Featured Review

98e:14010 14E15 32S45

Bierstone, Edward (3-TRNT; Toronto, ON);
Milman, Pierre D. (3-TRNT; Toronto, ON)

Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. **(English summary)**

Invent. Math. **128** (1997), *no.* 2, 207–302.

Hironaka's theorem on the existence of resolutions of singularities for any algebraic or analytic variety V over a field of characteristic zero [H. Hironaka, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 (1964), 205–326; MR **33** #7333; J. M. Aroca, H. Hironaka and J. L. Vicente, *Desingularization theorems*, Consejo Sup. Inv. Cient., Madrid, 1977; MR 80h:32027] is an outstanding achievement of twentieth-century mathematics, by virtue of the depth both of its proof and of its applications. For compact complex-analytic V, a resolution is a proper modification $f\colon V' \to V$ from a manifold V' onto V, where "modification" means "map inducing an isomorphism $V' - f^{-1}S \xrightarrow{\sim} V - S$ for some nowhere dense subvariety S of V" (for example, $S :=$ singular locus of V). If V is embedded in a manifold M, one also considers "embedded resolutions", where V' is embedded in a manifold M' and f is induced by a proper modification $g\colon M' \to M$ such that $g^{-1}V$ ($\supset V'$) has only normal crossing singularities (points where there is a local coordinate system such that each local component of $g^{-1}V$ is defined by the vanishing of a subset of the coordinate functions). Often—not always—one asks for g to be a composition of blowups with smooth centers. Hironaka also dealt with, e.g., real-analytic, non-compact, non-reduced varieties, where further technicalities come into play. Moreover, one may ask for a *canonical* realization of g—one which commutes with open immersions, and in particular with group actions, or more generally with smooth base change.

The history of the problem of existence of resolutions goes back more than a century [see, e.g., J. Lipman, in *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, 187–230, Amer. Math. Soc., Providence, R.I., 1975; MR **52** #10730]. Zariski brought about the transition from the classical characteristic-0 approaches—which succeeded only up to dimension 2—to the modern ones. In the 1940's he proved local uniformization (resolution "along a valuation") in any dimension, and was able to globalize in dimension 3. In positive characteristic, the 2- and 3-dimensional cases were dealt with in the 1950's and 1960's by S. S. Abhyankar [see, e.g., *Resolution of singularities of embedded algebraic surfaces*, Academic Press, New York, 1966; MR **36** #164], who, like Zariski, made heavy use of valuations, a technique which has only recently come back into vogue. And the history is ongoing: work continues on the still-open positive-characteristic case (the total effort diminished in its diversity by a premature announcement of success); and novel approaches to global desingularization have just been developed by A. J. de Jong et al. (see, e.g., his paper with D. Abramovich [J. Algebraic Geom. **6** (1997), no. 4, 789–802], and also the paper by F. A. Bogomolov and T. G. Pantev [Math. Res. Lett. **3** (1996), no. 3, 299–307; MR 97e:14024]), leading to a new generation of short, but non-constructive, proofs.

Hironaka's proof is lengthy, difficult, and non-constructive. Influential as the proof has been, few people can have checked it through entirely, even after some subsequent enhancements of the machinery—Hilbert-Samuel stratification (B. Bennett), maximal contact (J. Giraud, ...). Simplified, more algorithmic proofs are important not only for imparting better understanding of what is really involved in this great theorem, but also for their potential value in unearthing basic features of singularities and their classification. The challenge of finding more straightforward algorithmic approaches was taken up by Zariski, Abhyankar, and others, and successfully met only in the past decade by Bierstone, Milman, and Villamayor.

The paper under review presents—with care, precision, and examples—an essentially self-contained account of such an algorithm. One of the stated goals is to lead the reader along the road by which the authors have come to understand desingularization rather than simply "know" it is true. They write in the language of analysis, not in that of modern algebraic geometry. This not only enlarges the potential readership, but leads to a unified treatment covering algebraic and analytic spaces—complex, real, p-adic—as well as some categories where desingularization theorems, and applications such as Łojasiewicz's inequalities, were not previously known—for example "quasi-Noetherian" spaces and "quasi-analytic" hypersurfaces.

Let us indicate some salient features of the paper, with reference to the cultural context in which they arose.

In the early 1980's, Abhyankar announced a "canonical desingularization" procedure [*Weighted expansions for canonical desingularization*, Springer, Berlin, 1982; MR 84m:14013]. His fundamental idea was, very roughly, to construct a desingularization as a succession of blowups, specifying at every point of each variety appearing in the process a function, the "resolution invariant", which is locally invariant in the sense of compatibility with open immersions, which depends on the preceding history of the process, which takes a finite number of values in a totally ordered set, and whose maximum value locus is a smooth subvariety, upon blowing up which one lowers the said maximum value, thereby "improving the worst singularities". Eventually, by finiteness, the resolution invariant can no longer decrease—so it is constant everywhere and then the original variety has been transformed into a smooth one. (This description is not only vague but also oversimplified, since normal-crossings conditions also play a basic role; nevertheless it may suggest what everyone since has thought a constructive resolution should be.) Abhyankar stated in the preface of those notes that by going through the 220+ pages "consisting mostly of definitions ..., the reader could get an idea of the proof"; but complete details have never appeared. Abhyankar's paper [Adv. in Math. **68** (1988), no. 2, 87–256; MR 89e:14012], giving his 1966 proof of surface desingularization by "good points", was intended to serve as "a good introduction to the higher-dimensional canonical desingularization". (It is, in principle, but U. Orbanz's earlier exposition of that proof, in a book by V. Cossart, Giraud and Orbanz [*Resolution of surface singularities*, Lecture Notes in Math., 1101, Springer, Berlin, 1984; MR 87e:14032 (pp. 1–49)], is more readable.)

A local version of constructive resolution for analytic spaces, intermediate between uniformization and global resolution, was published by Bierstone and Milman in 1989 [J. Amer. Math. Soc. **2** (1989), no. 4, 801–836; MR 91c:32033]. (An earlier version for the hypersurface case was in §4 of another paper of theirs [Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 5–42; MR 89k:32011], hereafter referred to as [BM 1988].) The first published proof of global constructive resolution was Villamayor's [O. E. Villamayor U., Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32; MR 90b:14014], subsequently refined and clarified by Villamayor [Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677; MR 93m:14012] and Villamayor and S. Encinas ["Good points and constructive resolution of singularities", Acta Math., to appear]. Villamayor's papers all assume Hironaka's reduction of the general problem to the "hypersurface-like" problem of "simplification of coherent sheaves of ideals" (see below), and constructivity must be interpreted accordingly. Bierstone and Milman's globalization for hypersurfaces appeared in another paper [in *Effective methods in algebraic geometry (Castiglioncello, 1990)*, 11–30, Progr. Math., 94, Birkhäuser Boston, Boston, MA, 1991; MR 92h:32053]. The present paper treats fully the general—not necessarily hypersurface—case. Another helpful exposition by Bierstone and Milman of the hypersurface case is presented elsewhere ["Resolution of singularities", *Current developments in several complex variables*, edited by Y.-T. Siu and M. Schneider, Cambridge Univ. Press, to appear].

The Bierstone-Milman approach is "bottom-up": it begins with an explicit, elementary construction of a resolution invariant with properties as above. Carefully as the invariant is motivated and described, nevertheless a good deal of patience will be required to master it. This seems inevitable: given what it accomplishes, no resolution algorithm is likely to be all that simple. The description of the invariant, depending initially on a choice of local coordinates, involves numerous formulas, not easy to absorb, at least for the reviewer. Except for the rare reader with sufficient experience and intuition, considerable time and effort will likely be needed to understand fully what to do with, say, the surface $Z^n = X^a Y^b$—indeed, the paper spends $1\frac{2}{3}$ pages on just part of the case $Z^2 = X^2 Y^3$. (The point of that example is to illustrate the workings of the invariant, not to give the complete resolution of the surface.) This is not to say that such an effort will go unrewarded; the invariant and its behavior with respect to certain blowups computably encode the essential local features of the algorithm in a concise, efficient manner; and moreover their local properties, whose establishment in Chapter II is a basic part of the paper, are strong enough to open up an easy passage from local to global resolution—a fundamental problem in the work of Zariski, Hironaka, and Abhyankar. (This passage is negotiated in Chapter II for the hypersurface case and in Chapter IV for the general case.)

In contrast, Villamayor's approach is "top-down": he conceptualizes each step of his process, and ultimately converges to the construction of a resolution invariant—not, however, as explicitly described as Bierstone-Milman's. Thus the two approaches reinforce each other, illuminating various aspects of resolution procedures in different ways. On the other hand, the respective controlling invariants are built ultimately from the same ingredients, by similar processes. For instance, Encinas has informed me that in the above example $Z^2 = X^2Y^3$, Villamayor's algorithm blows up the same centers described by Bierstone and Milman, though the order for the blowups in "years two and three" is reversed. (By incorporating a variant of Abhyankar's "good point" notion, the algorithm in the most recent work of Villamayor and Encinas becomes shorter in some cases, though not in the preceding example. According to Bierstone and Milman, the algorithm in the present paper sometimes becomes more efficient too if one makes use of additional information—beyond the maximum value—carried by the resolution invariant.)

Let us go lightly over some of those ingredients, keeping in mind that having raw ingredients is one thing, and coming up with a recipe for refining and combining them is something else again. Hironaka made clear that the basic resolution problems, even in characteristic > 0, can be reformulated in terms of resolving "idealistic exponents": pairs (J, b) (modulo some equivalence relation) with J a coherent ideal on a non-singular variety W and b an integer—together with a collection E of hypersurfaces in W having only normal crossings [see in *Algebraic geometry (J. J. Sylvester Sympos., Johns Hopkins Univ., Baltimore, Md., 1976)*, 52–125, Johns Hopkins Univ. Press, Baltimore, Md., 1977; MR **58** #16661]. Such pairs appear, in somewhat different guise, as "basic objects" in the work of Villamayor, and as "infinitesimal presentations" in the present paper. (There is a significant technical difference between infinitesimal presentations and idealistic exponents, having to do with what are called "exceptional blowups". Resolution invariants are defined in terms of presentations, idealistic exponents, basic objects ..., so are not a priori invariant, since the defining data depend on local coordinate systems. Proving independence from coordinates is a major issue, taken care of vis-à-vis presentations in Chapter II—this is where exceptional blowups are essential.) The singular locus $S = S(J, b)$ consists of all points where J has order $\geq b$. One associates with any blowup $W' \to W$ of a "permissible" subvariety C of S (i.e., having only normal crossings in W with E) a transformed basic object $(J', b) + E'$; and the problem is to find a sequence of such blowups at the end of which the transform has empty singular locus. Hironaka proved that this is always possible in characteristic zero. He formulated the fundamental inductive strategy: roughly speaking, if S itself is "large enough" then it is permissible, and blowing it up gives a resolution; otherwise there exists a basic object in a "maximal contact" space of dimension $< \dim W$, whose resolution is equivalent to that of (J, b).

Hironaka used an integer ω, the "residual exponent" (roughly the order of J minus that of E, all divided by b), which Villamayor calls "weighted order", to measure the progress of the resolution. (When the weighted order vanishes, one has a "monomial situation", where a simple combinatoric procedure suffices.) A similar integer is a basic component of Bierstone-Milman's invariant. A key to simplifying Hironaka's very complicated induction argument seems to have been the discovery of the right way to take history into account as well: roughly, E has to be divided into two parts, one of which consists of divisors arising from blowups which have occurred since the weighted order last improved; and only those divisors should enter into the definition of weighted order. Working this idea into an inductive scheme is actually more complicated: informally speaking, the resolution invariant is a sequence of up to $\dim V$ different pairs, one for each level of the induction on dimension, each one a weighted order together with the number of exceptional divisors which were present the last time in the resolution process that weighted order, or anything at a higher-dimensional level in the invariant, decreased.

Bierstone and Milman state in the Introduction that "Essential points [of our proof] include the way we encode the history of the resolution process, originating in [BM 1988] ...". It seems then worth noting that some striking formal similarities are shared by the [BM 1988] proof and the above-mentioned early 1980's exposition by Orbanz of Abhyankar's two-dimensional good points proof [see V. Cossart, J. Giraud and U. Orbanz, op. cit.; MR 87e:14032 (pp. 35–38)], especially in the enabling representation of a power series as $p_1 \cdots p_r f$ where $f = z^n + \alpha_2(x,y)z^{n-2} + \cdots + \alpha_n(x,y)$, $p_j = z + \beta_j(x,y)$, where the p_j represent some history, and where finally induction on dimension has been used to make the α's and β's into a set of monomials totally ordered by divisibility. Any such similarities notwithstanding, the [BM 1988] proof obviously accomplishes much more, not only in what it proves—more than local uniformization in any dimension—but in its revelation of the inductive possibilities inherent in the techniques. The authors' inductive procedure, further developed in the above-mentioned paper [J. Amer. Math. Soc. **2** (1989), no. 4, 801–836; MR 91c:32033], lies at the heart of their approach to desingularization.

Hironaka's reduction of resolution problems to those for idealistic exponents involves coming up with what he calls an idealistic space associated to the Hilbert-Samuel function of an arbitrary variety. Hironaka's construction works for the étale topology. (That topology presents no obstruction to Villamayor's approach, which is compatible with arbitrary smooth base change.) In the present paper a self-contained treatment of an analogous reduction has a prominent place (Chapter III). The reduction is to a situation which can be handled by the techniques of Chapter II, where there is a self-contained proof for simplification of coherent sheaves, in particular for desingularization of hypersurfaces. A noteworthy improvement here is that, in accordance with the goal of presenting a constructive, unified treatment of desingularization for several different categories, only "regular" functions within the initial class under consideration are needed, and not functions algebraic over them (such as would be required for étale coverings).

The authors freely acknowledge their debt to Hironaka's results and philosophy. What can be gleaned from this paper concerning the natural question of its relation to the previously published work of Villamayor? Besides the listing in the References of the first two of the above-mentioned papers of Villamayor (as [V1] and [V2]), the only direct allusion is a statement that "... the way we encode the history of the resolution process, originating in [BM 1988], [is] used in a similar way in [V1])". In [V1], it might be noted, Villamayor does much more than the local induction of [BM 1988], describing a globally valid inductive proof of desingularization by blowing up permissible centers.

Bierstone-Milman and Villamayor are certainly aware of each other's work, and have communicated about it. My own impression is that both approaches were worked out more or less independently in the late 1980's and early 1990's. All three authors deserve the gratitude of the mathematics community for making both the understanding and possible applications of resolution much more accessible than was the case before.

In conclusion, Bierstone and Milman have synthesized and lovingly exposed an approach to canonical resolution whose algorithmic basis should prove of great use in the study of both local and global properties of singular spaces in a variety of categories—for some of which no form of resolution was previously available. This is a very substantial, valuable contribution to an area of fundamental importance.

Joseph Lipman (1-PURD; West Lafayette, IN)