# RESIDUES, DUALITY, AND THE FUNDAMENTAL CLASS OF A SCHEME-MAP

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Abstract. The duality theory of coherent sheaves on algebraic varieties goes back to Roch's half of the Riemann-Roch theorem for Riemann surfaces (1870s). In the 1950s, it grew into Serre duality on normal projective varieties; and shortly thereafter, into Grothendieck duality for arbitrary varieties and more generally, maps of noetherian schemes. This theory has found many applications in geometry and commutative algebra.

We will sketch the theory in the reasonably accessible context of a variety $V$ over a perfect field $k$, emphasizing the role of differential forms, as expressed locally via residues and globally via the fundamental class of $V/k$. (These notions will be explained.)

As time permits, we will indicate some connections with Hochschild homology, and generalizations to maps of noetherian (formal) schemes. Even 50 years after the inception of Grothendieck's theory, some of these generalizations remain to be worked out.

## Contents



## Introduction

This talk, aimed at graduate students, will be about the duality theory of coherent sheaves on algebraic varieties, and a bit about its massive—and still continuing—development into Grothendieck duality theory, with a few indications of applications. The prerequisite is some understanding of what is meant by cohomology, both global and local, of such sheaves.

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## 1. Riemann-Roch and duality on curves

This section reviews the Riemann-Roch theorem for a smooth irreducible projective curve $V$ over an algebraically closed field $k$, in a way that motivates the sequel.

**1.1.** In sheaf-theoretic terms, Riemann's part of the theorem is as follows. For any coherent $\mathcal{O}_V$-module (i.e., sheaf of $\mathcal{O}_V$-modules) $\mathcal{F}$, and any integer $i \ge 0$, set

$$h^i(\mathcal{F}) := \dim_k\big(\mathrm{H}^i(V,\mathcal{F})\big) < \infty.$$

The *Euler-Poincaré characteristic* of $\mathcal{F}$ is the integer

$$\chi(\mathcal{F}) := h^0(\mathcal{F}) - h^1(\mathcal{F}).$$

For *invertible* $\mathcal{F}$—that is, every $v \in V$ has an open neighborhood $U$ such that the restriction $\mathcal{F}_{|U}$ is isomorphic to $\mathcal{O}_U$—the *degree* of $\mathcal{F}$ is the integer

$$\boxed{\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_V).} \tag{1.1.1}$$

This is the Riemann theorem, transformed into a definition. However, to give the definition substance, one needs to interpret it more concretely—which one does by showing, via simple manipulations of suitable exact sequences and their cohomology, that for invertible $\mathcal{O}_V$-modules $\mathcal{L}_1$ and $\mathcal{L}_2$,

$$\boxed{\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg(\mathcal{L}_1) + \deg(\mathcal{L}_2).} \tag{1.1.2}$$

It follows that if $\mathcal{L} \supset \mathcal{O}_V$ is invertible, and if $\mathcal{L}^{-1}$ is the invertible sheaf $\mathcal{H}om_{\mathcal{O}_V}(\mathcal{L},\mathcal{O}_V) \subset \mathcal{O}_V$, then

$$\deg(\mathcal{L}) = \dim_k(\mathcal{O}_V/\mathcal{L}^{-1}). \tag{1.1.3}$$

It results via the standard correspondence between divisors and invertible sheaves that "degree" has the usual interpretation .

(For a proof of (1.1.2) applicable to arbitrary one-dimensional schemes proper over an artin ring $A$, with dimension of $k$-vector spaces over fields replaced by length of $A$-modules, see, e.g., [L69, pp. 214–215] (where, in Lemma (10.1), $\mathcal{F} \oplus \mathcal{N}$ should be $\mathcal{F} \otimes \mathcal{N}$).)

The problem Riemann was addressing is how to find the dimension of a complete linear system, which translates into finding $h^0(\mathcal{L})$ for invertible $\mathcal{L}$.

Rewriting $\chi(\mathcal{O}_V)$ as $1 - g$, where $g = h^1(\mathcal{O}_V)$ is the *genus* of $V$, one gets from (1.1.1):

$$\boxed{h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g + h^1(\mathcal{L}).}$$

Now $\deg(\mathcal{L})$ can be determined via (1.1.2) and (1.1.3). So one needs some information about $h^1(\mathcal{L})$. This is where Roch and duality come in.

**1.2.** Let $\Omega = \Omega_{V/k}$ be the sheaf of differential 1-forms on $V/k$. This is an invertible sheaf.

In modern terms Roch showed the following **global duality theorem**:

*For any invertible $\mathcal{L}$ there is a natural $k$-linear isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{L},\Omega) \cong \mathrm{H}^0(\mathcal{L}^{-1}\otimes\Omega) \xrightarrow{\ \sim\ } \mathrm{Hom}_k(\mathrm{H}^1(V,\mathcal{L}),k).$$

Thus $h^1$ becomes a somehow less mysterious $h^0$.

More elaborately, $\Omega$ *is a dualizing sheaf*, in the following sense.

A pair $(\omega,\theta)$ with $\omega$ a coherent $\mathcal{O}_V$-module and $\theta\colon \mathrm{H}^1(V,\omega)\to k$ a $k$-linear map such that for all coherent $\mathcal{O}_V$-modules $\mathcal{F}$, the composition

$$\mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F},\omega) \xrightarrow{\text{natural}} \mathrm{Hom}_k\big(\mathrm{H}^1(V,\mathcal{F}),\mathrm{H}^1(V,\omega)\big) \xrightarrow{\text{via}\,\theta} \mathrm{Hom}_k(\mathrm{H}^1(V,\mathcal{F}),k)$$

is an isomorphism, is said to be *dualizing.* The first component of a dualizing pair is called a *dualizing sheaf.* A simple category-theoretic argument shows that there is a unique isomorphism between any two dualizing pairs. Hence the dualizing sheaves form an isomorphism class of $\mathcal{O}_V$-modules (a class which a priori might have been empty).

Moreover, as we will now indicate, there is a *canonical* $k$-linear map

$$\textstyle\int_V\colon\ \mathrm{H}^1(V,\Omega) \xrightarrow{\ \sim\ } k,$$

such that the pair $\big(\Omega,\int_V\big)$ is dualizing—standing out in the isomorphism class of all dualizing pairs.

**1.3.** The map $\int_V$ is defined via *residues,* as follows.

Let $k(V)$ be the field of rational functions on $V$, and $\Omega_{k(V)}$ its vector space (one-dimensional) of relative $k$-differentials. For any closed point $v\in V$, let $\mathrm{H}^i_v$ denote local cohomology supported at the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{O}_{V,v}$. As the $\mathcal{O}_{V,v}$-module $\Omega_{k(V)}$ is *injective,* the local cohomology sequence associated to the natural exact sequence

$$0\longrightarrow\Omega_v\longrightarrow\Omega_{k(V)}\longrightarrow\Omega_{k(V)}/\Omega_v\longrightarrow 0$$

gives an isomorphism

$$\Omega_{k(V)}/\Omega_v = \mathrm{H}^0_v(\Omega_{k(V)}/\Omega_v) \xrightarrow{\ \sim\ } \mathrm{H}^1_v(\Omega).$$

**Theorem-Definition 1.3.1.** *There is a unique $k$-linear map*

$$\mathrm{res}_v\colon\ \mathrm{H}^1_v(\Omega) = \Omega_{k(V)}/\Omega_v \to k$$

*such that for any local coordinate $t$ at $v$,*

$$\begin{aligned}\mathrm{res}_v(t^{-1}dt+\Omega_v) &= 1\\ \mathrm{res}_v(t^{a}dt+\Omega_v) &= 0 \qquad (a<-1).\end{aligned}$$

Classically (when $k=\mathbb{C}$), $\mathrm{res}_v$ sends $\nu+\Omega_v$ to $\frac{1}{2\pi i}\oint\nu$ (the integral of $\nu$ counterclockwise around a small path enclosing $v$).

There are algebraic proofs, valid in all characteristics, e.g., [S88, Chap. I].

With $\widehat{\ }$ denoting completion at the maximal ideal $\mathfrak{m}_v$ of the local ring $\mathcal{O}_{V,v}$, **canonical local duality** says:

*For all finitely generated $\widehat{\mathcal{O}}_{V,v}$-modules $F$, the composition*

$$\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(F, \widehat{\Omega}_v) \xrightarrow{\text{natural}} \mathrm{Hom}_k(\mathrm{H}^1_{\hat{\mathfrak{m}}_v} F, \mathrm{H}^1_{\hat{\mathfrak{m}}_v} \widehat{\Omega}_v) \xrightarrow{\text{via}\, \mathrm{res}_v} \mathrm{Hom}_k(\mathrm{H}^1_{\hat{\mathfrak{m}}_v} F, k)$$

*is an isomorphism.*

In other words, $(\widehat{\Omega}_v, \mathrm{res}_v)$ is a canonical “locally dualizing” pair.

Now let us globalize. The constant sheaf $\overline{\Omega}$ of *meromorphic differentials,* with sections $\Omega_{k(V)}$, is an *injective* $\mathcal{O}_V$-module.

The sections of $\Omega^* := \overline{\Omega}/\Omega$ over an open $U \subset V$ are given by

$$\Omega^*(U) := \oplus_{v\in U}\, \Omega_{k(V)}/\Omega_v \cong \oplus_{v\in U}\, \mathrm{H}^1_v(\Omega).$$

The cohomology sequence associated to the natural exact sequence

$$0 \longrightarrow \Omega \longrightarrow \overline{\Omega} \longrightarrow \Omega^* \longrightarrow 0$$

gives the exact row in the diagram

$$\begin{tikzcd}
\Omega_{k(V)} \arrow[r] & \oplus_{v\in V}\, \mathrm{H}^1_v(\Omega) \arrow[rr] \arrow[dr, "\oplus \mathrm{res}_v"'] & & \mathrm{H}^1(V,\Omega) \arrow[r] \arrow[dl, dashed, "\int_V"] & 0 \\
 & & k & &
\end{tikzcd}$$

The key **residue theorem** says that “the sum of the residues of a meromorphic differential is zero.”

Classically, by Stokes theorem, this sum is the integral around the (empty) boundary of $V$. For an algebraic proof, see again, e.g., [S88, Chap. I].

In other words, the map $\oplus\mathrm{res}_v$ annihilates the image of $\Omega_{k(V)}$; that is *there is a unique $k$-linear map $\mathrm{H}^1(V,\Omega) \to k$ making the preceding diagram commute.*

This map is defined to be $\int_V$.

For details, see, e.g., [S88, Chap. II]. There, it is first shown that there exists *some* dualizing pair on $V$; and then the residue theorem is used to show that $\big(\Omega, \int_V\big)$ is dualizing.

**Scholium.** For smooth projective curves, differentials and residues give a canonical realization of, and compatibility between, global and local duality.

## 2. Regular differentials on algebraic varieties

**2.1.** Our principal goal in this lecture is to describe a generalization of the canonical compatibility at the end of §1 to arbitrary reduced irreducible $n$-dimensional varieties $V$ proper over a perfect field $k$.

“Dualizing sheaf” is defined as above, with $n$ in place of 1. In this section, we describe a canonical such sheaf, the sheaf of *regular differential $n$-forms.*

Roughly, some historical highlights in the development of this generalization are as follows.

• Rosenlicht (1952 thesis): $V$ a curve ($n = 1$), dualizing sheaf a certain sheaf of "regular" meromorphic differentials, see below and [S88, Chap. IV].

• Serre (1955±): $V$ normal, dualizing sheaf $i_*i^*\wedge^n\Omega$, where $i\colon U \hookrightarrow V$ is the inclusion into $V$ of its (open) smooth part $U$.

• Grothendieck (1957): $V$ embedded in projective $N$-space, $V \subset \mathbb{P}^N_k$, with (noncanonical) dualizing sheaf $\mathcal{E}xt^{N-n}_{\mathbb{P}^N_k/k}(\mathcal{O}_V, \wedge^n\Omega_{\mathbb{P}^N_k/k})$, see [G57].

• Grothendieck (1958): $V$ arbitrary (proper over $k$), with existentially defined dualizing sheaf, the target of a canonical map (the fundamental class) with source $\wedge^n\Omega$, this map being an isomorphism over the smooth part of $V$, see [G58, pp. 112–115]—leading to vast generalization ([H66], [C00], [L09]).

• Kunz (1975): $V$ projective over $k$, with dualizing sheaf of regular differentials, as explained below. This dualizing sheaf agrees with those of Rosenlicht and Serre when $V$ is a curve or normal variety, respectively. It contains $\wedge^n\Omega$, with equality at smooth points. See [Ku08].

• Lipman (1984). $V$ arbitrary (proper over $k$), with dualizing sheaf of regular differentials. See [L84].

**2.2.** Let us now define "regular differentials." Let $C$ be an integral domain finitely generated over $k$, and $B \subset C$ a polynomial $k$-algebra in $n$ variables over which $C$ is *finite*, and such that the corresponding extension of fraction fields $k(B) \subset k(C)$ is *separable*. (Such $B$ exist, by Noether normalization.) Setting $\Omega^n := \wedge^n\Omega$, one has the *differential trace map*

$$\tau\colon \Omega^n_{k(C)/k} = k(C)\otimes_{k(B)}\Omega^n_{k(B)/k} \xrightarrow{\text{trace}\otimes 1} k(B)\otimes_{k(B)}\Omega^n_{k(B)/k} = \Omega^n_{k(B)/k}.$$

The *generalized Dedekind complementary module* is

$$\omega_{C/B} := \{\, \nu \in \Omega^n_{k(C)/k} \mid \tau(C\nu) \subset \Omega^n_{B/k} \,\}.$$

The $C$-module $\omega_{C/B}$ *does not depend on the choice of* $B$. Kunz proved:

**Theorem 2.2.1.** *Lying between $\Omega^n$ and the constant sheaf $\overline{\Omega}^n$ of meromorphic differential $n$-forms, there is a unique coherent $\mathcal{O}_V$-module $\omega$ such that for any affine open subset $U = \operatorname{Spec} C \subset V$ and any $B \subset C$ as above,*

$$\Gamma(U, \omega) = \omega_{C/B}.$$

Sections of this $\omega$ are called *regular differential $n$-forms*. As mentioned before, the stalks $\Omega^n_v$ and $\omega_v$ are equal at any smooth point $v \in V$.

**2.3.** As indicated before, the sheaf of regular differentials is dualizing. This is not a trivial result, especially for nonprojective varieties.

Moreover, generalizing the above-described case of curves, there is a *canonical* $k$-linear $\int_V\colon \mathrm{H}^n(V, \omega) \to k$, closely related to residues, such that the pair $(\omega, \int_V)$ is dualizing.

The canonical dualizing pair $(\omega, \int_V)$ is the main actor in this lecture.

**Example 2.4** (Regular differentials for a local complete intersection)**.** Let $V \subset X$, an $N$-dimensional variety, and let $v \in V$ be a smooth point of $X$. Suppose further that $\mathcal{O}_{V,v} = \mathcal{O}_{X,v}/(f_1, \dots, f_{N-n})$ with $(f_1, \dots, f_{N-n})$ a *regular sequence* in $\mathcal{O}_{X,v}$.

Let $(x_1, \dots, x_N)$ generate the maximal ideal of $\mathcal{O}_{X,v}$, the indexing being such that, with $\bar{x}_i$ the image of $x_i$ in $\mathcal{O}_{V,v}$, the differentials $d\bar{x}_1, \dots, d\bar{x}_n$ generate $\Omega_{k(V)}$.

Then $\omega_v$ is freely generated over $\mathcal{O}_{V,v}$ by the meromorphic $n$-form

$$d\bar{x}_1 d\bar{x}_2 \cdots d\bar{x}_n / [\partial(f_1, \dots, f_{N-n})/\partial(x_{n+1}, \dots, x_N)]^{-},$$

where the denominator is the image in $\mathcal{O}_{V,v}$ of a Jacobian determinant.

## 3. Higher-dimensional residues

For any closed point $v \in V$, $\mathrm{H}^n_v$ is the functor assigning to an $\mathcal{O}_{V,v}$-module its $n$-th local cohomology module with supports at the maximal ideal $\mathfrak{m}_v$. For any $\mathcal{O}_V$-module $\mathcal{F}$, set

$$\mathrm{H}^n_v \, \mathcal{F} := \mathrm{H}^n_v \, \mathcal{F}_v,$$

the local cohomology of the stalk of $\mathcal{F}$ at $v$.

As above, $\omega$ is the sheaf of regular $n$-forms on $V$. The $k$-linear *residue map*

$$\mathrm{res}_v \colon \ \mathrm{H}^n_v(\omega) \to k$$

is as follows, at least when the residue field degree $[\mathcal{O}_{V,v}/\mathfrak{m}_v : k] = 1$, which we assume here to avoid technicalities involving traces.

Suppose first that $v$ is a smooth point, so that $\omega_v = \Omega^n_v$.

If $\mathfrak{m}_v$ is generated by $(t_1, t_2, \dots, t_n)$ then the $\mathcal{O}_{V,v}$-module $\omega_v$ is free of rank one, with basis $dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n$ ($d \colon \mathcal{O}_{V,v} \to \Omega_{\mathcal{O}_{V,v}/k}$ being the universal derivation).

It is known that $\mathrm{H}^n_v(\Omega^n_v) = \mathrm{H}^n_v(\widehat{\Omega}^n_v)$ is the direct limit of the filtered family

$$(\Omega^n_v / \mathbf{t}^{\mathbf{a}} \Omega^n_v)_{\mathbf{a}} = (\widehat{\Omega}^n_v / \mathbf{t}^{\mathbf{a}} \widehat{\Omega}^n_v)_{\mathbf{a}}$$

where $\mathbf{a} = (a_1, \dots, a_n)$ runs through all $n$-vectors of positive integers, where $\mathbf{t}^{\mathbf{a}} := t_1^{a_1} \dots t_n^{a_n}$, and where the transition map from $\mathbf{a}$ to $\mathbf{a}'$ ($a'_i \ge a_i$) is given by multiplication by $\mathbf{t}^{\mathbf{a}' - \mathbf{a}}$. Thus with $\pi_{\mathbf{a}}$ the natural composition

$$\widehat{\Omega}^n_v \twoheadrightarrow \widehat{\Omega}^n_v / \mathbf{t}^{\mathbf{a}} \widehat{\Omega}^n_v \to \mathrm{H}^n_v(\widehat{\Omega}^n_v),$$

every element in $\mathrm{H}^n_v(\widehat{\Omega}^n_v)$ can be represented, non-uniquely, as

$$\begin{bmatrix} \nu \\ t_1^{a_1}, \dots, t_n^{a_n} \end{bmatrix} := \pi_{\mathbf{a}} \nu.$$

Note that $\widehat{\mathcal{O}}_{V,v}$ is a power-series ring $k[[t_1, \dots, t_n]]$, so that any $\nu \in \widehat{\Omega}^n_v$ can be represented as

$$\nu = \sum_{\mathbf{a} \ge (0, \dots, 0)} c_{\mathbf{a}} \mathbf{t}^{\mathbf{a}} dt_1 dt_2 \cdots dt_n \qquad (c_{\mathbf{a}} \in k).$$

**Theorem-Definition 3.1.** *There exists a unique $k$-linear map*

$$\mathrm{res}_v\colon\ \mathrm{H}^n_v(\widehat{\Omega}^n_v) \to k$$

*such that for any $(t_1,\dots,t_n)$ as above and $\nu = \sum_{\mathbf{a}} c_{\mathbf{a}}\mathbf{t}^{\mathbf{a}}\,dt_1dt_2\cdots dt_n$,*

$$\mathrm{res}_v\begin{bmatrix}\nu\\ t_1^{a_1},\dots,t_n^{a_n}\end{bmatrix} = c_{(a_1-1,\dots,a_n-1)}.$$

*Remarks.* 1. Classically, $\mathrm{res}_v$ is given by some integral , see [GH78, Chap. 5].

2. There are a number of algebraic proofs, valid in all characteristics. The difficulty is to show that $\mathrm{res}_v$ *does not depend on the choice of* $(t_1,\dots,t_n)$. See e.g., [L84, §7].

3. When $n=1$, this $\mathrm{res}_v$ is the same as the one in §1.3.

As for the general case, given $v\in V$, it follows from Noether normalization that $S := \mathcal{O}_{V,v}$ has a local $k$-subalgebra $R$ that is the local ring of a point $p\in\mathbb{P}^n_k$, and such that $S$ is a localization of a finite $R$-algebra. It follows easily from the considerations in §2.2 that the differential trace map induces a map $\mathrm{H}^n_v(\omega_{V,v}) \to \mathrm{H}^n_p(\Omega^d_{\mathbb{P}^n_k/k})$. Composing this with $\mathrm{res}_p$ (as just defined), one gets a $k$-linear map $\mathrm{res}_v\colon\ \mathrm{H}^n_v(\omega_{V,v}) \to k$ that seems to depend on the choice of $V$ and of a Noether normalization of $V$, but in fact doesn't. For example, $\omega_{V,v}$ depends only on $S$, and not on the choice of $V$; so we can denote this module by $\omega_v$. Moreover, there is a definition of $\mathrm{res}_v$ involving Hochschild homology, that doesn't make use of any choices [L87]. See also [L01, §5].

## 4. Residues, integrals and duality

Now here is the **main result**, expressing via residues and integrals a canonical realization of and compatibility between local and global duality.

**Theorem 4.1.** (i) (Canonical local duality). *For all finitely generated $\widehat{\mathcal{O}}_{V,v}$-modules $F$, the composition*

$$\mathrm{Hom}_{\widehat{\mathcal{O}}_{V,v}}(F,\widehat{\omega}_v) \xrightarrow{\text{natural}} \mathrm{Hom}_k(\mathrm{H}^n_vF, \mathrm{H}^n_v\widehat{\omega}_v) \xrightarrow{\text{via}\,\mathrm{res}_v} \mathrm{Hom}_k(\mathrm{H}^n_vF, k)$$

*is an isomorphism.*

*In other words, $(\widehat{\omega}_v, \mathrm{res}_v)$ is a canonical locally dualizing pair.*

(ii) (Globalization). *There exists for each proper $n$-dimensional $k$-variety $V$ a unique map*

$$\textstyle\int_V\colon\ \mathrm{H}^n(V,\omega_{V/k}) \to k$$

*such that for each closed point $v\in V$, with $\gamma_v\colon\ \mathrm{H}^n_v(\omega_v) \to \mathrm{H}^n(V,\omega_{V/k})$ the natural map (derived from the inclusion of the functor of sections supported* *at $v$ into the functor of all sections*), *the following diagram commutes.*

$$
\begin{tikzcd}
\mathrm{H}^n_v(\omega_v) \arrow[rr, "\gamma_v"] \arrow[dr, "\mathrm{res}_v"'] & & \mathrm{H}^n(V, \omega_{V/k}) \arrow[dl, "\int_V"] \\
& k &
\end{tikzcd}
$$

(iii) (Canonical global duality). *For each $V$ as in* (ii)*, the pair* $(\omega_V, \int_V)$ *is dualizing.*

A *proof* occupies the first nine sections in [L84].

Probably the theorem can be deduced from general Grothendieck duality theory; but no one has yet carried that out (or maybe even bothered trying).

The theorem has been generalized by Hübl and Sastry to certain maps of noetherian schemes, see [HS93].

A further generalization, to flat maps of formal schemes, has been partially worked out. (See [L01, §5.6].) In that context, local and global duality become unified, and residues and integrals become different instances of a single map. Working out the precise connections draws one through a rich vein of functorial relationships.

## 5. Closing remarks

**5.1.** (Fundamental class.) As in the case of curves, what one does when generalizing to, say, a proper flat map $f\colon X \to Y$ of schemes, of relative dimension $n$, is first to show (for example, via Grothendieck duality theory) the existence of *some* relative dualizing pair $\omega, \theta$), and then show (if possible) the existence of a canonical map $\mathbf{c}_f$—the *fundamental class of $f$*—from the sheaf $\Omega$ of highest order relative differential forms to $\omega$.

This $\mathbf{c}_f$ is the foundation of the role played by differential forms in the abstract Deligne-Verdier approach to Grothendieck duality theory.

The fundamental class should be uniquely determined by the requirements that it should be an isomorphism at all points where $f$ is smooth, and that it should be compatible with a certain trace map for differential forms, relative to a factorization of $f$ as smooth $\circ$ finite.

If $f$ is *proper* then by Grothendieck duality, $\mathbf{c}_f$ corresponds to a canonical map $\int_f\colon \mathsf{R}^n f_*\Omega \to \mathcal{O}_Y$. But even when $f$ is not proper, one can often construct a fundamental class with the above characteristic properties. There is, for instance, one approach via Hochschild homology.

The residue theorem will say, very roughly, that $\mathbf{c}_f$ corresponds at each point of $x$, via a suitable form of local duality, to an intrinsically defined *residue map*, depending only on the local ring in question.

Thus the fundamental class is a globalization of locally defined residues.

All this is necessarily quite vague, given our time constraints. Nor has it all been published yet in definitive form. But there are some substantial treatments in [AL89] and [Kd09].

**5.2.** (Why?) Why bother to learn this extensive theory? First of all duality theory has many applications in commutative algebra and algebraic geometry. In commutative algebra, there are—to mention just a few—the Briançon-Skoda theorem (nowadays best understood in equal characteristics via tight closure, but for which the only proof in mixed characteristic makes use of duality), results on "adjoint ideals" (a special case of multiplier ideals), and results on Cohen-Macaulay graded rings. In algebraic geometry—again, to mention just a couple—there is a proof of resolution of singularities of two-dimensional noetherian schemes, and just recently, Chatzistamatiou and Rülling used a beautiful combination of techniques from intersection theory and Grothendieck duality to prove the invariance of cohomology of the structure sheaf under proper birational maps of smooth varieties over fields of arbitrary characteristic (previously known only in characteristic zero, via resolution of singularities).

Then there is the purely aesthetic motivation of discovering and understanding, for their own sake, deep-lying relationships in a fertile mathematical landscape. Right now, that's what drives my interest; but a young person building a career should be aware that it plus two bucks may get you no more than a cup of coffee at Starbucks.

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