by substituting 0 for all of $T_2^{1/n},T_3^{1/n},\dots,T_d^{1/n}$.

Just as in (6.3.2), we also have

$$\text{(6.3.2)}'\quad H(T_1^{1/n},\dots,T_d^{1/n}) - H(\omega^j T_1^{1/n},\dots,T_d^{1/n}) = \eta' T_1^{b_1/n} \dots T_d^{b_d/n} \varepsilon'(T_1^{1/n},\dots,T_d^{1/n}).$$

And from (6.3.1), we see that if $1 \le j \le b$, then substitution of 0 for all of $T_2^{1/n},\dots,T_d^{1/n}$ in (6.3.2)' does *not* annihilate the left hand side, whence $\eta' = 1$ and $b_2 = \dots = b_d = 0$. It follows (if $1 \le j < b$) that of the two characteristic monomials $T_1^{a_1/n} \dots T_d^{a_d/n}$, $T_1^{b_1/n}$, of ζ, the second properly divides the first (cf.(5.6)). Thus (from (6.3.2), (6.3.2)'):

$$
\begin{aligned}
\text{(6.3.4)} \quad & H(T_1^{1/n},T_2^{1/n},\dots,T_d^{1/n}) - H(\omega_1\omega^j T_1^{1/n},\omega_2 T_2^{1/n},\dots,\omega_d T_d^{1/n}) \\
&= T_1^{b_1/n}\varepsilon'(T_1^{1/n},\dots,T_d^{1/n}) + \eta\omega^{ja_1}T_1^{a_1/n}\dots T_d^{a_d/n}\varepsilon(\omega^j T_1^{1/n},T_2^{1/n},\dots,T_d^{1/n}) \\
&= T_1^{b_1/n}\varepsilon''(T_1^{1/n},\dots,T_d^{1/n}) \qquad\qquad \varepsilon''(0,\dots,0) \not\equiv 0.
\end{aligned}
$$

Now in (6.3.4) just replace $T_i^{1/n}$ by sufficiently small s_i $(1 \le i \le d)$, to see that (6.3.3) is impossible when $s_1 \not\equiv 0$ and $1 \le j \le b$. Thus either $s_1 = 0$ or $j = b$, and φ does indeed map (X',x') homeomorphically to (X,x).

It remains to compare the branching sequences of ζ and ζ'. If $c \le 1$ then (5.7) easily implies that

$$\zeta \in L(H(T_1^{1/n},0,\dots,0)) = L(T_1^{1/b}),$$

whence $\zeta' \in L$, and ζ' has an empty branching sequence. So we may assume $c \ge 2$. Consider the commutative diagram

$$
\begin{array}{ccc}
(X',x') & \longrightarrow & (X,x) \\
\cap & & \cap \\
(\mathbb{C}^{d+1},0) & \xrightarrow{\ \varphi\ } & (\mathbb{C}^{d+1},0) \\
\downarrow & & \downarrow \\
(\mathbb{C}^{d},0) & \xrightarrow{\ \varphi_0\ } & (\mathbb{C}^{d},0)
\end{array}
$$

where the vertical arrows represent "projection onto the first d coordinates", and

$$\varphi_0(t_1,t_2,\ldots,t_d) = (t_1^b,t_2,\ldots,t_d).$$

Since, as we have just seen, the top arrow is a homeomorphism, it is evident (cf.(4.5.4)(i)) that (with self-explanatory notation):

$$m_1 = bm_1'$$

$$m_i = m_i' \qquad (i\geq 2).$$

Moreover, since (as mentioned above) we have a commutative diagram

$$\begin{array}{ccc} & (\bar{X},\bar{x}) & \\ \swarrow & & \searrow^{\nu} \\ (X',x') & \xrightarrow[\varphi]{} & (X,x) \end{array}$$

where ν is "normalization", therefore

$$n_1 = bn_1'$$

$$n_i = n_i' \qquad (i\geq 2).$$

So, to show that the branching sequence of ζ' is the same as the reduced branching sequence of ζ, it suffices to prove that:

(6.3.5) $$b = m_1/m_2.$$

Now we have seen (following (5.9.3)) that $\zeta^{(2)}$ is a quasi-ordinary branch whose characteristic monomials are the same as those of ζ which are not divisible by $T_2^{1/n}$; and by (5.7) these monomials generate $L(\zeta^{(2)})$ over L. But the proof of (5.10.1) makes it obvious that these monomials are just the characteristic monomials of ζ which have the form T_1^{λ}, i.e. just the characteristic monomials of $H(T_1^{1/n},0,\ldots,0)$; and, by (5.7) and (6.3.1), these monomials generate the fraction field of $\mathbb{C}\langle T_1^{1/b}\rangle$ over that of $\mathbb{C}\langle T_1\rangle$, a field extension of degree b. It follows that

$$[L(\zeta^{(2)}):L] = b.$$

Moreover,

$$[L(\zeta^{(2)}):L] = [L(\zeta):L]/[L(\zeta):L(\zeta^{(2)})]$$
$$= m_1/m_2$$

(cf.(5.9.3), (5.10.1)). This proves (6.3.5) and finishes the proof of Proposition (6.3).

* * *

Further preparation for the proof of Theorem (6.1) is provided by the next result, due to Gau (at least when $d = 2$) [G, (3.4.2), Remarks 3, 4].

Proposition (6.4) *Let* (X,x) *be parametrized by a quasi-ordinary branch* $\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$, *the variables being labelled as in* (5.10.1). *Then the following conditions are equivalent:*

(i) $A_{d-1}(X)_x \cong H_{2d-2}(X)_x = 0.$

(ii) *Every characteristic monomial of* ζ *is of the form* T_1^{λ}.(1)

(ii)' *The reduced branching sequence of* ζ *is empty.*

(iii) (X,x) *is topologically smooth, i.e. homeomorphic to the germ* $(\mathbb{C}^d,0)$.

Proof. (i) $\Rightarrow$ (ii). According to (5.10) we need to show that the integer c of (5.10.1) is ≤ 1; but this follows from (5.10.1) and the equality $a = m_2m_3 \ldots m_c$ of (5.10.3).

(ii) $\Leftrightarrow$ (ii)'. Both these conditions are equivalent to the condition $c \leq 1$.

(1) In other words the discriminant $\Delta(\zeta)$ is of the form εT_1^p, $\varepsilon(0) \neq 0$, cf.(5.10). In Zariski's terminology [Z, Theorem 4.4], this means that (X,x) has a singularity of *dimensionality type 1*; i.e. the singular locus Σ of X is a (d-1)-dimensional manifold (if not empty) near x, and X is *equisingular* along Σ at x.

(ii)' ⇒ (iii). The branching sequence of ζ' in (6.3) is empty, so $c = 0$ for ζ', i.e. $\zeta' \in \mathbb{C}\langle T_1,\ldots,T_d\rangle$, and the germ (X',x') parametrized by ζ' is isomorphic to $(\mathbb{C}^d,0)$; moreover, by (6.3), (X,x) is homeomorphic to (X',x').

(iii) ⇒ (i). Obvious, since by (1.1):

$$H_{2d-2}(\mathbb{C}^d)_0 = H_{2d-3}((2d-1)\text{-sphere}) = 0.$$

Q.E.D.

Proposition (6.4) shows that any part of the singular locus Σ where X is irreducible and equisingular will be "topologically invisible". Not surprisingly, however, in the proof of Theorem (6.1) we will need some subvarieties of Σ which *do* have topological significance. Two such subvarieties $\Sigma^{(2)} \subset \Sigma^{(1)}$ will now be described, in (6.5) and (6.6) respectively.

(6.5) Let (X,x) be parametrized by a quasi-ordinary branch ζ, and let $\pi:(X,x) \longrightarrow (\mathbb{C}^d,0)$ be the associated quasi-ordinary projection, corresponding to the inclusion

$$\mathbb{C}\langle T_1,\ldots,T_d\rangle \subset R = \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta].$$

Let $D \subset \mathbb{C}^d$ be the discriminant locus of π, so that (after restricting to a suitably small neighborhood of $0 \in \mathbb{C}^d$), D has only normal crossing singularities. In fact we may assume that D is given by the equation $T_1T_2 \ldots T_c = 0$ (cf.(5.10.1)). We set

$$\Sigma^{(2)} = \pi^{-1}(\mathrm{Sing}(D))$$

where $\mathrm{Sing}(D)$, the singular locus of D, consists of all points in $\mathbb{C}^d$ (near 0) where at least two of the $T_i (1 \le i \le c)$ vanish. Thus:

Lemma (6.5.1). *The subvariety* $\Sigma^{(2)} \subset X$ *has* $c(c-1)/2$ *irreducible components* Z_{ij} *at* x, *corresponding to the prime ideals*

$$p_{ij} = \sqrt{(T_i,T_j)R} \qquad 1 \le i < j \le c.$$

(The proof at the beginning of §4 that $\sqrt{T_i R}$ is prime can be modified to apply to p_{ij}.)

The definition of $\Sigma^{(2)}$ involves a choice of ζ. But $\Sigma^{(2)}$ can also be characterized topologically:

Proposition (6.5.2). *The variety* $\Sigma^{(2)}$ *consists of all points* $y \in X$ *such that* either $H_{2d-2}(Y,y) \neq 0$ *for some component* Y *of* X *at* y, or *some two such components have an intersection which is reducible at* y.

Remark (6.5.3). Proposition (6.5.2) shows that $\Sigma^{(2)}$ is *topologically invariant*, in the sense that any germ-homeomorphism $\varphi:(X,x) \longrightarrow (X',x')$ maps $\Sigma^{(2)}(X,x)$ onto $\Sigma^{(2)}(X',x')$: for, any such φ maps each component of X at a point $y \in X$ to a component of X' at $\varphi(y)$ (and similarly for intersections of such components) [GL, p.172, Lemma (A.8)]; and the groups $H_i(Y)_y$ are topological invariants (cf. (1.1)). It follows then from (6.5.1) that $c(c-1)$ *is a topological invariant of* (X,x) (whence so is c if $c \ge 2$, i.e. if $\Sigma^{(2)}$ is non-empty, i.e. if the equivalent conditions of (6.4) do not hold).

Proof of (6.5.2).[1] If the discriminant D has at most one component at $\pi(y)$, (i.e. $y \notin \Sigma^{(2)}$), then the same is true for the discriminant $D_Y \subset D$ of the restriction π_Y of π to any component Y of X at y, and so by (6.4), $H_{2d-2}(Y)_y = 0$. Moreover if Y' is another component of X at y, then $Y' \cap Y$ is a $d-1$ dimensional subvariety of the singular locus of X, whence $Y' \cap Y$ is irreducible at y by part (2a) of Theorem 4.5 in [Z]. [We can argue directly as follows. Clearly

$$Y' \cap Y \subset \pi^{-1}(D) \cap Y = \pi_Y^{-1}(D).$$

[1] May (should) be skipped on first reading.

If $\pi(y) \in D_Y$ then (D being irreducible) $D_Y = D$ near $\pi(y)$, and $\pi_Y^{-1}(D) = \pi_Y^{-1}(D_Y)$ near y. But $\pi_Y^{-1}(D_Y)$ is irreducible at y, of dimension $d-1$: in fact if T_i vanishes at $\pi(y)$, and if R' is the local ring of (Y,y), then $\pi_Y^{-1}(D_Y)$ is the zero set of the ideal $\sqrt{T_i R'}$, which is a prime ideal (beginning of §4). Hence, near y,

$$Y' \cap Y = \pi_Y^{-1}(D_Y)$$

and so $Y' \cap Y$ is irreducible at y. Finally if $\pi(y) \notin D_Y$ then π_Y is a local isomorphism at y, so $\pi_Y^{-1}(D)$ is irreducible at y, and again we see that $Y' \cap Y(=\pi_Y^{-1}(D))$ is irreducible at y.] This proves one half of (6.5.2).

For the other half, suppose that D has at least two components at $\pi(y)$ (i.e. $y \in \Sigma^{(2)}$). Suppose moreover that $H_{2d-2}(Y)_y = 0$ for every component Y of X at y. We need then to show that X has two components Y' and Y at y whose intersection is reducible at y.

Assume that T_i and T_j vanish at y (where $1 \le i < j \le c$). We can choose n-th roots of unity $\omega_1, \ldots, \omega_d$ such that, with θ the $\mathbb{C}\langle T_1, \ldots, T_d\rangle$-automorphism of $\mathbb{C}\langle T_1^{1/n}, \ldots, T_d^{1/n}\rangle$ taking $T_k^{1/n}$ to $\omega_k T_k^{1/n}$ $(1 \le k \le d)$, we have

$$\zeta - \theta\zeta = M\varepsilon(T_1^{1/n}, \ldots, T_d^{1/n}) \tag{6.5.4}$$

where $\varepsilon(0, \ldots, 0) \neq 0$ and M is a characteristic monomial of ζ divisible by both $T_i^{1/n}$ and $T_j^{1/n}$ (e.g. the largest characteristic monomial of ζ). We can rewrite (6.5.4) as

$$H(T_1^{1/n}, \ldots, T_d^{1/n}) - H(\omega_1 T_1^{1/n}, \ldots, \omega_d T_d^{1/n}) = M\varepsilon. \tag{6.5.4)'$$

Consider the map $\psi:U \to X$ defined on a neighborhood U of 0 in $\mathbb{C}^d$ by

$$\psi(s_1,\ldots,s_d) = (s_1^n,\ldots,s_d^n, H(s_1,\ldots,s_d))$$

cf. (5.3.2). Choose $(u_1,\ldots,u_d) \in U$ so that

$$y = \psi(u_1,\ldots,u_d) \qquad (\Rightarrow u_i=u_j=0)$$

and set

$$y' = \psi(\omega_1u_1,\ldots,\omega_du_d).$$

When we substitute u_k for $T_k^{1/n}$ in (6.5.4)' $(1\le k\le d)$, M vanishes (because $u_i = u_j = 0$), and hence $y = y'$.

Now ψ maps a neighborhood of $(u_1,\ldots,u_d)$ (respectively $\omega_1u_1,\ldots,\omega_du_d$)) onto a component Y (respectively Y') of the germ (X,y). Moreover if $(v_1,\ldots,v_d)$ is close to $(u_1,\ldots,u_d)$ and if *either* $v_i = 0$ *or* $v_j = 0$ then substitution of v_k for $T_k^{1/n}$ in (6.5.4)' $(1\le k\le d)$ shows that

$$\psi(v_1,\ldots,v_d) = \psi(\omega_1v_1,\ldots,\omega_dv_d).$$

Thus the locus $T_iT_j = 0$ is contained in $\pi(Y\cap Y')$ (in a neighborhood of $\pi(y)$), so that $Y \cap Y'$ will be reducible at y, as asserted, *provided that* $Y \neq Y'$.

Let us complete the proof by showing that

(6.5.5) *if* $Y = Y'$, *then the locus* $T_iT_j = 0$ *is contained in* D_Y *near* $\pi(y)$

contradicting (6.4) (recall that, by assumption, $H_{2d-2}(Y)_y = 0$).

If $Y = Y'$, then for each $v = (v_1,\ldots,v_d)$ very close to $(u_1,\ldots,u_d)$ there exists $v' = (v_1',\ldots,v_d')$ very close to $(\omega_1u_1,\ldots,\omega_du_d)$ such that $\psi(v') = \psi(v)$; i.e. there exist n-th roots of unity $\omega_1',\ldots,\omega_d'$ such that

(i) $(\omega_1' v_1, \ldots, \omega_d' v_d)$ is very close to $(\omega_1 u_1, \ldots, \omega_d u_d)$, and

(ii) $H(\omega_1' v_1, \ldots, \omega_d' v_d) = H(v_1, \ldots, v_d)$.

From (i) it follows that:

(iii) $\omega_k' = \omega_k$ for all k such that $u_k \neq 0$.

Supposing, as we may, that no v_k vanishes, setting

$$H(T_1^{1/n}, \ldots, T_d^{1/n}) - H(\omega_1' T_1^{1/n}, \ldots, \omega_d' T_1^{1/n}) = M'\varepsilon'(T_1^{1/n}, \ldots, T_d^{1/n})$$

(where M', if not identically zero, is a characteristic monomial of ζ, and $\varepsilon'(0, \ldots, 0) \neq 0$), and substituting v_k for $T_k^{1/n} (1 \leq k \leq d)$ we see from (ii) that M' vanishes, so that M' must be identically zero. Thus H doesn't change when, for all $k = 1, 2, \ldots, d$, $T_k^{1/n}$ is multiplied by ω_k', and so from (6.5.4)' we conclude that

$$(6.5.6) \qquad H(\ldots, T_k^{1/n}, \ldots) - H(\ldots, (\omega_k')^{-1} \omega_k T_k^{1/n}, \ldots) = M\varepsilon.$$

We will use (6.5.6) to show that $\pi_Y(y') \in D_Y$ for any $y' \in Y$ where either T_i or T_j vanishes (thereby proving (6.5.5)).

Let V be a neighborhood in $\mathbb{C}^d$ of $(u_1, \ldots, u_d)$. If V is sufficiently small, then, for $v, v^* \in V$, $\pi \circ \psi(v) = \pi \circ \psi(v^*)$ if and only if there exist n-th roots of unity $\omega_1^*, \ldots, \omega_d^*$ such that

(i)* $\qquad v_k^* = \omega_k^* v_k \qquad 1 \leq k \leq d$

and

(ii)* $\qquad \omega_k^* = 1$ if $u_k \neq 0$.

Suppose we have another sequence (ω_k^{**}) of n-th roots of unity with $\omega_k^{**} = 1$ whenever $u_k \neq 0$, and that

$$H(\ldots, \omega_k^* T_k^{1/n}, \ldots) \neq H(\ldots, \omega_k^{**} T_k^{1/n}, \ldots).$$

Then

$$(6.5.7) \quad H(\ldots, \omega_k^* T_k^{1/n}, \ldots) - H(\ldots, \omega_k^{**} T_k^{1/n}, \ldots) = M^* \varepsilon^*(\ldots, T_k^{1/n}, \ldots)$$

where M^* is a characteristic monomial of ζ, and $\varepsilon^*(0,\ldots,0) \neq 0$. If we substitute v_k for $T_k^{1/n}$ in (6.5.7) $(1 \le k \le d)$, where $(v_1,\ldots,v_k) \in V$ and no $v_k = 0$, we see that

$$H(\ldots,\omega_k^* v_k,\ldots) \neq H(\ldots,\omega_k^{**} v_k,\ldots).$$

We conclude easily from this that

(6.5.8) *the branching order of* π_Y *at* y *is equal to the number of distinct conjugates of* ζ *of the form* $H(\ldots,\omega_k^* T_k^{1/n},\ldots)$ *with* ω_k^* *satisfying* (ii)* *above.*

Finally, from (6.5.6), noting that $\omega_k^* = (\omega_k')^{-1}\,\omega_k$ satisfies (ii)* (cf. (iii) preceding (6.5.6)) and recalling that M is divisible by both $T_i^{1/n}$ and $T_j^{1/n}$, we see that if $y' = \psi(v_1,\ldots,v_d)$ with either v_i or $v_j = 0$, then the number of distinct points in $\pi_Y^{-1}(\pi(y'))$ is *less* than the number of conjugates of ζ as in (6.5.8), so that $\pi_Y(y') \in D_Y$.

Q.E.D.

Example (6.5.9). Let $(X,x) \subseteq (\mathbb{C}^4,0)$ be the three-dimensional germ parametrized by the quasi-ordinary branch

$$\zeta = T_1^{3/2}T_2 + T_1^2 T_2^{3/2} T_3.$$

Both monomials on the right are characteristic for ζ, and $c = 3$. By (5.9.3), $m_1 = m_2 = 4$, $m_3 = 2$. Hence (cf.(5.10.3)) $H_4(X)_x$ has order 8. (In fact by (5.9), $H_4(X)_x \cong (\mathbb{Z}/2\mathbb{Z})^3$).

Here is a brief description of the behavior of (X,y) along the three components of $\Sigma^{(2)}$.

At any point y where $T_1 = T_2 = 0$, $T_3 \neq 0$, (X,y) is irreducible, and $H_4(X)_y$ has order 4. (In fact $H_4(X)_y \cong (\mathbb{Z}/2\mathbb{Z})^2$.)

At any point y where $T_1 = T_3 = 0$, $T_2 \neq 0$, (X,y) has two components (Y,Y') each of which is equisingular along the plane consisting of all points

(t_1,t_2,t_3,z) with $t_1 = z = 0$. So

$$H_4(Y)_y = H_4(Y')_y = 0.$$

The intersection of the two components is reducible, viz.

$$Y \cap Y' = \{t_1=0,\ z=0\} \cup \{t_3=0,\ z^2=t_1^3t_2^2\}.$$

Finally, at any y where $T_2 = T_3 = 0$, $T_1 \not\equiv 0$, (X,y) has two components Y, Y', with $H_4(Y)_y$ and $H_4(Y')_y$ both having order 2; and $Y \cap Y'$ is the plane $t_2 = z = 0$.

Remark (6.5.10). Using Mayer-Vietoris for Borel-Moore homology, and the fact that the singular locus of X is a manifold (or empty) at any point $y \notin \Sigma^{(2)}$ [Z, Theorem (4.5)], one deduces from (6.4) that at such a y:

$$H_{2d-2}(X)_y = 0.$$

I don't know whether it is possible to have a trivial $H_{2d-2}(X)_y$ for some $y \in \Sigma^{(2)}$. (If not, we would have a more elegant topological characterization of $\Sigma^{(2)}$.)

(6.6) We will also need the *topological singular locus* :

$$\Sigma^{(1)} = \{y \in X \mid X \text{ is not a topological manifold at } y\}.$$

This locus is clearly "topologically invariant", in the sense that it is respected by germ-homeomorphisms (cf.(6.5.3)).

Because of (6.4), we have

(6.6.1) $\Sigma^{(1)} = \Sigma^{(2)} \cup \{y \in X \mid X \text{ is reducible at } y\}.$

Proposition (6.6.2) *The topological singular locus* $(\Sigma^{(1)},x)$ *is an analytic subgerm of* (X,x). *Its codimension one components are the zero-sets* Z_i *of the prime ideals* $p_i = \sqrt{T_i R}$ $(1 \le i \le c)$ *for those* i *such that* $m_i \neq n_i$ *(cf. (4.5)). Its remaining components are components of* $\Sigma^{(2)}$ *(cf. (6.5.1)).*

Proof. Let $\pi:(X,x) \to (\mathbb{C}^d,0)$ be as before, with discriminant locus D. In view of (6.6.1) and the fact that $y \in X$ is smooth if $\pi(y) \notin D$, we see that

$$y \in \Sigma^{(1)} - \Sigma^{(2)} \Leftrightarrow \begin{cases} \pi(y) \text{ is a smooth point of } D \\ \text{and } X \text{ is reducible at } y. \end{cases}$$

So for any $y \in \Sigma^{(1)} - \Sigma^{(2)}$, there is a unique i such that $y \in Z_i$ and X is equisingular along Z_i at y. Since Z_i is irreducible, the number of components of X at *any* point $z \in Z_i$ where X is equisingular is the same [Z, §4][1], namely m_i/n_i (cf.(4.5.4)(iii)). Since X is reducible at y we have that $m_i > n_i$, and so

$$\Sigma^{(1)} - \Sigma^{(2)} \subset \bigcup_{m_i \neq n_i} Z_i.$$

Conversely if $m_i \neq n_i$, then X is equisingular and hence reducible everywhere along $Z_i - \bigcup_{j \neq i} Z_{ij}$, and so

$$\bigcup_{m_i \neq n_i} Z_i \subset \Sigma^{(1)}.$$

The conclusion follows.

---

[1] Zariski works in the "algebroid" category; but his results carry over to the analytic context. For example, if $\nu:\bar{X} \to X$ is normalization, then the induced map $\nu^{-1}(Z_i) \to Z_i$ is *étale* over any $z \in Z_i$ where X is equisingular, and so the fibre cardinality (= number of components of X) stays constant over the equisingular part of Z_i. All this, if not explicitly stated in [Z,§4], is easily extracted therefrom. Or, one could use the fact that equisingularity along Z_i implies that X is locally a topological product of Z_i with some plane curve germ, so that, again, the number of components of X at $y \in Z_i$ is locally constant.

* * *

(6.7) We proceed now to prove Theorem (6.1) by showing that the reduced branching sequence of ζ is determined by topological data associated with the *topologically invariant* subvarieties $\Sigma^{(2)} \subset \Sigma^{(1)}$ of the d-dimensional germ (X,x) parametrized by ζ (cf.(6.5.3),(6.6)).

The variables T_i will be labelled, again, as in (5.10.1). Because of (6.3), we may assume that ζ is reduced. Moreover, as in (6.4) and its proof, $c \leq 1$ is equivalent to (X,x) being topologically smooth and also to ζ having an empty branching sequence. So we may assume that $c \geq 2$.

For each $i = 1, 2,\ldots,c$, we let $\hat{Z}_i \subset X$ be the subvariety defined by

$$\hat{Z}_i\colon T_1 = T_2 = \ldots = T_{i-1} = T_{i+1} = \ldots = T_c = 0.$$

Arguing as at the beginning of §4, we see that $\hat{Z}_i$ is irreducible. *The collection of* c *distinct subvarieties* $\{\hat{Z}_1,\ldots,\hat{Z}_c\}$ *is topologically invariant*: its members are all the (d-c+1)-dimensional subvarieties of X which are intersections of components of $\Sigma^{(2)}$ (6.5.1), and, as in (6.5.3), $\Sigma^{(2)}$ is topologically invariant, hence so are its components, as well as intersections of those components and dimensions of such intersections [GL, p.172, Lemma (A.8)]. (This is the only place $\Sigma^{(2)}$ is needed.)

It is now evident that Theorem (6.1) will follow from:

Theorem (6.8). *Let* ζ' *be a reduced quasi-ordinary branch parametrizing a germ* (X',x'), *and suppose that there exists a germ-homeomorphism* $\varphi\colon(X,x) \to (X',x')$. *As noted above, for each* $i = 1,2,\ldots,c$ *there is an* i' *with* $1 \leq i' \leq c' = c$ *such that (with self-explanatory notation):*

$$\varphi(\hat{Z}_i) = \hat{Z}'_{i'}.$$

The assertion is that

$$m_i = m'_{i'}$$

and

$$n_i = n'_{i'} \qquad (1 \le i \le c).$$

Proof. Let $\nu:(\bar{X},\bar{x}) \to (X,x)$ be the normalization map. For any $y \in X$, let $\bar{y} \in \nu^{-1}(\{y\})$, and let

$$a_{\bar{y}} = \text{order of } H_{2d-2}(\bar{X})_{\bar{y}}.$$

We also set

$$e_y = \text{number of components of } X \text{ at } y.$$

Lemma (6.8.1) *If*

$$y \in \hat{Z}_i - \bigcup_{j \neq i} \hat{Z}_j$$

then

$$e_y(a_{\bar{x}}/a_{\bar{y}}) = n_i.$$

The *proof* of (6.8.1) will be given below.

Now if $y' = \varphi(y)$ then $e_y = e'_{y'}$ (topological invariance of components, cf. (6.5.3)). Since local homology groups are topological invariants (1.1), to see that $a_{\bar{y}} = a'_{\bar{y}'}$ (and similarly $a_{\bar{x}} = a'_{\bar{x}'}$) - whence, by (6.8.1), $n_i = n'_{i'}$ - it suffices to note that, with ν and ν' the respective normalizations, there is a germ-homeomorphism $\bar{\varphi}$ making the following diagram commute:

$$\begin{array}{ccc} (\bar{X},\bar{x}) & \xrightarrow{\bar{\varphi}} & (\bar{X}',\bar{x}') \\ \nu \downarrow & & \downarrow \nu' \\ (X,x) & \xrightarrow[\varphi]{} & (X',x'). \end{array}$$

The existence of such a $\bar{\varphi}$ is immediate from the above-mentioned "topological invariance of components" and the following topological description of ν (cf. e.g.[W, p.258]): The points of $\bar{X}$ are equivalence classes of pairs (y,Y) where $y \in X$ and Y is a locally closed subvariety of X containing, and irreducible at, y, and where $(y,Y) \equiv (y^*,Y^*)$ if and only if $y = y^*$ and $Y = Y^*$ near y; the map ν is defined by $\nu(y,Y) = y$; and a basis of open sets on $\bar{X}$ is given by $\nu^{-1}(V)$ as V runs through all everywhere irreducible locally closed subvarieties of X.

Having thus shown that $n_i = n'_{i'}$, we now prove that $m_i = m'_{i'}$, considering two cases.

(A) *Every codimension one component of the topological singular locus* $\Sigma^{(1)}$ *(6.6) contains* $\hat{Z}_i$.

In this case Z_i (the zero-set of $p_i = \sqrt{T_i R}$) cannot be a component of $\Sigma^{(1)}$, and hence, by (6.6.2), $m_i = n_i$. The condition (A) is of course "preserved" by φ, so we see similarly that

$$m'_{i'} = n'_{i'} \quad (= n_i = m_i).$$

(B) *Some codimension one component of* $\Sigma^{(1)}$ *does not contain* $\hat{Z}_i$.

In this case there is a *unique* codimension one component of $\Sigma^{(1)}$ not containing $\hat{Z}_i$, namely Z_i. By (4.5.4)(iii)

$$m_i = e_i n_i$$

where e_i is the number of components of X at any point z in some open dense subset of Z_i. Similarly ((B) being preserved by φ)

$$m'_{i'} = e'_{i'} n_{i'}.$$

But by topological invariance of components

$$e'_{i'} = e_i.$$

Since $n'_{i'} = n_i$ (see above) therefore $m'_{i'} = m_i$.

Q.E.D.

It remains to prove Lemma (6.8.1).

For any component Y of X at y, π induces a quasi-ordinary projection

$$\pi_Y\colon (Y,y) \to (\mathbb{C}^d,\pi(y))$$

whose discriminant at $\pi(y)$ is contained in the zero-set of $T_1 \dots T_{i-1}T_{i+1} \dots T_c$. The corresponding integers

$$n_j(y) \qquad 1 \le j \le c, \quad j \neq i$$

are the same as the integers $n_j = n_j(x)$: indeed, by (4.5.4), if $\nu\colon \bar{X} \to X$ is normalization, and $\bar{z}$ is a generic point of

$$\nu^{-1}(\pi_Y^{-1}\{T_j=0\} \cap V)$$

where V is some neighborhood of y on Y, then

$$n_j(y) = \deg((\pi_Y \circ \nu)_{\bar{z}}) = \deg(\bar{\pi}_{\bar{z}}) = n_j(x).$$

It follows then from (4.1.3) that

$$a_{\bar{y}} = n_1 \dots n_{i-1}n_{i+1} \dots n_c/\deg((\pi_Y)_y),$$

and that

$$a_{\bar{x}} = n_1n_2 \dots n_c/\deg(\pi_x).$$

Thus

$$(6.8.2) \qquad n_i a_{\bar{y}}/a_{\bar{x}} = \deg(\pi_x)/\deg((\pi_Y)_y);$$

and we need to prove that the right hand side is e_y.

We show first that

$$(6.8.3) \qquad \pi^{-1}\{\pi(y)\} = \{y\}.$$

By (5.3.2) setting

$$\zeta = H(T_1^{1/n},\dots,T_d^{1/n})$$

we can write

$$y = (s_1^n,\dots,s_d^n,H(s_1,\dots,s_d))$$

for some complex numbers $s_1,\dots,s_d$ with

$$(6.8.4) \qquad s_1 = \dots = s_{i-1} = s_{i+1} = \dots = s_c = 0;$$

and then the number of points in $\pi^{-1}\pi(y)$ is equal to the number of distinct values of

$$H(\omega_1 s_1,\dots,\omega_d s_d)$$

as $(\omega_1,\dots,\omega_d)$ runs through all d-tuples of n-th roots of unity. However, by (5.2.2), (5.5.1), we have, for any other such d-tuple $(\omega_1',\dots,\omega_d')$:

$$H(\omega_1 s_1,\dots,\omega_d s_d) - H(\omega_1' s_1,\dots,\omega_d' s_d) = M(s_1,\dots,s_d)\varepsilon(s_1,\dots,s_d)$$

where, if the right hand side is not zero, then $M(T_1^{1/n},\dots,T_d^{1/n})$ is a characteristic monomial of ζ. Since ζ is, by assumption, *reduced*, it follows that M is divisible by $T_j^{1/n}$ for some $j \neq i$ (cf.(6.2)). But then (6.8.4) shows that $M(s_1,\dots,s_d) = 0$. So (6.8.3) holds.

Finally, for a generic ξ near $\pi(y)$, the fibre $\bar{\pi}^{-1}(\xi)$ has cardinality equal to $\deg(\pi_X)$ (where $\bar{\pi} = \pi \circ \nu$, cf.(4.5.3)). Also, as in the proof of (4.5.4), the degree $\deg(\bar{\pi}_{\bar{y}})$ is the same for *all* $\bar{y} \in \nu^{-1}(y)$, and is in fact equal to $\deg((\pi_Y)_y)$, since ν maps a neighborhood of some $\bar{y}$ onto a neighborhood of y in Y, almost everywhere bijectively. Since the fibre $\nu^{-1}(y)$ has cardinality e_y, we conclude that

$$e_y \deg((\pi_Y)_y) = \deg(\pi_X),$$

i.e. the right hand side of (6.8.2) is e_y.

This completes the proof of (6.8.1), and of Theorem (6.1).

§7 Appendix: The singular locus

We substantiate the importance of the characteristic monomials of a quasi-ordinary parametrization

$$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$$

of a germ (X,x) by showing how these monomials determine the multiplicity of a point $y \in X$ as soon as we know which of the T_i vanish at y (Theorem (7.2)). There results a fairly detailed description of the singular locus $\Sigma(X,x)$ (Theorem (7.3))[(1)]. In particular, every component of $\Sigma(X,x)$ is of the form Z_i (codimension one, cf.(6.6.2)) or Z_{ij} (codimension two, cf. (6.5.1)); and for components of the latter form, we prove for any point $y \in Z_{ij}$ where no T_k with $k \neq i,j$ vanishes, that (X,y) is analytically isomorphic to a subgerm of $(\mathbb{C}^{d+1},0)$ defined by an equation of the form $Z^b = T_iT_j$.

We conclude with some examples to illustrate both the results of this section and the computation (as in (5.9)) of $A_{d-1}(X)_x \cong H_{2d-2}(X)_x$.

(7.1) As usual we label the variables T_i in such a way that the discriminant $\Delta(\zeta)$ (cf. (5.10)) is divisible by T_i if and only if $i \leq c$. As in (6.6.2) we let $Z_i \subset X$ be the zero-set of T_i; and for any sequence

$$I: 1 \leq i_1 < i_2 < \ldots < i_e \leq c$$

we set

$$Z_I = Z_{i_1i_2\ldots i_e} = Z_{i_1} \cap Z_{i_2} \cap \ldots \cap Z_{i_e}.$$

At the beginning of §4 we saw that Z_i is irreducible at x. A similar argument shows that:

---

(1) Such a description will be needed, at least in part, to prove (0.2.1).

Z_I *is irreducible at* x, *of codimension* e *in* X. *The prime ideal*

$$p_I \subset R = \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta]$$

(cf. (5.2)) consisting of germs of functions vanishing on Z_I *is*

$$p_I = (T_{i_1}^{1/n},\ldots,T_{i_e}^{1/n})\,\mathbb{C}\langle T_1^{1/n},\ldots,T_d^{1/n}\rangle \cap R.$$

We define

$$Z_I^0 = Z_I - \bigcup_{\substack{1\le j\le c\\ j\notin I}} Z_j\,.$$

Then Z_I^0 is a connected open dense subvariety of Z_I; and clearly

(7.1.1) $$Z_I = \bigcup_{I'\supseteq I} Z_{I'}^0, \qquad \text{(disjoint union).}$$

We also set

$$\zeta^I = \begin{cases} \text{the fractional power series obtained} \\ \text{from } \zeta = H(T_1^{1/n},\ldots,T_d^{1/n}) \text{ by substituting } 0 \\ \text{for every one of } T_{i_1}^{1/n},\ldots,T_{i_e}^{1/n}. \end{cases}$$

As usual, $\pi\colon (X,x) \longrightarrow (\mathbb{C}^d,0)$ denotes the projection corresponding to the inclusion

$$\mathbb{C}\langle T_1,\ldots,T_d\rangle \subset \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta] = R,$$

and L denotes the field of fractions of $\mathbb{C}\langle T_1,\ldots,T_d\rangle$.

Proposition (7.1.2) *With preceding notation, let* D_I *be the submanifold of* $\mathbb{C}^d$ *defined by* $T_{i_1} = \ldots = T_{i_e} = 0$, *and let*

$$D_I^0 = \{z \in D_I \mid T_j \text{ does not vanish at } z \text{ if } 1 \le j \le c \text{ and } j \notin I\},$$

so that $\pi^{-1}(D_I^0) = Z_I^0$. *Then*

$$\pi\colon Z_I^0 \longrightarrow D_I^0$$

is an étale covering of degree $[L(\zeta^I):L]$. *In particular* Z_I^0 *is smooth.*

Remark (7.1.3). The space $D \subset \mathbb{C}^d$ defined by $T_1T_2 \dots T_c = 0$ has an obvious stratification, with strata D_I^0. Together with (7.1.1), (7.1.2) shows that the inverse image of this stratification under π is a stratification of X. By [Spd, p. 574, Lemma 2], for example, *the multiplicity of* X *is constant along each stratum.* We will prove this in (7.2) below by actually calculating the multiplicities.

Proof of (7.1.2) For convenience, we may assume that I is the sequence $1 < 2 <\dots< e$. The assertion is then that for *any* sufficiently small complex numbers $t_{e+1}, t_{e+2},\dots,t_d$ with $t_i \neq 0$ for $e < i \le c$, the number τ of points in the fibre

$$\pi^{-1}(0,\dots,0,t_{e+1},\dots,t_d)$$

satisfies

$$\tau = [L(\zeta^I):L] = [L(H(0,\dots,0,T_{e+1}^{1/n},\dots,T_d^{1/n})):L].$$

According to (5.3.2), τ is equal to the number of distinct values of $H(0,\dots,0,s_{e+1},\dots,s_d)$ as $(s_{e+1},\dots,s_d)$ runs through all $(d-e)$-tuples of complex numbers such that

$$s_k^n = t_k \qquad e < k \le d.$$

So we must show that *this last number is equal to the number of distinct conjugates of* ζ^I *over* L (*i.e. to* $[L(\zeta^I):L]$).

Now (cf.(5.5.1)) the conjugates of ζ^I are all of the form

$$\zeta_a^I = H_a(0,\dots,0,\omega^{a_{e+1}}T_{e+1}^{1/n},\dots,\omega^{a_d}T_d^{1/n})$$

where $\omega = e^{2\pi i/n}$, a primitive n-th root of unity, and

$$a = (a_{e+1},\dots,a_d) \in \mathbb{Z}^{d-e};$$

and $\zeta_a^I \neq \zeta_{a'}^I$ implies that

$$\zeta_a^I - \zeta_{a'}^I = T_{e+1}^{b_{e+1}/n} \cdots T_c^{b_c/n}\,\varepsilon(T_{e+1}^{1/n},\dots,T_d^{1/n}) \qquad (7.1.2.1)$$

for suitable integers $b_{e+1},\dots,b_c$ and a power series ε with $\varepsilon(0,\dots,0) \neq 0$ (we use here the fact that for $j > c$, $T_j^{1/n}$ does not divide any characteristic monomial of ζ, cf.(5.10)). So, substituting (sufficiently small) s_k for $T_k^{1/n}$ in (7.1.2.1), with $s_{e+1}s_{e+2}\cdots s_c \neq 0$, we see that there is indeed a one-one correspondence between conjugates of ζ^I and values of $H(0,\dots,0,s_{e+1},\dots,s_d)$ as above. Q.E.D.

(7.1.4) The fractional power series ζ^I is a quasi-ordinary branch whose characteristic monomials are those of ζ which are not divisible by $T_i^{1/n}$ for any $i \in I$ (cf.(5.2),(5.5.1)). From (5.7) it follows then that $L(\zeta^I) \subset L(\zeta)$. We set

$$m_{i_1 i_2 \cdots i_e} = m_I = [L(\zeta):L(\zeta^I)].$$

As in the proof of (5.10.1), for each $i = 1,2,\dots,c$ there is an integer e_i such that $T_i^{1/n}$ divides the characteristic monomial M_j if and only if $j > e_i$. From this it follows that

$$m_I = \max_{i\in I} m_i\ .$$

Recall from (4.5) the definition of the *branching order* deg (π_y) at a point $y \in X$. The following result generalizes (5.9.3).

Proposition (7.1.5). *For any* $y \in Z_I^0$, *the branching order* $\deg(\pi_y)$ *is equal to* m_I.

Proof. We show first that $\deg(\pi_y) \geq m_I$. Choose $i \in I$ such that $m_I = m_i$, and apply semicontinuity of branching orders: let (y_α) be a sequence of points in Z_i approaching y; from (the proof of) (4.5.4), we see that for any given $\delta > 0$, there are points ξ_α in any neighborhood of $\pi(y_\alpha)$ such that m_i distinct members of the fibre $\pi^{-1}(\xi_\alpha)$ are at a distance less than δ from y_α; and letting $\alpha \to \infty$, we find that indeed $\deg(\pi_y) \geq m_i$.

Now by (7.1.2), the fibre $\pi^{-1}\pi(y) \subset Z_I^0$ has $[L(\zeta^I):L]$ members. Since $\deg(\pi_x) = [L(\zeta):L]$, we have

$$[L(\zeta):L] = \sum_{\pi(y')=\pi(y)} \deg(\pi_{y'}) \geq m_I[L(\zeta^I):L] = [L(\zeta):L].$$

Hence

$$\deg(\pi_{y'}) = m_I$$

for all y'. Q.E.D.

Now we can specify the multiplicity of any $y \in Z_I^0$.

Theorem (7.2). *With preceding notation, if there exist characteristic monomials of* ζ *in which the exponents of* $T_{i_1},\ldots,T_{i_e}$ *are not all integers, then let*

$$M^*_{i_1 i_2 \ldots i_e} = M^*_I = T_1^{\lambda_1} \ldots T_e^{\lambda_e}$$

be the smallest such monomial (cf. (5.6)). Set

$$\begin{aligned} \lambda_I &= \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_e} && \text{if } M_I^* \text{ exists} \\ &= 1 && \text{otherwise.} \end{aligned}$$

Then, for any $y \in Z_I^0$, *the multiplicity* $\mu(X)_y$ *of* X *at* y *satisfies*

$$\mu(X)_y = \min(m_I, \lambda_I m_I).$$

with m_I *as in (7.1.4), (7.1.5).*

Remark. In (7.2) we could have defined M_I^* to be the smallest characteristic monomial of ζ in which the exponents of $T_{i_1},\ldots,T_{i_e}$ are not all zero. This might change the value of λ_I, but *not* the value of $\min(1,\lambda_I)$.

We begin the *proof of (7.2)* with

Lemma (7.2.1). *Let*

$$M^* = T_1^{\lambda_1} \ldots T_c^{\lambda_c}$$

be the smallest characteristic monomial of ζ *(if* ζ *has any characteristic monomial, i.e. if* $[L(\zeta):L] > 1$*); and set*

$$\begin{aligned} \lambda = \lambda_\zeta &= \lambda_1 + \lambda_2 + \ldots + \lambda_c && \text{if } M^* \text{ exists} \\ &= 1 && \text{otherwise.} \end{aligned}$$

Also, set

$$m = \deg(\pi_X) = [L(\zeta):L].$$

Then

$$\mu(X)_x = \min(m, \lambda m).$$

Proof. With $\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$ as before, let

$$H_0(T_1,\ldots,T_d) = \begin{cases} \text{sum of all those terms } aT_1^{\rho_1} \ldots T_d^{\rho_d} \ (a\neq 0) \\ \text{appearing in } H \text{ for which every} \\ \text{exponent } \rho_i \text{ is an integer.} \end{cases}$$

Nothing of interest changes if we replace ζ by $\zeta - H_0$; so *we may assume that* $H_0 = 0$.

Then every monomial M_0 appearing in H is "moved" by some automorphism θ as in (5.5), i.e. it appears, multiplied by $1-\omega$ for some n-th root of unity $\omega \neq 1$, in

$$\zeta - \theta\zeta = M\varepsilon(T_1^{1/n},\ldots,T_d^{1/n}) \qquad (\text{cf.}(5.2.2)).$$

Thus M_0 is divisible by the characteristic monomial M of ζ, which is in turn divisible by M^*. Hence we have *either* $\zeta = 0$, in which case (7.2.1) is obvious; *or*

$$\zeta = M^*\eta(T_1^{1/n},\ldots,T_d^{1/n}) \qquad \eta(0,\ldots,0) \neq 0;$$

and then if $\zeta_1, \zeta_2,\ldots,\zeta_m$ are the distinct conjugates of ζ over L, we can write (by(5.5.1)):

$$\zeta_i = M^*\eta_i(T_1^{1/n},\ldots,T_d^{1/n}) \qquad \eta_i(0,\ldots,0) \neq 0.$$

In this case, the local ring R of (X,x) satisfies

$$R \cong \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta] = \mathbb{C}\langle T_1,\ldots,T_d\rangle[Z]/F(Z)$$

where

$$(7.2.1.1) \qquad F(Z) = \prod_{i=1}^{m} (Z-\zeta_i) = \prod_{i=1}^{m} (Z-M^*\eta_i)$$

is the minimal polynomial of ζ. The multiplicity $\mu(X)_x$ of the local ring R is the order of vanishing, at the origin, of the power series

$$F(Z) \in \mathbb{C}\langle T_1,\ldots,T_d,Z\rangle;$$

and it is apparent from (7.2.1.1) that this order is precisely $\min(m,\lambda m)$, as asserted. Q.E.D.

Now consider an arbitrary $y \in Z_I^0$. Let $Y_1,\dots,Y_a$ be the local components of X at y. Then for any $i = 1,2,\dots,a$ and any analytic isomorphism of germs

$$\psi_*:(\mathbb{C}^d,\pi(y)) \to (\mathbb{C}^d,0)$$

the map

$$\pi_i = \psi_*\circ\pi:(Y_i,y) \to (\mathbb{C}^d,0)$$

is clearly a quasi-ordinary projection, and correspondingly the germ (Y_i,y) will have a quasi-ordinary parametrization $\zeta_{(i)}$. Setting

$$\mu_{(i)} = \mu(Y_i)_y$$

$$m_{(i)} = \deg((\pi_i)_y)$$

$$\lambda_{(i)} = \lambda_{\zeta_{(i)}} \qquad \text{(cf.(7.2.1))}$$

we have

$$\mu(X)_y = \sum_{i=1}^{a} \mu_{(i)}$$

$$m_I = \sum_{i=1}^{a} m_{(i)}.$$

In view of (7.2.1), therefore, (7.2) *will be proved if we show that:*

(7.2.2) $$\lambda_{(i)} = \lambda_I \qquad \textit{for all } i = 1,2,\dots,a.$$

For this purpose, let us describe the appropriate $\zeta_{(i)}$ more explicitly.

We note first that for $j > c$, T_j appears in H only with integral exponents, i.e. *there is a convergent power series* $H^* = H^*(U_1,\dots,U_d)$ *such that*

$$H(U_1,\dots U_c,U_{c+1},\dots,U_d) = H^*(U_1,\dots,U_c,U_{c+1}^n,\dots,U_d^n),$$

whence

$$\zeta = H^*(T_1^{1/n},\dots,T_c^{1/n},T_{c+1},\dots,T_d).$$

The reason is this: if in H there appears a term

$$M_0 = tT_1^{\lambda_1} \dots T_j^{\lambda_j} \dots T_d^{\lambda_d} \qquad 0 \neq t \in \mathbb{C}$$

then as in the proof of (5.7) we have

$$M_0 \in L(\zeta) = L(M_2,\dots,M_m)$$

where the M_i are the characteristic monomials of ζ; and since no such M_i is divisible by $T_j^{1/n}$ (cf.(5.10)), it follows that $\lambda_j \in \mathbb{Z}$.

For convenience we may assume that I is the sequence $1 < 2 <\dots< e$. The patient reader will have no difficulty in deducing (7.2.2) from the following Lemma.

Lemma (7.2.3). *Let*

$$y = (0,\dots,0,t_{e+1},\dots,t_d,z) \in X \subset \mathbb{C}^{d+1} \qquad (e \leq c)$$

where

$$t_{e+1}t_{e+2} \dots t_c \neq 0 \qquad (\textit{i.e.}\ \ y \in Z_I^0).$$

Then, with H^* *as above and* y *sufficiently close to* $0 \in \mathbb{C}^{d+1}$, *for each component* (Y_i,y) *of the germ* (X,y) *there is an analytic isomorphism*

$$\psi_*:(\mathbb{C}^d,\pi(y)) \to (\mathbb{C}^d,0)$$

such that the projection $\psi_*\circ\pi$ *is associated with a quasi-ordinary parametrization of* (Y_i,y) *of the form*

$$\zeta_u = H^*(T_1^{1/n},\dots,T_e^{1/n},T_{e+1} + u_{e+1},\dots,T_d + u_d) - z$$

where the $u_i \in \mathbb{C}$ *satisfy*

$$u_k^n = t_k \qquad e < k \le c$$

$$u_\ell = t_\ell \qquad c < \ell \le d$$

and

$$H^*(0,\ldots,0,u_{e+1},\ldots,u_d) = z.$$

Moreover, if the monomial

$$M_I^* = T_1^{\lambda_1} T_2^{\lambda_2} \ldots T_c^{\lambda_c}$$

of Theorem (7.2) exists, then the smallest characteristic monomial of ζ_u *is* $T_1^{\lambda_1} T_2^{\lambda_2} \ldots T_e^{\lambda_e}$; *and otherwise* ζ_u *has no characteristic monomials.*

Proof. As in (5.3.2) we may take X to be the image of a neighborhood V of $0 \in \mathbb{C}^d$ under the map $\varphi\colon V \to \mathbb{C}^{d+1}$ given by

$$\varphi(v_1,\ldots,v_d) = (v_1^n,\ldots,v_c^n,v_{c+1},\ldots,v_d,H^*(v_1,\ldots,v_d)),$$

the map

$$\pi\colon(X,y) \to (\mathbb{C}^d,\pi(y))$$

being given by projection to the initial d coordinates.

The preceding map φ factors as $V \to \overline{X} \xrightarrow{\nu} X$, where ν is the normalization map. For any sufficiently small neighborhood X_y of y in X, ν separates the components of X_y at y in the sense that $\nu^{-1}(X_y)$ consists of disjoint open sets, each mapped by ν onto an irreducible component of X_y passing through y; and hence a similar statement holds for $\varphi^{-1}(X_y)$.[1] Thus φ induces parametrizations of the germs (Y_i,y).

[1] We didn't really need ν here, just that $\varphi\colon V \to X$ is proper and surjective, with finite fibres.

More specifically, let $u \in \varphi^{-1}(y)$, say

$$u = (0,0,\dots,0,u_{e+1},\dots,u_d).$$

Since, for $e < k \le c$, $u_k^n = t_k \neq 0$, we can choose $\varepsilon > 0$, and holomorphic functions $\psi_k(\tau)$ defined for $|\tau - t_k| < \varepsilon$ such that

$$\psi_k(\tau) = \sqrt[n]{\tau} - u_k, \quad \psi_k(t_k) = 0;$$

i.e. for some $\delta > 0$ and all v_k with $|v_k| < \delta$:

$$\psi_k((v_k+u_k)^n) = v_k \qquad (e < k \le c).$$

Then we can define an *analytic isomorphism* of germs

$$\psi:(\mathbb{C}^{d+1},y) \xrightarrow{\sim} (\mathbb{C}^{d+1},0)$$

by

(7.2.3.1) $\psi(\tau_1,\dots,\tau_d,\xi) = (\tau_1,\dots,\tau_e,\psi_{e+1}(\tau_{e+1}),\dots,\psi_c(\tau_c),\tau_{c+1}-t_{c+1},\dots,\tau_d-t_d,\xi-z).$

Since φ is actually a *proper* map of V onto X, with finite fibres, we can choose the above neighborhood X_y such that

$$\varphi^{-1}(X_y) \subset \bigcup_{u \in \varphi^{-1}(y)} N(u,\delta)$$

where $N(u,\delta)$ is the polydisc neighborhood of u with radii all equal to δ. As remarked above, the components of X at y are then represented by the irreducible varieties

$$\{\varphi(\varphi^{-1}(X_y) \cap N(u,\delta))\}_{u \in \varphi^{-1}(y)}.$$

(Some repetition of the same component may occur here.)

But for all $(v_1,\dots,v_d)$ in $N(0,\delta)$ we have

$$\psi\,\varphi(v_1,\dots,v_e,v_{e+1}+u_{e+1},\dots,v_d+u_d)$$

$$=(v_1^n,\dots,v_e^n,v_{e+1},\dots,v_c,v_{c+1},\dots,v_d,H^*(v_1,\dots,v_e,v_{e+1}+u_{e+1},\dots,v_d+u_d)-z).$$

Thus each (Y,y) has a parametrization of the asserted form ζ_u, associated with the quasi-ordinary projection $\psi_*\circ\pi$, where ψ_* is the restriction of ψ to $(\mathbb{C}^d,\pi(y))\subset(\mathbb{C}^{d+1},y)$ (inclusion via the initial d coordinates).

For the last assertion in (7.2.3) let $\omega_1,\dots,\omega_e$ be n-th roots of unity such that

$$\text{(7.2.3.2)}\quad H^*(T_1^{1/n},\dots,T_e^{1/n},T_{e+1}+u_{e+1},\dots)-H^*(\omega_1T_1^{1/n},\dots,\omega_eT_e^{1/n},T_{e+1}+u_{e+1},\dots)\neq 0.$$

Then this difference can be written in the form

$$\text{(7.2.3.3)}\quad T_1^{\lambda_1'}\dots T_e^{\lambda_e'}(T_{e+1}+u_{e+1})^{n\lambda_{e+1}'}\dots(T_c+u_c)^{n\lambda_c'}\eta(T_1^{1/n},\dots,T_e^{1/n},T_{e+1}+u_{e+1},\dots)$$

where

$$\text{(7.2.3.4)}\quad T_1^{\lambda_1'}T_2^{\lambda_2'}\dots\ T_c^{\lambda_c'}=T_1^{\lambda_1'}\dots\ T_e^{\lambda_e'}(T_{e+1}^{1/n})^{n\lambda_{e+1}'}\dots(T_c^{1/n})^{n\lambda_c'}$$

is a characteristic monomial of ζ (cf.(5.2.2), (5.5.1), and recall that $T_{c+1},\dots,T_d$ do not appear in the characteristic monomials), and where η is a convergent power series not vanishing at $0\in\mathbb{C}^d$, so that (y being sufficiently close to $0\in\mathbb{C}^{d+1}$)

$$\eta(0,\dots,0,u_{e+1},\dots,u_d)\neq 0.$$

By assumption

$$(u_{e+1}\dots u_c)^n=t_{e+1}\dots t_c\neq 0$$

and so (7.2.3.3) can be rewritten as

(7.2.3.3)' $T_1^{\lambda_1'} \ldots T_e^{\lambda_e'}\eta'(T_1^{1/n},\ldots,T_e^{1/n},T_{e+1},\ldots,T_d) \qquad \eta'(0,\ldots,0) \neq 0.$

This shows again that ζ_u is a quasi-ordinary branch; and the characteristic monomial $T_1^{\lambda_1'} \ldots T_e^{\lambda_e'}$ of ζ_u is obtained from a characteristic monomial (7.2.3.4) of ζ by substituting 1 for $T_{e+1}^{1/n},\ldots,T_c^{1/n}$. Moreover not all of $\lambda_1',\ldots,\lambda_e'$ can be integers, since otherwise the monomial $T_1^{\lambda_1'} \ldots T_e^{\lambda_e'}$ could not appear in the difference (7.2.3.2). Thus every characteristic monomial of ζ_u is divisible by the monomial $T_1^{\lambda_1} \ldots T_e^{\lambda_e}$ described in (7.2.3), provided that M_I^* exists; and ζ_u has no characteristic monomials if M_I^* doesn't exist.

It remains to be shown, when $M_I^* = T_1^{\lambda_1} \ldots T_c^{\lambda_c}$ exists, that $T_1^{\lambda_1} \ldots T_e^{\lambda_e}$ is a characteristic monomial of ζ_u.

Write $\lambda_i = b_i/n$ with $b_i \in \mathbb{Z}$ $(1 \le i \le c)$. Since $\lambda_1,\ldots,\lambda_e$ are not all integers, there exist n-th roots of unity $\omega_1,\ldots,\omega_e$ such that

$$(\omega_1 T_1^{1/n})^{b_1} \ldots (\omega_e T_e^{1/n})^{b_e} \neq T_1^{b_1/n} \ldots T_e^{b_e/n}.$$

Then the monomial M_I^* appears (with non-zero coefficient) in

$$H(\omega_1 T_1^{1/n},\ldots,\omega_e T_e^{1/n},T_{e+1}^{1/n},\ldots,T_d^{1/n}) - H(T_1^{1/n},\ldots,T_d^{1/n}) = M\varepsilon(T_1^{1/n},\ldots,T_d^{1/n})$$

where $\varepsilon(0,\ldots,0) \neq 0$ and

$$M = T_1^{\lambda_1'} \ldots T_c^{\lambda_c'}$$

is a characteristic monomial of ζ, with at least one of $\lambda_1',\lambda_2',\ldots,\lambda_e'$ not an integer. Since M divides M_I^*, it follows from the definition of M_I^* that $M = M_I^*$. As above ((7.2.3.2) etc.) we find that $T_1^{\lambda_1'} \ldots T_e^{\lambda_e'}$ is a characteristic monomial of ζ_u, and we are done.

(7.3) Let (X,x) be parametrized by the quasi-ordinary branch

$$\zeta = H(T_1^{1/n},\ldots,T_d^{1/n})$$

where, as usual, the variables are labelled so that T_i divides the discriminant $\Delta(\zeta)$ if and only if $i \le c$ (cf.(5.10)).

We are ready now to describe the *singular locus*

$$\Sigma = \{y \in X \mid X \text{ is not smooth at } y\}$$

$$= \{y \in X \mid \mu(X)_y > 1\}$$

where $\mu(X)_y$ is the multiplicity of X at y.

Theorem (7.3). (a) *The singular locus* (Σ,x) *is an analytic subgerm of* (X,x) *whose components (at* x*) all have the form* Z_i $(1\le i\le c)$ *or* Z_{ij} $(1\le i<j\le c)$, *cf. (7.1). Moreover if neither* Z_i *nor* Z_j *is a component (where* $1\le i<j\le c$*) then* Z_{ij} *is a component.*

(b) *For* $1 \le i \le c$, Z_i *is not a component of* Σ *if and only if* $M_i^* = T_1^{\lambda_1} \ldots T_c^{\lambda_c}$ *exists (cf.(7.2)) and is the largest characteristic monomial of* ζ, *and* $\lambda_i = 1/b$ *for some integer* $b \ge 2$ *such that*

$$(M_i^*)^b \in L(\zeta^i) = L(H(T_1^{1/n},\ldots,T_{i-1}^{1/n},0,T_{i+1}^{1/n},\ldots,T_d^{1/n}))$$

(where L *is the fraction field of* $\mathbb{C}\langle T_1,\ldots,T_d\rangle$*).*

(c) *For* $1 \le i < j \le c$, Z_{ij} *is a component of* Σ *if and only if for some integer* $b \ge 2$ *and for some* $y \in Z_{ij}$, *the germ* (X,y) *is isomorphic to the subgerm of* $(\mathbb{C}^{d+1},0)$ *defined by the equation* $Z^b = T_iT_j$. *If* Z_{ij} *is a component, then in fact this last condition holds for all* $y \in Z_{ij}^0$ *(cf.(7.1)).*

Remarks. (7.3.1) Concerning (b), it is not hard to show, using (5.7), that if $M_1 < M_2 < \ldots < M_a$ are the characteristic monomials of ζ which are not divisible by $T_i^{1/n}$ (i.e. they are the characteristic monomials of ζ^i) then $(M_i^*)^b \in L(\zeta^i)$ if and only if there are integers $p_j (1\le j\le d)$ and $q_k (1\le k\le a)$ such that

$$(M_i^*)^b = T_1^{p_1} \ldots T_d^{p_d} M_1^{q_1} \ldots M_a^{q_a}.$$

Thus (or by relations similar to (5.9.4)) the conditions in (b) can be checked computationally.

Also, the integer b in (c) can be determined easily from the characteristic monomials of ζ (cf. proof of (7.3.3) below).

(7.3.2) Suppose that (X,x) is *normal*. Then no Z_i is a component of Σ, and (b) shows that ζ has either no characteristic monomials (i.e. (X,x) is smooth) or exactly one characteristic monomial, this being of the form $T_1^{1/b}T_2^{1/b} \ldots T_c^{1/b}$. Using (5.7), we see that $R = \mathbb{C}\langle T_1,\ldots,T_d\rangle[\zeta]$ is then the integral closure of $\mathbb{C}\langle T_1,\ldots,T_d\rangle$ in $L(T_1^{1/b}T_2^{1/b} \ldots T_c^{1/b})$, i.e.

$$R \cong \mathbb{C}\langle T_1,\ldots,T_d\rangle[(T_1T_2 \ldots T_c)^{1/b}] \cong \mathbb{C}\langle T_1,\ldots,T_d,Z\rangle/(Z^b-T_1T_2 \ldots T_c).$$

In other words, (X,x) is isomorphic to the subgerm $X_{b,c}$ of $(\mathbb{C}^{d+1},0)$ given by the equation $Z^b = T_1T_2 \ldots T_c$ ($b=1$ if X is smooth).

Conversely any such $X_{b,c}$ is normal, and b, c are uniquely determined by $X_{b,c}$: in fact $X_{b,c}$ is a quotient singularity with local fundamental group $(\mathbb{Z}/b\,\mathbb{Z})^{c-1}$, and reduced branching sequence consisting of c copies of the pair (b,b) if $c \geq 2$ (cf. §6).

Proof of (7.3). (a) Let $\pi:(X,x) \to (\mathbb{C}^d,0)$ be as usual, with discriminant locus D given by $T_1T_2 \ldots T_c = 0$. Then

$$\Sigma \subset \pi^{-1}(D) = \bigcup_{1\leq i\leq c} Z_i.$$

Also, by (6.6),

$$\Sigma^{(2)} = \bigcup_{1\leq i<j\leq c} Z_{ij} \subset \Sigma^{(1)} \subset \Sigma.$$

If Σ meets Z_i^0, then, since $\mu(X)_y$ is the same for all $y \in Z_i^0$ (by Theorem (7.2), or by equisingularity of X along Z_i^0) it follows that $Z_i \subset \Sigma$. Hence Σ is the union of $\Sigma^{(2)}$ and of all the Z_i such that Σ meets Z_i^0; and (7.3)(a) results.

(b) If $i \le c$, then $m_i > 1$ (5.10.1). So by (7.2), if Z_i is not a component of Σ, then

$$\lambda_i = m_i^{-1} < 1$$

and hence M_i^* exists. Let us show that then M_i^* is the largest characteristic monomial of ζ, and that

$$(M_i^*)^b \in L(\zeta^i) \quad \textit{where} \quad b = m_i = \lambda_i^{-1}.^{(1)}$$

Assume, for convenience, that $i = 1$. Since $\lambda_1 = b^{-1}$, the orbit of M_1^* under the group of $\mathbb{C}\langle T_1, T_2^{1/n}, \ldots, T_d^{1/n}\rangle$ - automorphisms of $\mathbb{C}\langle T_1^{1/n}, T_2^{1/n}, \ldots, T_d^{1/n}\rangle$ consists of b elements, namely all the elements ωM_1^* with $\omega^b = 1$. Since such automorphisms leave ζ^1 fixed, therefore

$$[L(\zeta^1, M_1^*):L(\zeta^1)] \ge b = m_1 = [L(\zeta):L(\zeta^1)].$$

But by (5.7), $M_1^* \in L(\zeta)$, and so

$$L(\zeta) = L(\zeta^1, M_1^*).$$

Hence M_1^* has exactly b conjugates over $L(\zeta^1)$, necessarily the above b elements ωM_1^*, and the norm of M_1^* in $L(\zeta^1)$ is

$$\mathrm{Norm}(M_1^*) = \prod_{\omega^b=1} (\omega M_1^*) = (M_1^*)^b \in L(\zeta^1).$$

Furthermore, since (by (5.7)) $L(\zeta^1)$ is generated by characteristic monomials of ζ which are not divisible by $T_1^{1/n}$, hence are smaller than M_1^*, and since every characteristic monomial M of ζ lies in $L(\zeta) = L(\zeta^1, M_1^*)$ (again by (5.7)), therefore (5.8.1) shows that $M \le M_1^*$, i.e. M_1^* is the largest characteristic monomial.

---

(1) Also $b = n_i$, by (5.9.2) or (6.6.2).

Conversely, supposing that

(i) M_1^* exists and is the largest characteristic monomial,

(ii) $\lambda_1 = 1/b$, $b \in \mathbb{Z}$, and

(iii) $(M_1^*)^b \in L(\zeta^1)$,

let us deduce that

$$\lambda_1^{-1} = b = [L(\zeta):L(\zeta^1)] = m_1$$

whence, by (7.2), $\mu(X)_y = 1$ for $y \in Z_1^0$, so that $Z_1 \not\subset \Sigma$. (This will complete the proof of (7.5)(b)).

Indeed, from (i) and (ii) and the definition of M_1^* we find that the characteristic monomials of ζ other than M_1^* are not divisible by $T_1^{1/n}$, i.e. they are just the characteristic monomials of the quasi-ordinary branch ζ^1. Lemma (5.7) then yields

$$L(\zeta) = L(\zeta^1, M_1^*).$$

From (ii), we see as before that M_1^* has at least b conjugates over $L(\zeta^1)$, viz, $\{\omega M_1^*\}_{\omega^b=1}$. Hence

$$b \le [L(\zeta^1, M_1^*):L(\zeta^1)].$$

The opposite inequality follows from (iii). Thus

$$b = [L(\zeta^1, M_1^*):L(\zeta^1)] = [L(\zeta):L(\zeta^1)]$$

as desired.

(c) Suppose that for some $y \in Z_{ij}$, (X,y) is defined (up to isomorphism) by $Z^b = T_iT_j$. Then X is normal at y, and therefore neither of the codimension one subvarieties Z_i and Z_j passing through y can be entirely in Σ; so by (a), Z_{ij} is a component of Σ.

Conversely, if Z_{ij} is a component of Σ, then there are no codimension one components of Σ passing through any $y \in Z_{ij}^0$ (by (a), such a component would have to be Z_i or Z_j - as always we assume that y is sufficiently close to x). So (X,y) is normal, and in particular irreducible. The discriminant of π has two components at $\pi(y)$ (viz. $T_i = 0$ and $T_j = 0$). Hence by Remark (7.3.2), the germ (X,y) is given by $Z^b = T_iT_j$ for some b; and $b \geq 2$ since X is not smooth at y.

Q.E.D.

Corollary (7.3.3). *If* Σ *has components of the form* Z_{ij}, *then the integer* b *in* (7.3)(c) *depends only on* (X,x), *and not on* i,j.

Proof. First of all, since $Z^b = T_iT_j$ defines a quotient singularity whose local fundamental group is cyclic of order b (cf. §2), therefore b is uniquely determined by Z_{ij}.

Next, since neither Z_i nor Z_j are components of Σ, (7.3)(b) shows that for some integer $b' \geq 2$ the largest characteristic monomial of ζ is

$$M_i^* = M_j^* = M_{ij}^* = T_1^{\lambda_1} \dots T_c^{\lambda_c}$$

with

$$\lambda_i = \lambda_j = 1/b';$$

and hence no other characteristic monomial of ζ is divisible either by $T_i^{1/n}$ or by $T_j^{1/n}$. So according to (7.2.3), for $y \in Z_{ij}^0$ the irreducible (normal) germ (X,y) has a quasi-ordinary parametrization whose smallest characteristic monomial is $T_i^{1/b'}T_j^{1/b'}$. But then Remark (7.3.2) shows that (X,y) is given by $Z^{b'} = T_iT_j$, and so $b' = b$.

Finally, as in the proof of (7.3)(b), if $L^1 \subset L(\zeta)$ is the field generated over L by all characteristic monomials of ζ other than the largest one, then

$$b' = [L(\zeta):L^1],$$

and so b' does not depend on i, j.

Example (7.4) (d=7). For convenience, let

$$(t,u,v,w,x,y,z) = (T_1,T_2,T_3,T_4,T_5,T_6,T_7).$$

Let L be the fraction field of $\mathbb{C}\langle t,u,v,w,x,y,z\rangle$, and let ζ be the quasi-ordinary branch

$$\zeta = t^{1/2} + t^{5/4}u^{3/2} + t^{13/8}u^{19/12}v^{1/6}w^{1/3}x^{1/6}y^{1/6} + t^{5/2}u^{5/3}v^{4/3}w^{8/3}x^{1/3}y^{1/3}z.$$

Using the relation

$$(t^{1/2})(t^{5/4}u^{3/2})^3(t^{13/8}u^{19/12}v^{1/6}w^{1/3}x^{1/6}y^{1/6})^2$$

$$= (t^{5/2}u^{5/3}v^{4/3}w^{8/3}x^{1/3}y^{1/3}z)(t^5u^6v^{-1}w^{-2}z^{-1})$$

we see that ζ has three characteristic monomials. Moreover

(7.4.1) $$(t^{13/8}u^{19/12}v^{1/2}w^{1/3}x^{1/6}y^{1/6})^6 = t^8u^8t^{1/2}(t^{5/4}u^{3/2}) \in L(t^{1/2},t^{5/4}u^{3/2}).$$

By (5.7), we have then

$$[L(\zeta):L] = [L(t^{1/2},t^{5/4}u^{3/2},t^{13/8}u^{19/12}v^{1/6}w^{1/3}x^{1/6}y^{1/6}):L] = 24.$$

(The last equality can be deduced from (7.4.1) or via (5.9.4).) Using (5.9.3), we calculate further:

$$(m_1,m_2,m_3,m_4,m_5,m_6,m_7) = (24,12,6,6,6,6,1).$$

And by (5.9.2):

$$(n_1,n_2,n_3,n_4,n_5,n_6,n_7) = (8,12,6,3,6,6,1).$$

So the reduced branching sequence (§6) is

$$\{(12,12),(12,4),(6,6),(6,6),(6,6),(6,3)\}.$$

By (7.2.1), the multiplicity of the germ (X,x) parametrized by ζ is

$$\mu(X)_x = 12.$$

By (7.3) the components of the singular locus Σ are:

(Z_1): $t = \zeta = 0$ (generic multiplicity $= (1/2)m_1 = 12$, cf. (7.2))

(Z_2): $u = 0,\ \zeta = t^{1/2}$ (generic multiplicity $= m_2 = 12$)

(Z_4): $w = 0,\ \zeta = t^{1/2} + t^{5/4}u^{3/2}$ (generic multiplicity $= (1/3)m_4 = 2$)

(Z_{35}): $v = x = 0,\ \zeta = t^{1/2} + t^{5/4}u^{3/2}$ (generically $Z^6 = vx$)

(Z_{36}): $v = y = 0,\ \zeta = t^{1/2} + t^{5/4}u^{3/2}$ (generically $Z^6 = vy$)

(Z_{56}): $x = y = 0,\ \zeta = t^{1/2} + t^{5/4}u^{3/2}$ (generically $Z^6 = xy$).

As explained in (5.9), the group $A_6(X)_x \cong H_{12}(X)_x$ is presented by the 9×6 matrix obtained by setting the following 3×6 matrix on top of the 6×6 diagonal matrix with entries (24,12,6,6,6,6).

$$\begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 \\ 30 & 18 & 0 & 0 & 0 & 0 \\ 39 & 19 & 1 & 2 & 1 & 1 \end{pmatrix}$$

A standard reduction of this 9×6 matrix to canonical form yields the decomposition

$$A_6 \cong \mathbb{Z}/12 \oplus (\mathbb{Z}/6)^4. \tag{7.4.2}$$

Example (7.5). In (7.4.2), the invariant factors of $A_{d-1}(X)_x$ are $m_c | m_{c-1} | \ldots | m_2$. This is not always true. For the three-dimensional germ (X,x) parametrized by

$$\zeta = x^{3/2}y^{2/3} + x^{7/4}y^{7/6} + x^{9/4}y^{9/6}z^{1/4} + x^{19/8}y^2z^{11/8}$$

one calculates (as in (5.9)):

$$(n_x,n_y,n_z) = (8,6,8) \qquad (5.9.2)$$

$$A_2(\overline{X})_{\overline{x}} = \mathbb{Z}/2\,\mathbb{Z}. \qquad (5.9.1)$$

$$[L(\zeta):L] = 8.6.8/2 = 192 \qquad (5.9.4)$$

$$(m_x,m_y,m_z) = (192,192,16) \qquad (5.9.3)$$

and finally

$$A_2(X)_x \cong \mathbb{Z}/2\,\mathbb{Z} \oplus \mathbb{Z}/16\,\mathbb{Z} \oplus \mathbb{Z}/96\,\mathbb{Z}. \qquad (5.9.1)$$

The invariant factors are $2|16|96$.

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APPENDIX

by J. Lipman [7]

INVERSION LEMMA. *If a hypersurface germ in* $\mathbb{C}^3$ *has a quasi-ordinary parametrization* $\zeta = X^{u/n}H(X^{1/n}, Y^{1/n})$ *with* $0 < u < n$ *and* $H(0,0) \neq 0$, *then it also has one of the form* $\zeta' = X^{n/u}H'(X^{1/u}, Y^{1/n})$ *where* $H'(0,0) \neq 0$.

Proof: Let $f(X,Y,Z)$ be the minimum polynomial of ζ over $\mathbb{C}[[X,Y]]$. The conjugates of ζ over $\mathbb{C}[[X,Y]]$ are of the form $\zeta_i = X^{u/n}H_i(X^{1/n}, Y^{1/n})$ with $H_i(0,0) \neq 0$ ($i = 1, 2, \dots, m$ where $m =$ degree of f in Z). Thus

$$f(X,Y,Z) = \prod_{i=1}^{m}[Z - X^{u/n}H_i(X^{1/n}, Y^{1/n})]$$

and $f(X,0,0) = X^{mu/n}\widetilde{H}(X^{1/n}, 0)$ with $\widetilde{H}(0,0) \neq 0$. Hence mu/n is a positive integer, and by the Weierstrass preparation theorem there is a power series $E(X,Y,Z)$, $E(0,0,0) \neq 0$, such that $Ef = g$ where g is a polynomial of degree mu/n in X over $\mathbb{C}[[Y,Z]]$. It will clearly be sufficient to show that the discriminant of g has the form $Z^pY^q\cdot$(unit in $\mathbb{C}[[Y,Z]]$), and that g has a root of the form $Z^{n/u}H'(Z^{1/u}, Y^{1/n})$, $H'(0,0) \neq 0$.

Let $\overline{H} = X^uH$, so that $\zeta = \overline{H}(X^{1/n}, Y^{1/n})$. We shall construct a power series G in two variables (over $\mathbb{C}$), $G(0,0) \neq 0$, such that

$$\overline{H}(Z^{1/u}G(Z^{1/u}, Y^{1/n}), Y^{1/n}) = Z.$$

Assuming that such a G exists we set $\xi = G(Z^{1/u}, Y^{1/n})$. Since $f(X, Y, \overline{H}(X^{1/n}, Y^{1/n})) = 0$, we have, upon substituting $Z^{1/u}\xi$ for $X^{1/n}$, $f(Z^{n/u}\xi^n, Y, Z) = 0$; hence $Z^{n/u}\xi^n$ is a root of g. Therefore, the discriminant of g is the product of all the conjugates over $\mathbb{C}[[Y,Z]]$ of the element $g_X(Z^{n/u}\xi^n, Y, Z)$. Now

$$g_X(Z^{n/u}\xi^n, Y, Z) \cdot \frac{\partial}{\partial Z}(Z^{n/u}\xi^n) + g_Z(Z^{n/u}\xi^n, Y, Z) = 0.$$

But

$$\frac{\partial}{\partial Z}(Z^{n/u}\xi^n) = Z^{(n/u)-1}\epsilon'(Z^{1/u}, Y^{1/n}) \qquad \epsilon'(0,0) \neq 0$$

and

$$g_Z(Z^{n/u}\xi^n, Y, Z) = E(Z^{n/u}\xi^n, Y, Z) f_Z(Z^{n/u}\xi^n, Y, Z).$$

The product of all the conjugates (over $\mathbb{C}[[X,Y]]$) of the element $f_Z(X, Y, \zeta)$ is the discriminant of f, which is, by assumption, of the form $X^a Y^b \epsilon(X, Y)$, $\epsilon(0,0) \neq 0$; hence

$$f_Z(X, Y, \overline{H}(X^{1/n}, Y^{1/n})) = X^{c/n} Y^{d/n} \epsilon''(X^{1/n}, Y^{1/n})$$

(c, d integers, $\epsilon''(0,0) \neq 0$). Thus

$$\begin{aligned} g_Z(Z^{n/u}\xi^n, Y, Z) &= E(Z^{n/u}\xi^n, Y, Z) Z^{c/u}\xi^c Y^{d/n} \epsilon''(Z^{1/u}\xi, Y^{1/n}) \\ &= Z^{c/u} Y^{d/n} \epsilon'''(Z^{1/u}, Y^{1/n}) \qquad \epsilon'''(0,0) \neq 0. \end{aligned}$$

So

$$g_X(Z^{n/u}\xi^n, Y, Z) \cdot Z^{(n/u)-1}\epsilon' + Z^{c/u} Y^{d/n} \epsilon''' = 0$$

and

$$g_X(Z^{n/u}\xi^n, Y, Z) = Z^{s/u} Y^{t/n} \epsilon^*(Z^{1/u}, Y^{1/n})$$

(s, t integers, $\epsilon^*(0,0) \neq 0$). Hence the product of the conjugates of $g_X(Z^{n/u}\xi, Y, Z)$ is of the form $Z^p Y^q \cdot$ (unit in $\mathbb{C}[[Y, Z]]$) where p and q are integers, necessarily non-negative since $g_X(Z^{n/u}\xi, Y, Z)$ is integral over $\mathbb{C}[[Y, Z]]$. Thus g is as desired.

To prove the existence of G, we remark that if W is an indeterminate, then $\overline{H}(X, Y) - W^u = X^u H(X, Y) - W^u$ has a factor (in $\mathbb{C}[[X, Y, W]]$) of the form $X\overline{G}(X, Y) - W$ with $\overline{G}(0,0) \neq 0$ (for $\overline{G}$ we take any power series such that $\overline{G}^u = H$). By the preparation theorem, there is a unit $\overline{E}(X, Y, W)$ such that

$$\overline{E}(X, Y, W)(X\overline{G}(X, Y) - W) = X - G'(W, Y)$$

and, setting $X = 0$, we see that $G'(W, Y) = W\overline{E}(0, Y, W)$.

Let $G(W, Y) = \overline{E}(0, Y, W)$; then $G(0,0) \neq 0$, and setting $X = G'(W, Y) = WG$ in the above relation, we have $WG \cdot \overline{G}(WG, Y) - W = 0$, whence $\overline{H}(WG, Y) - W^u = 0$. Our conclusion follows on substituting $Z^{1/u}$ for W and $Y^{1/n}$ for Y. ∎