Detailed Plan of Lectures for MA 165

Lesson 1

Topics: Exponential functions and logarithmic functions
Section Number: 1.3
Lecture Plan:

(1) Use the first 20 minutes to explain the ground rules.
- Structure of the course
- MyLabMath Homework
- Recitation Class (Pre-Quiz Exercise Problems)

(2) Exponential function
- Explanation using a picture
  - Show how the graph of $y = b^x$ changes as $b > 0$ changes.
  - Observation that all the graphs pass through $(0, 1)$.
  - **The characterization of** e: The tangent to the graph of $y = b^x$ at $(0, 1)$ has slope 1 when $b = e$.

(3) One-to-One function
- Definition using some set-theoretic and easy examples
  - Basic relations $\begin{cases} f^{-1}(f(x)) & = & x \\ f(f^{-1}(y)) & = & y \end{cases}$
- Explanation using a picture
  - Horizontal line test
- Examples:
  Example ① $y = f(x) = x^2$ over $(-\infty, \infty)$ (NOT one-to-one)
  Example ② $y = f(x) = x^2$ over $[0, \infty)$ (one-to-one)

(4) Inverse function
- Definition of f^{-1} (when a one-to-one function f is given)
- How to find the formula for f^{-1} given the one for f
  - Present the recipe using examples.
- Examples:
  Example ③ $f(x) = 2x + 6$ over $(-\infty, \infty)$
  Example ④ $f(x) = x^2$ over $[0, \infty)$
- Relation between the graph of $y = f(x)$ and that of $y = f^{-1}(x)$
  - Observation and explanation: symmetric w.r.t. $y = x$

(5) Logarithmic function
- Definition as the inverse of the exponential function
- Show the graph of the logarithmic function using the relation

Lesson 2

Topics: Trigonometric functions and their inverses
Section Number: 1.4
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 1. This should serve as a review of Lesson 1.
- Review of Lesson 1
- MyLabMath Homework for Lesson 1

(2) Trigomometric functions as a whole
- Definition of $\sin\theta$, $\cos\theta$, and $\tan\theta$
  - **Visualization using the unit circle**

Warning: Be aware that most of the students have only seen the definition using a right triangle. Force them to grow out of SOH CAH TOH.
  - Discussion and explanation of how to visualize $\tan\theta$
- Disscussion of the basic relation $\sin^2\theta + \cos^2\theta = 1$.

(3) Sine function
- The graph of the sine function
  - how to visualize the graph using the unit circle definition
- Inverse sine function:
  - Discussion of how to define the inverse sine function

by restricting its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - The graph of the inverse sine function

(4) Cosine function
- The graph of the cosine function
  - how to visualize the graph using the unit circle definition
- Inverse cosine function:
  - Discussion of how to define the inverse cosine function

by restricting its domain to $[0, \pi]$
  - The graph of the inverse cosine function

(5) Exact values for the inverse functions
- Examples

Example ① $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Example ② $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

Example ③ $\cos^{-1}(\cos(3\pi)) = \pi \neq 3\pi$

Example ④ $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4} \neq \frac{3\pi}{4}$

Warning: Make them aware that, using Examples ③ and ④, the answer is different from what is naively expected, because of the convention for the choice of the domain above. Explain using the picture of the unit circle and corresponding angles.

If time permits:

(6) Tangent function
- The graph of the tangent function
  - how to visualize the graph using the unit circle definition
- Inverse tanget function:
  - Discussion of how to define the inverse tangent function

by restricting its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$
  - The graph of the inverse tangent function

Lesson 3

Topics: The idea of limits, Definition of limits
Section Number: 2.1, 2.2
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 2. This should serve as a review of Lesson 2.
- Review of Lesson 2
- MyLabMath Homework for Lesson 2

(2) Introduction to the notion of the limit
- Discussion of the average velocity vs instantaneous velocity using Example 1 on Page 56 of the textbook
  - Average velocity
  - Instantaneous velocity $\longrightarrow$ slope of the tangent

(3) Definition of the limit $\lim_{x\to a} f(x) = L$ in general.
- Explanation using the picture
- Emphasize that $\lim_{x\to a} f(x) \neq f(a)$ in genral.
- Discussion of the examples where the denominator becomes 0 if you plug in $x = a$

  Example ① $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$

  Example ② $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

(4) Definition of the right hand limit and left hand limit
- Discussion of the difficult examples involving the absolute value

  Example ③ $\lim_{x\to 1^-} \dfrac{2x^2 - 6x + 4}{|x - 1|}$ vs. $\lim_{x\to 1^+} \dfrac{2x^2 - 6x + 4}{|x - 1|}$

- Discussion of the "confusing" example (the true meaning of the right hand limit may contradict the superficial understanding) using the picture for explanation

  Example ④ $\lim_{x\to \left(\frac{\pi}{2}\right)^+} \tan x$

If time permits:

(5) More of some difficult examples

  Example ⑤ $\lim_{x\to 0} \cos\left(\dfrac{1}{x}\right)$ vs. $\lim_{x\to 0} x \cdot \cos\left(\dfrac{1}{x}\right)$

  Example ⑥ $\lim_{x\to 1^+} \dfrac{x - 1}{|x - 1|}$ vs. $\lim_{x\to 1^-} \dfrac{x - 1}{|x - 1|}$ vs. $\lim_{x\to 1} \dfrac{x - 1}{|x - 1|}$

Lesson 4

Topics: Techniques of computing the limits
Section Number: 2.3
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 3. This should serve as a review of Lesson 3.
- Review of Lesson 3
- MyLabMath Homework for Lesson 3

(2) Factoring the denominaor, which becomes 0 when you plug in $x = a$
- Work out some examples

  Example ① $\lim_{x\to 2} \dfrac{x^2 - 6x + 8}{x^2 - 4}$

  ∘ Mention the fact that
  A polynomial P(x) becomes 0 when $x = a \iff x - a$ divides $P(x)$
- Work out some examples

  Example ② $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

(3) Piecewise defined function
Example ③

$$f(x) = \begin{cases} -2x + 4 & \text{if} \quad x \le 1 \\ \sqrt{x - 1} & \text{if} \quad x > 1 \end{cases} \quad \text{Then} \begin{cases} \lim_{x\to 1^-} f(x) & = & ? \\ \lim_{x\to 1^+} f(x) & = & ? \\ \lim_{x\to 1} f(x) & = & ? \end{cases}$$

(4) **Squeeze Theorem**
- Statement

$$\begin{array}{cccccc} & f(x) & \le & g(x) & \le & h(x) \\ x \to a & \downarrow & & \downarrow & & \downarrow \\ & L & & L & & L \end{array}$$

- Work out some examples

  Example ④ $\lim_{x\to 0} x^2 \cdot \sin\left(\dfrac{1}{x}\right)$

  Exercise: $\lim_{x\to 0} x \cdot \sin\left(\dfrac{1}{x}\right)$

  Important Example ⑤ $\lim_{x\to 0} \dfrac{\sin x}{x}$

  ∘ Discuss the detail with the picture.

(5) Discussion of some easy trig limits

Example ⑥ $\lim_{x\to 0} \dfrac{\sin^2 x}{1 - \cos x}$

Example ⑦ $\lim_{x\to 0} \dfrac{1 - \cos 2x}{\sin x}$

∘ Review of the double angle formula

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Lesson 5

Topics: Infinite limits and limits at infinity
Section Number: 2.4, 2.5
Note: In the original syllabus these topics were to be discussed in two seperate lessons. However, there is not much to be discussed in each topic by itself. We find it better to discuss both of them together in one lesson.
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 4. This should serve as a review of Lesson 4.
- Review of Lesson 4
- MyLabMath Homework for Lesson 4

(2) Show some examples of the limits which approach $\pm\infty$ by drawing the graph of the function.
- Examples

Example ① $y = f(x) = \dfrac{x-2}{(x-1)^2(x-3)}$

$$\begin{cases} \lim_{x\to 1^-} f(x) = \infty \\ \lim_{x\to 1^+} f(x) = \infty \\ \lim_{x\to 1} f(x) = \infty \end{cases} \quad \begin{cases} \lim_{x\to 3^-} f(x) = -\infty \\ \lim_{x\to 3^+} f(x) = \infty \\ \lim_{x\to 3} f(x) \ \text{ DNE} \end{cases}$$

Example ② $y = f(x) = \dfrac{x^2-4x+3}{x^2-1}$

$\{\lim_{x\to 1} f(x) = -1$, even though $f(1)$ is undefined

$$\begin{cases} \lim_{x\to -1^-} f(x) = +\infty \\ \lim_{x\to -1^+} f(x) = -\infty \\ \lim_{x\to -1} f(x) \ \text{ DNE} \end{cases}$$

(3) Show a typical example where a (seemingly correct) geometrical interpretation of the "right hand side" limit may be deceiving.
- Examples

Example ③ $\lim\limits_{\theta\to\left(\frac{\pi}{2}\right)^-} \tan x$ vs. $\lim\limits_{\theta\to\left(\frac{\pi}{2}\right)^+} \tan x$

(4) Explain how the limits can be computed when $x\to\infty$ or $x\to-\infty$ by examples.
Warning: Emphasize that "plugging in $\pm\infty$ will NOT work.
- Examples

Example ④ $\lim_{x\to-\infty} \left(2 + \dfrac{10}{x^2}\right)$

Example ⑤ $\lim_{x\to\infty} \left(3x^4 - 5x^3 + 6x^2 - x + 10\right)$

Example ⑥ $\lim_{x\to-\infty} \left(2x^3 + 3x^2 - 12\right)$

Example ⑦ $\lim_{x\to\infty} \dfrac{3x+2}{x^2-1}$

Example ⑧ $\lim_{x\to\infty} \dfrac{5x^4+4x^2-1}{3x^4+8x^2+4}$

Example ⑨ $\lim_{x\to-\infty} \dfrac{x^3-2x+1}{2x+4}$

Example ⑩ $\lim_{x\to-\infty} \dfrac{x^4-2x+1}{2x+4}$

Example ⑪ $\lim_{x\to\infty} \dfrac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$ vs. $\lim_{x\to-\infty} \dfrac{10x^3-3x^2+8}{\sqrt{25x^6+x^4+2}}$ (**TRICKY !**)

Lesson 6

Topics: Continuity
Section Number: 2.6
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 5. This should serve as a review of Lesson 5.
- Review of Lesson 5
- MyLabMath Homework for Lesson 5

(2) Definition of a function $y = f(x)$ being continuous at $x = a$
- Explanation (visualization) using a picture
- Formal definition

f is continuous at $x = a$

$\Longleftrightarrow$

$\lim_{x\to a} f(x) = f(a)$

$$\Longleftrightarrow \begin{cases} \text{①}\ f(a) \text{ is defined.} \\ \text{②}\ \lim_{x\to a} f(x) \text{ exists.} \\ \text{③}\ \lim_{x\to a} f(x) = f(a). \end{cases}$$

- Example

$$f(x) = \frac{x(x-1)}{(x^2 - 7x + 12)(x-1)} = \frac{x(x-1)}{(x-3)(x-4)(x-1)}.$$

Q1: Where is f defined ?
A1: When $x \neq 3, 4, 1$.
Q2: At what point(s) over (∞, ∞), is f NOT continuous ?
A1: At $x = 3, 4$ (infinite discontinuity),
and at $x = 1$ (removable discontinuity).
Draw the graph and explain.

(3) **Intermediate Value Theorem**
- Picture
- Statement

f continuous over $[a, b]$
$f(a) < L < f(b)$

$\Longrightarrow$

$\exists\, c \in (a, b)$ s.t. $f(c) = L$.

(4) Example Problems

Example ①: Find over which interval, the equation

$$x^3 - 2x = 3$$

has a solution

$$(-1, 0)(0, 1)(1, 2)(2, 3)(3, 4).$$

x	-1	0	1	2	3	4
f(x)	1	0	-1	4	21	56

Since the number 3 lies between -1 and 4, i.e.,

$$-1 < 3 = L < 4,$$

we conclude by I. M. V. Th. that there is a number $c \in (1, 2)$ such that $f(c) = L$, i.e.,

$$c^3 - 2c = 3.$$

So we conclude that the equation has a solution over the interval $(1, 2)$.

Example ②: Determine the constants a & b so that the piece-wise defined function

$$f(x) = \begin{cases} \dfrac{x^2+3x+a}{x+1} & \text{when} \quad x < -1 \\ b & \text{when} \quad x \geq -1 \end{cases}$$

is continuous everywhere.
Solution.
Step 0.

Observe that the function f is continuous (no matter what a & b are) when $x \neq -1$.

So we only have to check the continuity at $x = -1$.

Step 1.

$\lim_{x\to-1^-} f(x)$ has to exist, i.e.,

$\lim_{x\to-1^-} \dfrac{x^2+3x+a}{x+1} = L$ where L is some finite number.

$\longrightarrow$

$$\begin{cases} \lim_{x\to-1^-} x^2+3x+a \\ = \dfrac{x^2+3x+a}{x+1} \cdot (x+1) \\ = L \cdot 0 = 0. \end{cases}$$

i.e.,

$\lim_{x\to-1^-} x^2+3x+a = 0$

$\longrightarrow$

$(-1)^2 + 3(-1) + a = 0$

$\longrightarrow$

$a = 2.$

And in fact, when $a = 2$, we have

$$\begin{aligned} \lim_{x\to-1^-} f(x) &= \lim_{x\to-1^-} \frac{x^2+3x+a}{x+1} \\ &= \lim_{x\to-1^-} \frac{x^2+3x+2}{x+1} \\ &= \lim_{x\to-1^-} \frac{\cancel{(x+1)}(x+2)}{\cancel{x+1}} = 1. \end{aligned}$$

Step 2.

In order for f to be continuous, we have to have

$$\begin{array}{ccccc} \lim_{x\to-1^-} f(x) & = & f(-1) & = & \lim_{x\to-1^+} f(x) \\ \| & & \| & & \| \\ 1 & & b & & b \end{array}$$

$\longrightarrow$

$b = 1.$

Lesson 7

Topics: Introducing the Derivative
Section Number: 3.1
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 6. This should serve as a review of Lesson 6.
- Review of Lesson 6
- MyLabMath Homework for Lesson 6

(2) Derivative as the slope of tangent
- Draw the picture with
  - the graph of the function $y = f(x)$
  - the points $P(a, f(a))$ and $Q(x, f(x))$, and the secant joining P and Q
- Main Idea
  m_{PQ}: the slope of the secant
  $m_{\tan}$: the slope of the tangeny

As the point Q gets closer to P, the slope m_{PQ} gets closer to $m_{\tan}$.
- Formula

$$\begin{aligned} m_{\tan} &= \lim_{Q\to P} m_{PQ} \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \end{aligned}$$

Note: Slightly different notation with the same picture
- the graph of the function $y = f(x)$
- the points $P(a, f(a))$ and $Q(a+h, f(a+h))$, and the secant joining P and Q

$$\begin{aligned} m_{\tan} &= \lim_{Q\to P} m_{PQ} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \end{aligned}$$

- Examples

Example Problem ①: Find the euqation of the tangent to $y = f(x) = \dfrac{3}{x}$ at $\left(2, \dfrac{3}{2}\right)$.

Solution.
Using the first formula, we compute

$$\begin{aligned} m_{\tan} &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{3}{x}-\frac{3}{2}}{x-2} = \lim_{x\to 2} \frac{\frac{3\cdot 2-3\cdot x}{x\cdot 2}}{x-2} = \lim_{x\to 2} \frac{\frac{3(2-x)}{x\cdot 2}}{2x(x-2)} \\ &= \lim_{x\to 2} \frac{-3}{2x} = -\frac{3}{4}. \end{aligned}$$

Using the second formula, we also compute

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{2+h} - \frac{3}{2}}{h} = \lim_{h\to 0} \frac{\frac{3\cdot 2 - 3\cdot(2+h)}{(2+h)\cdot 2}}{h} = \lim_{h\to 0} \frac{\frac{3\{2-(2+h)\}}{(2+h)\cdot 2}}{2(2+h)h} \\ &= \lim_{h\to 0} \frac{-3}{2(2+h)} = -\frac{3}{4}. \end{aligned}$$

<u>Conclusion</u>

The tangent line has the slope $m_{\tan} = -\frac{3}{4}$ passing the point $\left(2, \frac{3}{2}\right)$.

Therefore, its equation is given by $y - \frac{3}{2} = \left(-\frac{3}{4}\right)(x-2)$.

Example Problem ②: Given the function $f(x) = \sqrt{2x} + 3$, find $f'(5)$.

Solution.

Using the first formula, we compute

$$\begin{aligned} f'(5) &= \lim_{x\to 5} \frac{f(x)-f(5)}{x-5} \\ &= \lim_{x\to 5} \frac{((\sqrt{2x}+3) - (\sqrt{2\cdot 5}+3)}{x-5} = \lim_{x\to 5} \frac{\sqrt{2x}-\sqrt{10}}{x-5} \\ &= \lim_{x\to 5} \frac{(\sqrt{2x}-\sqrt{10})(\sqrt{2x}+\sqrt{10})}{(x-5)(\sqrt{2x}+\sqrt{10})} \\ &= \lim_{x\to 5} \frac{2x-10}{(x-5)(\sqrt{2x}+\sqrt{10})} = \lim_{x\to 5} \frac{2\cancel{(x-5)}}{\cancel{(x-5)}(\sqrt{2x}+\sqrt{10})} \\ &= \lim_{x\to 5} \frac{2}{\sqrt{2x}+\sqrt{10}} = \frac{2}{\sqrt{2\cdot 5}+\sqrt{10}} = \frac{\cancel{2}}{\cancel{2}\sqrt{10}} = \frac{\sqrt{10}}{10} \end{aligned}$$

Exercise: Compute $f'(5)$ using the second formula.

Example Problem ③ (Optional !): Given the function $g(x) = \frac{1}{x^2}$, compute $g'(3)$ using the second formula.

(3) Challenge Problem

• Statement of the problem: We have the function $y = f(x)$ whose slope of the tangent at $(3, f(3))$ is equal to 2, i.e., $f'(3) = 2$.

Evaluate the following limit

$$\lim_{h\to 0} \frac{f(3+4h)-f(3-7h)}{5h}.$$

• Point of the problem: We know, using the second formula,

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = f'(3) = 2.$$

But how the h*ll could we compute the above limit, which seems to have nothing to do with the definition of the derivative ?

Solution.

Use the picture to explain

$$\lim_{h\to 0} \frac{f(3+4h)-f(3-7h)}{(3+4h)-(3-7h)} = f'(3) = 2.$$

Now we compute

$$\begin{aligned} \lim_{h\to 0} \frac{f(3+4h)-f(3-7h)}{5h} &= \lim_{h\to 0} \frac{f(3+4h)-f(3-7h)}{(3+4h)-(3-7h)} \cdot \frac{11h}{5h} \\ &= f'(3)\cdot\frac{11}{5} = 2\cdot\frac{11}{5} = \frac{22}{5}. \end{aligned}$$

Lesson 8

Topics: Derivative as a function
Section Number: 3.2
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 7. This should serve as a review of Lesson 7.
- Review of Lesson 7
- MyLabMath Homework for Lesson 7

(2) Derivative as a function
- Main Idea

f: a function which sends x to the value $f(x)$ of of the function f

$$f : x \mapsto f(x)$$

f': a function which sends x to the value $f'(x)$ of the slope of the tangent at $(x, f(x))$

$$f : x \mapsto f'(x)$$

- Formula (How to compute $f'(x)$)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

∘ Example Problem: Given $f(x) = -x^2 + 6x$, compute $f'(x)$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\{-(x+h)^2 + 6(x+h)\} - \{-x^2 + 6x\}}{h} \\ &= \lim_{h\to 0} \frac{-2hx + h^2 + 6h}{h} = \lim_{h\to 0}(-2x + h + 6) \\ &= -2x + 6. \end{aligned}$$

- Warning: Domain of f' maybe different from the one for f.

∘ Example

f: $f(x) = \sqrt{x}$ has domain $[0, \infty)$

f':

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{\not{h}}{\not{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

has domain $(0, \infty)$.

That is to say, unlike the original function $f(x) = \sqrt{x}$, the derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$ is NOT defined when $x = 0$.

- Graph of the derivative
  - ◦ Draw the graph of the following piecewise defined function

$$f(x) = \begin{cases} -3x - 5 & \text{when} \quad x \leq -2 \\ x + 3 & \text{when} \quad -2 < x \leq 0 \\ -\dfrac{1}{2}x + 3 & \text{when} \quad 0 < x \end{cases}$$

  - ◦ Show how to draw the graph of the derivative f', knowing the graph of f, even when one does not have the explicit formula for f (or f')

(3) Continuity vs Differentiability

- Formal definitions (Skip for the moment, and come back later)
- Geometrical meaning (down-to-earth explanation)

Continuous $\leftrightarrow$ the graph is without jump

$\leftrightarrow$ You can draw the graph of the function without leaving the paper

differentiable $\leftrightarrow$ the graph is smooth

Draw the graph of the function, and describe where the function is continuous/differentialble.

- Examples

Example ①: $y = f(x) = |x + 2|$

Example ②: $y = f(x) = \sqrt{|x|}$

Example ③: $y = f(x) = \sqrt[3]{x}$

(4) Differentialble $\Longrightarrow$ Continuous

Warning: Differentialble $\not\Longleftarrow$ Continuous

Lesson 9

Topics: Rules of differentiation

- Linearity
- Power Rule
- Product Rule
- Quotient Rule

Section Number: 3.3, 3.4

Note: In the original syllabus these four topics were to be discussed in two seperate lessons. However, We can easily cover these four topics in one lesson and the students easily understand them. We actually find it even better to discuss all of these in one lesson.

Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 8. This should serve as a review of Lesson 8.
- Review of Lesson 8
- MyLabMath Homework for Lesson 8

(2) Lineraity
- $(cf)' = cf'$ (c constant)
- $(f+g)' = f' + g'$

(3) Power Rule
- $(x^n)' = n \cdot x^{n-1}$

Warning: "n" does not have to be an integer.

Example:

$$\begin{aligned}(\sqrt{x})' &= \left(x^{\frac{1}{2}}\right)' \quad (n = \tfrac{1}{2}) \\ &= \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}}.\end{aligned}$$

- Verification of Power Rule when $n = 3$

$f(x) = x^3$
$f'(x) = (x^3)' = ?$

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\ &= 3x^2 \\ &= nx^{n-1}.\end{aligned}$$

- Special case of Power Rule

$$\begin{aligned}(1)' &= (x^0)' \quad (n = 0) \\ &= 0 \cdot x^{0-1} = 0.\end{aligned}$$

Application

$(c)' = (c \cdot 1)' = c(1)' = 0 \qquad (c \text{ constant})$

(4) Examples (of Linearity and Power Rule)

Example ①:

$$\begin{aligned}(3x^3-5x+12)' &= (3x^3)'-(5x)'+(12)' \\ &= 3(x^3)'-5(x)'+(12)' \\ &= 3\cdot 3x^2-5\cdot 1+0 \\ &= 9x^2-5.\end{aligned}$$

Example ②: $y=3t+2e^t$

$$\frac{dy}{dt}=3+2e^t$$

(5) Product Rule

- $(fg)'=f'g+fg'$
- Verification of Product Rule

$$\begin{aligned}(fg)' &= (fg)'(x) \\ &= \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+f(x)\cdot\frac{g(x+h)-g(x)}{h}\right\} \\ &= f'(x)g(x)+f(x)g'(x)=f'g+fg'\end{aligned}$$

- Example

$$\begin{aligned}(x^7)' &= 7x^6 \quad \text{(by Power Rule)} \\ \| \quad & \\ (x^3\cdot x^4)' &= (x^3)'(x^4)+(x^3)(x^4)' \quad \text{(by Product Rule)} \\ &= 3x^2\cdot x^4+x^3\cdot 4x^3 \\ &= 7x^6 \quad \text{(the same as above obtained by Power Rule)}\end{aligned}$$

(6) Examples of Product Rule

Example ③:

$$\begin{aligned}(x^2e^x)' &= (x^2)'e^x+x^2(e^x)' \\ &= 2xe^x+x^2e^x \\ &= (2x+x^2)e^x.\end{aligned}$$

Example ③:

$$\begin{aligned}\frac{d}{dv}\left\{v^2(2\sqrt{v}+1)\right\} &= \frac{d}{dv}(v^2)(2\sqrt{v}+1)+v^2\frac{d}{dv}(2\sqrt{v}+1) \\ &= 2v(2\sqrt{v}+1)+v^2\left(2\frac{1}{2\sqrt{v}}+1\right)\end{aligned}$$

(7) Quotient Rule

- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Verification of Quotient Rule

$$\begin{aligned}
\left(\frac{f}{g}\right)' &= \left(\frac{f}{g}\right)'(x) \\
&= \lim_{h\to 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} \cdot g(x) - f(x) \cdot \frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{f'g - fg'}{g^2}
\end{aligned}$$

COMPLAINTS: This is so hard !!!

- Easier way to verify Quotient Rule

Product Rule $\longrightarrow$ Quotient Rule

Step 1. Show $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$.

Observe $g \cdot \frac{1}{g} = 1$.

$\longrightarrow$

$$\left(g \cdot \frac{1}{g}\right)' = (1)' = 0$$

$\|$ by Product Rule

$$g'\left(\frac{1}{g}\right) + g\left(\frac{1}{g}\right)'$$

$\longrightarrow$

$$g'\left(\frac{1}{g}\right) + g\left(\frac{1}{g}\right)' = 0$$

$\longrightarrow$

$$g\left(\frac{1}{g}\right)' = -g'\left(\frac{1}{g}\right) = -\frac{g'}{g}$$

$\longrightarrow$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

Step 2. Use Step 1 and Product Rule once more.

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f\cdot\frac{1}{g}\right)' \\ &\overset{\text{Product Rule}}{=} f'\left(\frac{1}{g}\right)+f\left(\frac{1}{g}\right)' \\ &\overset{\text{Step 1}}{=} f'\left(\frac{1}{g}\right)+f\left(-\frac{g'}{g^2}\right)' \\ &= \frac{f'g-fg'}{g^2}.\end{aligned}$$

- Examples

Example ⑤:

$$\begin{aligned}\left(\frac{x^2+3x+4}{x^2-1}\right)' &= \frac{(x^2+3x+4)'(x^2-1)-(x^2+3x+4)(x^2-1)'}{(x^2-1)^2} \\ &= \frac{(2x+3)(x^2-1)-(x^2+3x+4)\cdot 2x}{(x^2-1)^2} \\ &= \frac{-3x^2-110x-3}{(x^2-1)^2}\end{aligned}$$

Example ⑥:

$$\begin{aligned}(e^{-x})' &= \left(\frac{1}{e^x}\right)' \\ &= -\frac{(e^x)'}{(e^x)^2}=-\frac{e^x}{e^{2x}} \\ &= -\frac{1}{e^x}=-e^{-x}.\end{aligned}$$

Example ⑦:

$$\begin{aligned}\left(\frac{4xe^x}{x^2+1}\right)' &= \frac{(4xe^x)'(x^2+1)-(4xe^x)(x^2+1)'}{(x^2+1)^2} \\ &= \frac{(4e^x+4xe^x)(x^2+1)-(4xe^x)\cdot 2x}{(x^2+1)^2} \\ &= -\frac{4e^x(x^3-x^2+x+1}{(x^2+1)^2}\end{aligned}$$

Lesson 10

Topics: Derivatives of trigonometric functions
Section Number: 3.5
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 9. This should serve as a review of Lesson 9.
- Review of Lesson 9
- MyLabMath Homework for Lesson 9

(2) Basic limits involving trig functions
- Basic Limits

Limit (i): $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

Note: It was shown using the Squeeze Theeorem.
Meaning: When x is small (i.e., ~ 0), $\sin x$ and x are almost equal (i.e., $\sin x \sim x$).

Limit (ii): $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

∘ Verification

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1}{x} &= \lim_{x\to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x - 1} \\
&= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} \\
&= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)} \\
&= \lim_{x\to 0} -\frac{\sin x}{x} \cdot \sin x \cdot \frac{1}{\cos x + 1} \\
&= -1 \cdot 0 \cdot \frac{1}{2} = 0.
\end{aligned}$$

- Exercises

Exercise ①:

$\lim_{x\to 0} \dfrac{\sin 4x}{x} = \lim_{x\to 0} \dfrac{\sin 4x}{4x} \cdot \dfrac{4x}{x} = 1 \cdot 4 = 4.$

Exercise ②:

$\lim_{x\to 0} \dfrac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \dfrac{\sin 3x}{3x} \cdot \dfrac{5x}{\sin 5x} \cdot \frac{3x}{5x} = 1 \cdot 1 \cdot \dfrac{3}{5} = \dfrac{3}{5}.$

(3) Derivatives of the trigonometric functions
- Formulas

$$\begin{cases}
\dfrac{d}{dx}(\sin x) = \cos x \\
\dfrac{d}{dx}(\cos x) = -\sin x \\
\dfrac{d}{dx}(\tan x) = \sec^2 x
\end{cases}$$

- Verifications

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left\{ \sin x \cdot \underset{\substack{\downarrow \\ 0}}{\frac{\cos h - 1}{h}} + \cos x \cdot \underset{\substack{\downarrow \\ 1}}{\frac{\sin h}{h}} \right\} \\
&= \cos x.
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\left\{\cos x\cdot \underset{\substack{\downarrow\\0}}{\frac{\cos h-1}{h}}-\sin x\cdot \underset{\substack{\downarrow\\1}}{\frac{\sin h}{h}}\right\} \\
&= -\sin x.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{(\sin x)'\cos x-\sin x(\cos x)'}{\cos^2 x} \\
&= \frac{\cos x\cdot\cos x-\sin x(-\sin x)}{\cos^2 x} \\
&\quad \frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\
&\quad \frac{1}{\cos^2 x}=\sec^2 x.
\end{aligned}
$$

(4) Other Derivatives

- Formulas

$$
\begin{cases}
\frac{d}{dx}(\csc x) = -\csc x\cot x \\
\frac{d}{dx}(\sec x) = \sec x\tan x \\
\frac{d}{dx}(\cot x) = -\cot^2 x
\end{cases}
$$

- Verifications

$$
\begin{aligned}
\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) \\
&= \frac{(1)'\cos x-1(\cos x)'}{\cos^2 x} \\
&= \frac{-(-\sin x)}{\cos^2 x} \\
&\quad \frac{\sin x}{\cos^2 x} \\
&\quad \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\sec x\tan x.
\end{aligned}
$$

Note: Verfications of the other formulas are left to the students as an exercise.

Lesson 11

Topics: Derivatives as Rate of Change, Chain Rule Part I
Section Number: 3.6, 3.7
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 10. This should serve as a review of Lesson 10.
- Review of Lesson 10
- MyLabMath Homework for Lesson 10

(2) Position function, velocity, acceleration
- General Idea
  $s(t) = f(t)$: position as a function of time t
  $v(t) = s' = f'(t)$: velocity Note. $|v|$: speed
  $a(t) = v' = f''(t)$: acceleration
- Example
  $s(t) = f(t) = t^2 - 5t$
  $v(t) = s' = f'(t) = 2t - 5$ Note. $|v|(t) = |2t - 5|$
  $a(t) = v' = f''(t) = 2$
- ∘ with the accompanying graphs

(3) Chain Rule
- Situation

$$\begin{cases} A \xrightarrow{g} B \xrightarrow{f} C \\ A \xrightarrow{g\circ f} C \\ x \mapsto u = g(x) \mapsto y = f(u) = f(g(x)) = (f\circ g)(x) \end{cases}$$

Remark:
(i) Draw the diagram.
(ii) The order matters, i.e., $f \circ g \neq g \circ f$.
- Formula

$$\begin{aligned} (f\circ g)'(x) &= f'(u)\cdot g'(x) = f'(g(x))\cdot g'(x) \\ \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \end{aligned}$$

- Examples
  Example ①: $y = \sin^3 x = (f\circ g)(x)$
  $x \mapsto u = g(x) = \sin x \mapsto y = f(u) = u^3$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= 3u^2\cdot\cos x \\ &= 3(\sin x)^2\cdot\cos x \\ &= 3\sin^2 x\cdot\cos x \end{aligned}$$

Note: Draw the diagram.
  Example ②: $y = (\tan x + 10)^21 = (f\circ g)(x)$
  $x \mapsto u = g(x) = \tan x + 10 \mapsto y = f(u) = u^21$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= 21u^{20}\cdot\sec^2 x \\ &= 21(\tan x + 10)^{20}\cdot\sec^2 x \end{aligned}$$

Note: Draw the diagram.

(4) Meaning of the Chain Rule

- Meaning of the Derivative
  - Situation
    $y = f(x) \quad \& \quad f'(x) = 3$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = 3 \end{aligned}$$

  - Meaning
    Δy is about 3 times as much as Δx (when $|Deltax$ is small (i.e., $\Delta x \sim 0$)).
- Meaning of the Chain Rule
  - Situation
    $x \quad \mapsto \quad u = g(x) \quad \mapsto \quad y = f(u) = f(g(x)) = (f \circ g)(x)$
    $g'(x) = 3$ i.e., $\lim_{h\to 0} \dfrac{\Delta u}{\Delta x} = 3$
    $f'(u) = 7$ i.e., $\lim_{h\to 0} \dfrac{\Delta y}{\Delta u} = 7$
  - Formula

$$\begin{aligned} (f\circ g)'(x) &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\ &= f'(u)\cdot g'(x) = 7\cdot 3 = 21 \\ &= \frac{dy}{du}\cdot\frac{du}{dx} \end{aligned}$$

  - Meaning

Δy is about 7 times as much as Δu, and Δu is about 3 times as much as Δx (when Δx is small (i.e., $\Delta \sim 0$), and hence Δu is small (i.e., $\Delta u \sim 0$)).
Therefore, Δy is about $7 \cdot 3 = 21$ times as much as Δx
(when Δx is small (i.e., $\Delta \sim 0$).

Lesson 12

Topics: Chain Rule Part II
Section Number: 3.7
Lecture Plan:

(1) Use the first 15 minutes to discuss some difficult problems from MyLabMath HW for Lesson 11. This should serve as a review of Lesson 11.
- Review of Lesson 11
- MyLabMath Homework for Lesson 11

(2) Chain Rule for Composition of three or more functions
- Examples

Example ①: $y = \sin\left(e^{\cos x}\right)$

$$x \mapsto u = \cos x \mapsto t = e^u \mapsto y = \sin t$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx} \\ &= \cos t \cdot e^u \cdot (\sin x) \\ &= \cos\left(e^{\cos x}\right) \cdot e^{\cos x} \cdot (-\sin x)\end{aligned}$$

Example ②: $y = \sin^5\left(\cos(8x)\right)$

$$x \mapsto u = 8x \mapsto t = \cos u \mapsto s = \sin t \mapsto y = s^5$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx} \\ &= 5s^4 \cdot \cos t \cdot (-\sin u) \cdot 8 \\ &= 5\sin^4(\cos(8x)) \cdot \cos(\cos(8x)) \cdot (-\sin(8x)) \cdot 8\end{aligned}$$

(3) 2nd derivative of composition
- Question

$$\frac{d^2}{dx^2}\left[f(g(x))\right] = \,?$$

- Answer

$$\begin{aligned}\frac{d^2}{dx^2}\left[f(g(x))\right] &= \frac{d}{dx}\left\{\frac{d}{dx}\left[f(g(x))\right]\right\} \\ &= \frac{d}{dx}\left\{\left[f'(g(x)) \cdot g'(x)\right]\right\} && \text{(by Chain Rule)} \\ &= \frac{d}{dx}\left\{f'(g(x))\right\} \cdot g'(x) + f'(g(x)) \cdot \frac{d}{dx}\left\{g'(x)\right\} && \text{(by Product Rule)} \\ &= \left\{f''(g(x)) \cdot g'(x)\right\} \cdot g'(x) + f'(g(x)) \cdot g''(x) && \text{(by Chain Rule plus Definition)} \\ &= f''(g(x)) \cdot \left[g'(x)\right]^2 + f'(g(x)) \cdot g''(x)\end{aligned}$$

Lesson 13

Topics: Implicit Differentiation
Section Number: 3.8
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 12. This should serve as a review of Lesson 12.
- Review of Lesson 12
- MyLabMath Homework for Lesson 12

(2) Explicit Differentiation vs Implicit Differentiation

Example 1: $x^2 + y^2 = 1$

• **Explicit Differentiation**
Warning: There is no such terminology as "Explicit Differentiation". I made it up merely to emphasize the importance of "Implicit Differentiation" in contrast.

Solve for y.

$$y^2 = 1 - x^2$$
$$\longrightarrow$$
$$y = \pm\sqrt{1 - x^2}$$

Now compute the differentiation.

$$\frac{dy}{dx} = \pm\frac{-2x}{2\sqrt{1-x^2}} = \pm\frac{-x}{\sqrt{1-x^2}}.$$

• **Implicit Differentiation**

Write down the equation.

$$x^2 + y^2 = 1.$$

Differentiate both sides of the equation.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$
$$\longrightarrow$$
$$2x + 2y \cdot \frac{dy}{dx} = 0.$$

Now compute the differentiation.

$$2y \cdot \frac{dy}{dx} = -2x.$$
$$\longrightarrow$$
$$\frac{dy}{dx} = -\frac{-2x}{2y} = -\frac{x}{y}$$

• Example Problem: Find the equation of the tangent line to $x^2 + y^2 = 1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
Solution: Draw the picture.
Slope of the tangent: $\frac{dy}{dx} == -\frac{x}{y} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$.
Equation of the tangent: $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

Example 2: $x^2 + xy - y^3 = 7$

• **Explicit Differentiation**

Solve for y ... $y^3 - xy + 7 - x^2 = 0$ ... You can't !
(unless you know how to solve the cubic equation !)
$\longrightarrow$
($y =$??? Now compute the differentiation $\dfrac{dy}{dx} =$???.)

• **Implicit Differentiation**

Write down the equation.

$$x^2 + xy - y^3 = 7.$$

Differentiate both sides of the equation.

$$\frac{d}{dx}(x^2 + xy - y^3) = \frac{d}{dx}(7)$$
$\longrightarrow$
$$2x + 1 \cdot y + x \cdot \frac{dy}{dx} - 3y^2 \cdot \frac{dy}{dx} = 0.$$

Now compute the differentiation.

$$2x + y + (x - 3y^2)\frac{dy}{dx} = 0.$$
$\longrightarrow$
$$\frac{dy}{dx} = -\frac{-2x - y}{x - 3y^2}$$

• Example Problem: Find the equation of the tangent line to $x^2 + xy - y^3 = 7$ at the point $(3, 2)$.
Solution: (Draw the picture. Use Desmos ($\wedge \circ \wedge$)))
Slope of the tangent: $\dfrac{dy}{dx} == \dfrac{-2x - y}{x - 3y^2} = \dfrac{-2 \cdot 3 - 2}{3 - 3 \cdot 2^2} = \dfrac{-8}{-9} = \dfrac{8}{9}$
Equation of the tangent: $y - 2 = \dfrac{8}{9}(x - 3)$

(3) One More Example of Implicit Differentiation

$$\sin(xy) = x^2 + y$$

• **Implicit Differentiation**

Write down the equation.

$$\sin(xy) = x^2 + y.$$

Differentiate both sides of the equation.

$$\frac{d}{dx}(\sin(xy)) = \frac{d}{dx}(x^2 + y)$$
$\longrightarrow$
$$\cos(xy)\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = 2x + \frac{dy}{dx}.$$

Now compute the differentiation.

$$x\cos(xy)\frac{dy}{dx} - \frac{dy}{dx} = y\cos(xy) + 2x.$$
$\longrightarrow$
$$\frac{dy}{dx} = \frac{y\cos(xy) + 2x}{x\cos(xy) - 1}$$

Lesson 14

Topics: Derivatives of Logarithmic and Exponential Functions
Section Number: 3.9
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 13. This should serve as a review of Lesson 13.
- Review of Lesson 13
- MyLabMath Homework for Lesson 13

(2) Derivative of a logarithmic function $y = \ln x$
- Problem:

$$\frac{dy}{dx} = ?$$

- Solution:

$y = \ln x$
$\longrightarrow$
$e^y = e^{\ln x} = x$

Differentiate both sides of the equation.

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$
$\longrightarrow$
$e^y \cdot \frac{dy}{dx} = 1$

Now compute the differentiation.

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$

Grand Conclusion

$$\boxed{\frac{d}{dx}(\ln x) = \frac{dy}{dx} = \frac{1}{x}}$$

(3) Variation: Derivative of a logarithmic function $y = \ln |x|$
- Problem:

$$\frac{dy}{dx} = ?$$

- Solution:

$$|x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

$$\ln |x| = \begin{cases} \ln(-x) & \text{if} \quad x < 0 \\ \ln x & \text{if} \quad x > 0 \end{cases}$$

Case: $x < 0$

$$\frac{d}{dx}(\ln |x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}$$

Case: $x > 0$

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

<u>Grand Conclusion</u>

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{dy}{dx} = \frac{1}{x}}$$

• Exercise

$$y = \ln|\sin x|$$

$$\frac{dy}{dx} = ?$$

$$x \mapsto u = \sin x \mapsto y = \ln|u|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot \cos x \\ &= \frac{1}{\sin x} \cdot \cos x = \cot x. \end{aligned}$$

• Extra: Derive

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

using the fact that $\ln x$ & e^x are inverse to each other.

$$x \mapsto y = \ln x \mapsto z = e^y = e^{\ln x} = x$$

$$\begin{aligned} \frac{dz}{dx} &= \frac{dz}{dy} \cdot \frac{dy}{dx} \\ \| \\ \frac{dx}{dx} &= 1 \end{aligned}$$

$\longrightarrow$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

(4) Derivative of an exponential function $y = b^x$ $(b > 0:$ some constant$)$

- Problem:

$$\frac{dy}{dx} = ?$$

- Solution:

Case: $b = e$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$$

Case: b general

Observe

$$y = b^x = (e^{\ln b})^x = e^{\ln b \cdot x}$$

Now use the Chain Rule.

$$x \mapsto u = \ln b \cdot x \mapsto y = e^u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot \ln b \\ &= e^{\ln b \cdot x} \cdot \ln b \\ &= b^x \cdot \ln b \end{aligned}$$

<u>Grand Conclusion</u>

$$\boxed{\frac{d}{dx}(b^x) = b^x \cdot \ln b}$$

- Exercise

$$y = 3^x$$

$$\frac{dy}{dx} = \frac{d}{dx}(3^x) = 3^x \cdot \ln 3.$$

Note:

$$y = e^x$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x \cdot \ln e = e^x \cdot 1 = e^x$$

(5) Review

$y = f(x)$	$\frac{dy}{dx}$
$y = \ln x$	$\frac{1}{x}$
$y = \ln \lvert x \rvert$	$\frac{1}{x}$
$y = \log_5 x = \frac{\ln x}{\ln 5}$	$\frac{1}{\ln 5}\frac{1}{x}$
$y = e^x$	e^x
$y = 5^x$	$5^x \cdot \ln 5$

• Exercise

$$y = 5^{\sin x}$$

$$x \mapsto u = \sin x \mapsto y = 5^u$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5^u \cdot \ln 5 \cdot \cos x \\ &= 5^{\sin x} \cdot \ln 5 \cdot \cos x \end{aligned}$$

(6) Lagarithmic Differentiation

• Example
Problem:

$$y = \frac{(x^3-1)^4\sqrt{3x-1}}{x^2+4}$$

$$\frac{dy}{dx} = ?$$

Solution 1: Use Quotient Rule ... Mess !
Solution 2: Slicker way ! Loagrithmic Differentiation !

Take "ln" of both sides of the equation.

$$\begin{aligned} \ln y &= \ln\left(\frac{(x^3-1)^4\sqrt{3x-1}}{x^2+4}\right) \\ &= 4\ln(x^3-1) + \frac{1}{2}\ln(3x-1) - \ln(x^2+4) \end{aligned}$$

Differentiate both sides of the equation.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(4\ln(x^3-1) + \frac{1}{2}\ln(3x-1) - \ln(x^2+4)\right)$$

$\longrightarrow$

$$\frac{1}{y}\frac{dy}{dx} = 4 \cdot \frac{3x^2}{x^3-1} + \frac{1}{2} \cdot \frac{3}{3x-1} - \frac{2x}{x^2+4}$$

$\longrightarrow$

$$\begin{aligned} \frac{dy}{dx} &= y \cdot \left\{4 \cdot \frac{3x^2}{x^3-1} + \frac{1}{2} \cdot \frac{3}{3x-1} - \frac{2x}{x^2+4}\right\} \\ &= \frac{(x^3-1)^4\sqrt{3x-1}}{x^2+4} \cdot \left\{\frac{12x^2}{x^3-1} + \frac{3}{2(3x-1)} - \frac{2x}{x^2+4}\right\} \end{aligned}$$

Lesson 15

Topics: Derivatives of the functions of the form $f(x)^{g(x)}$
Section Number: 3.9
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 14. This should serve as a review of Lesson 14.
- Review of Lesson 14
- MyLabMath Homework for Lesson 14

(2) Derivatives of the functions of the form $f(x)^{g(x)}$
Example 1: $y = x^x$
- Problem

$$\frac{dy}{dx} = ?$$

- Answer
  - ○ **Wrong Answer ①** (Exponential Function version $(b^x)' = b^x \cdot \ln b$)

$$\frac{d}{dx}(x^x) \;\not=\; x^x \cdot \ln x$$

Remark:
Why is it wrong ?
Because, in order to use the formula for the derivative of an exponential function, "b" has to be a constant, while "x" is NOT a constant.

  - ○ **Wrong Answer ②** (Power Rule version $(x^n)' = nx^{n-1}$)

$$\frac{d}{dx}(x^x) \;\not=\; x \cdot x^{x-1} = x^x$$

Remark:
Why is it wrong ?
Because, in order to use the Power Rule, "n" has to be a constant, while "x" is NOT a constant.

  - ○ **Genuine Answer**

Method 1

$$y = x^x$$
$$\longrightarrow$$
$$\ln(y) = \ln(x^x) = x \ln x$$
$$\longrightarrow$$
$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x \ln x)$$
$$\longrightarrow$$
$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$
$$\longrightarrow$$
$$\frac{dy}{dx} = y \cdot (\ln x + 1) = x^x(\ln x + 1)$$

Method 2

$$y = x^x = \left(e^{\ln x}\right)^x = e^{\ln x \cdot x} = e^{x \ln x}$$

Now use the Chain Rule.

$$x \mapsto u = x \ln x \mapsto y = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= e^u\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right)\\ &= e^{x\ln x}\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right)\\ &= x^x(\ln x+1)\end{aligned}$$

- **Funny (!?) Observation**

$$\begin{aligned}\frac{dy}{dx} &= x^x(\ln x+1)\quad\text{(genuine answer)}\\ &= x^x\cdot\ln x + x^x\\ &= \text{Wrong Answer①} + \text{Wrong Answer②}\end{aligned}$$

Is this a sheer coincidene ?
NO. (⟵ Chain Rule in Calculus of several variables !)

Example 2: $y = x^{\sin x}$
Method 1

$$y = x^{\sin x}$$
$\longrightarrow$
$$\ln(y) = \ln(x^{\sin x}) = \sin x\ln x$$
$\longrightarrow$
$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\sin x\ln x)$$
$\longrightarrow$
$$\frac{1}{y}\frac{dy}{dx} = \cos x\ln x + \sin x\cdot\frac{1}{x}$$
$\longrightarrow$
$$\frac{dy}{dx} = y\cdot\left(\cos x\ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$$

Method 2

$$y = x^{\sin x} = \left(e^{\ln x}\right)^{\sin x} = e^{\ln x\cdot\sin x} = e^{\sin x\ln x}$$

Now use the Chain Rule.

$$x \mapsto u = \sin x\ln x \mapsto y = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= e^u\cdot\left(\cos x\cdot\ln x + \sin x\cdot\frac{1}{x}\right)\\ &= e^{\sin x\ln x}\cdot\left(\cos x\cdot\ln x + \sin x\cdot\frac{1}{x}\right)\\ &= x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)\end{aligned}$$

Lesson 16

Topics: Derivatives of the Inverse Trigonometric Finctions
Section Number: 3.10
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 15. This should serve as a review of Lesson 15.
- Review of Lesson 15
- MyLabMath Homework for Lesson 15

(2) Derivative of the Inverse Sine

$$y = \sin^{-1} x = \arcsin x \left(\neq \frac{1}{\sin x} \right)$$

- Question: $\dfrac{dy}{dx} = ?$
- Answer:

$y = \sin^{-1} x$
$\longrightarrow$
$\sin y = x$

Differentiate both sides of the equation.

$\dfrac{d}{dx}(\sin y) = \dfrac{d}{dx}(x)$
$\longrightarrow$
$\cos y \cdot \dfrac{dy}{dx} = 1$
$\longrightarrow$
$\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$

Explanation of $\cos y = \sqrt{1-x^2}$ **drawing the picture.**

Grand Conclusion

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

(3) Derivative of the Inverse Cosine

$$y = \cos^{-1} x = \arccos x \left(\neq \frac{1}{\cos x} \right)$$

- Question: $\dfrac{dy}{dx} = ?$
- Answer:

$y = \sin^{-1} x$
$\longrightarrow$
$\cos y = x$

Differentiate both sides of the equation.

$\dfrac{d}{dx}(\cos y) = \dfrac{d}{dx}(x)$
$\longrightarrow$
$-\sin y \cdot \dfrac{dy}{dx} = 1$
$\longrightarrow$
$\dfrac{dy}{dx} = \dfrac{1}{-\sin y} = \dfrac{1}{-\sqrt{1-x^2}}$

Explanation of $\sin y = \sqrt{1-x^2}$ **drawing the picture.**

<u>Grand Conclusion</u>

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{dy}{dx} = \frac{1}{-\sqrt{1-x^2}}}$$

(4) Derivative of the Inverse Tangent

$y = \tan^{-1} x = \arctan x \left(\neq \dfrac{1}{\tan x} \right)$

- Question: $\dfrac{dy}{dx} = ?$
- Answer:

$y = \tan^{-1} x$

$\longrightarrow$

$\tan y = x$

Differentiate both sides of the equation.

$\dfrac{d}{dx}(\tan y) = \dfrac{d}{dx}(x)$

$\longrightarrow$

$\sec^2 y \cdot \dfrac{dy}{dx} = 1$

$\longrightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{\tan^2 y + 1} = \dfrac{1}{x^2+1}$

<u>Grand Conclusion</u>

$$\boxed{\frac{d}{dx}(\tan^{-1} x) = \frac{dy}{dx} = \frac{1}{x^2+1}}$$

Lesson 17

Topics: Related Rates, Part I
Section Number: 3.11
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 16. This should serve as a review of Lesson 16.
- Review of Lesson 16
- MyLabMath Homework for Lesson 16

(2) Typical Example

Step 0. Picture

Monkey George hanging onto the baloon

Step 1. Given

$\frac{dV}{dt} = 100 \quad \text{cm}^3/\text{s}$

Step 2. Unknown

$\frac{dr}{dt} = ?$ when $D = 50$ cm ($\longrightarrow r = 25$)

Step 3. Relation

$V = \frac{4}{3}\pi r^3$

Step 4. Solution

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

$100 = \frac{4}{3}\pi \cdot 3(25)^2 \cdot \frac{dr}{dt}$

$\longrightarrow$

$\frac{dr}{dt} = \frac{100}{\frac{4}{3}\pi \cdot 3 \cdot (25)^2} = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}$

(3) More Examples

Example ①:

Step 0. Picture

Ladder leaning on the wall

Step 1. Given

$\frac{dx}{dt} = 1$

Step 2. Unknown

$\frac{dy}{dt} = ?$ when $x = 6$

Step 3. Relation

$x^2 + y^2 = 10^2$

Step 4. Solution

$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(10^2)$

$\longrightarrow$

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

$2 \cdot 6 \cdot \frac{dx}{dt} + 2 \cdot 8 \cdot \frac{dy}{dt} = 0$

$\longrightarrow$

$\frac{dy}{dt} = -\frac{2 \cdot 6 \cdot 1}{2 \cdot 8} = -\frac{3}{4}$

Example ②:

Step 0. Picture

Reversed conical tank with water pouring in

Step 1. Given

$\frac{dV}{dt} = 2$

Step 2. Unknown

$\frac{dh}{dt} = ?$ when $h = 3$

Step 3. Relation

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{4}h\right)^2 h = \frac{1}{12}\pi h^3$$

Step 4. Solution

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$2 = \frac{1}{12}\pi \cdot 3 \cdot 3^2 \cdot \frac{dh}{dt}$$

$$\longrightarrow \frac{dh}{dt} = \frac{2}{\frac{1}{12}\pi \cdot 3 \cdot 3^2} = \frac{8}{9\pi}.$$

Example ③:

Step 0. Picture

Gravel piling up in a conical shape

Step 1. Given

$\frac{dV}{dt} = 30$

Step 2. Unknown

$\frac{dh}{dt} = ?$ when $h = 10$

Step 3. Relation

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

Step 4. Solution

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$30 = \frac{1}{12}\pi \cdot 3 \cdot 10^2 \cdot \frac{dh}{dt}$$

$$\longrightarrow \frac{dh}{dt} = \frac{30}{\frac{1}{12}\pi \cdot 3 \cdot 10^2} = \frac{6}{5\pi}.$$

Example ④:

Step 0. Picture

Snowball melting

Step 1. Given

$$\frac{dS}{dt} = -1$$

Step 2. Unknown

$$\frac{dD}{dt} = ? \text{ when } D = 10$$

Step 3. Relation

$$S = 4\pi r^2 = 4\pi \left(\frac{D}{2}\right)^2 = \pi D^2$$

Step 4. Solution

$$S = \pi D^2$$

$$\frac{dS}{dt} = \pi \cdot 2D \cdot \frac{dD}{dt}$$

$$-1 = \pi \cdot 2 \cdot 10 \cdot \frac{dD}{dt}$$

$\longrightarrow$

$$\frac{dD}{dt} = \frac{-1}{\pi \cdot 2 \cdot 10} = -\frac{1}{20\pi}.$$

Lesson 18

Topics: Related Rates, Part II
Section Number: 3.11
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 17. This should serve as a review of Lesson 17.
- Review of Lesson 17
- MyLabMath Homework for Lesson 17

(2) Brief Review
Step 0. Picture with variables
Step 1. Given
Step 2. Unknown
Step 3. Relation
Step 4. Solution

(3) More Examples
Example ⑤:
Step 0. Picture
Two cars running toward weat and north, respectively.
Step 1. Given
$\frac{dx}{dt} = -50, \frac{dy}{dt} = -60$
Step 2. Unknown
$\frac{dz}{dt} = ?$ when $x = 0.3, y = 0.4$
Step 3. Relation
$x^2 + y^2 = z^2$
Step 4. Solution
$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(z^2)$
$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2z\frac{dz}{dt}$
$2 \cdot 0.3 \cdot (-50) + 2 \cdot 0.4 \cdot (-60) = 2 \cdot 0.5 \cdot \frac{dz}{dt} \quad (0.5 = \sqrt{0.3^2 + 0.4^2})$
$\longrightarrow$
$\frac{dz}{dt} = -78.$

Example ⑥:
Step 0. Picture
Shadow of a person walking away from the streetlight
Step 1. Given
$\frac{dx}{dt} = 5$
Step 2. Unknown
$\frac{dy}{dt} = ?$ when $x = 40$
Step 3. Relation
$y : 15 = (y - x) : 6$ i.e., $\frac{y}{15} = \frac{y - x}{6}$
$\longrightarrow$
$6y = 15(y - x) \longrightarrow 15x = 9y$
$\longrightarrow$
$y = \frac{15}{9}x = \frac{5}{3}x$
Step 4. Solution
$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3}.$

Example ⑦:

Step 0. Picture

Boat approaching a pier

Step 1. Given

$$\frac{dl}{dt} = -1$$

Step 2. Unknown

$$\frac{dx}{dt} = ?\text{ when } x = 8$$

Step 3. Relation

$$l^2 = x^2 + 1^2$$

Step 4. Solution

$$\frac{d}{dt}(l^2) = \frac{d}{dt}(x^2 + 1^2)$$

$$2l \cdot \frac{dl}{dt} = 2x \cdot \frac{dx}{dt}$$

$$2 \cdot \sqrt{65} \cdot (-1) = 2 \cdot 8 \cdot \frac{dx}{dt} \qquad (\sqrt{65} = \sqrt{1^2 + 8^2})$$

$\longrightarrow$

$$\frac{dx}{dt} = \frac{2 \cdot \sqrt{65} \cdot (-1)}{2 \cdot 8} = -\frac{\sqrt{65}}{8}$$

Example ⑧:

Step 0. Picture

Flying a kite

Step 1. Given

$$\frac{dx}{dt} = 8$$

Step 2. Unknown

$$\frac{d\theta}{dt} = ?\text{ when } s = 200$$

Step 3. Relation

$$x \tan\theta = 100$$

Step 4. Solution

$$\frac{d}{dt}(x \tan\theta) = \frac{d}{dt}(100)$$

$$\frac{dx}{dt} \cdot \tan\theta + x \cdot \sec^2\theta \frac{d\theta}{dt} = 0$$

$$\begin{cases} s &= 200 \\ x &= \sqrt{200^2 - 100^2} = 100\sqrt{3} \\ \tan\theta &= \dfrac{100}{100\sqrt{3}} = \dfrac{1}{\sqrt{3}} \\ \sec\theta &= \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{3}/2} = \dfrac{2}{\sqrt{3}} \end{cases}$$

Picture of the right triangle

$$8 \cdot \frac{1}{\sqrt{3}} + 100\sqrt{3} \cdot \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{\theta}{dt} = 0$$

$\longrightarrow$

$$\frac{\theta}{dt} = \frac{-8 \cdot \dfrac{1}{\sqrt{3}}}{100\sqrt{3} \cdot \left(\dfrac{2}{\sqrt{3}}\right)^2} = -\frac{1}{50}$$

Lesson 19

Topics: Maxima & Minima
Section Number: 4.1
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 18. This should serve as a review of Lesson 18.
- Review of Lesson 18
- MyLabMath Homework for Lesson 18

(2) Explanation of Absolute (Local) Max & Min using the picture
- Example

Look at
$f(x) = 3x^4 - 16x^3 + 18x^2$ over $[-1, 4]$
Draw the graph of the function.

Absolute Max	37	at $x = -1$
Absolute Min	-27	at $x = 3$
Local Max	5	at $x = 1$
but NOT	37	at $x = -1$
or	32	$x = 4$
Local Min	0	at $x = 0$
	-27	at $x = 3$

Big Warning: Exclude the end points from the consideration of local max and local min.

(3) **Extreme Value Theorem**
- Statement

$y = f(x)$ a function continuous on $[a, b]$ (which is a closed interval).
$\Longrightarrow$
f has abs. max and abs. min.
- Explanation using the picture
  - ◦ Typical Case
  - ◦ Cases where some assumption E.V. Th. does not hold, and the assertion fails ot hold

(4) Recipe to find abs. max and abs. min
- Situation

$y = f(x)$ a function continuous over $[a, b]$

Step 1.
Look at
① end points a, b
① critical numbers c
Meaning of "critical numbers":
$$\begin{cases} f'(c) = 0 \\ \text{or} \\ f'(c) \text{ does not exist.} \end{cases}$$

Step 2.
Compare $f(a), f(b)$ and $f(c)$'s.
biggest $\longrightarrow$ abs. max
smallest $\longrightarrow$ abs. min

Note: We allow the tie for the biggest and/or smallest.

(5) Explanation using the picture of
$$\left\{ \begin{array}{l} f(c) \text{ local max or local min} \\ \& \\ f'(c) \text{ exists} \end{array} \right\} \Longrightarrow f'(c) = 0$$

(6) Example Problems

Example Problem 1: Find abs. max and abs. min of

$$y = f(x) = x^3 - 3x^2 + 1 \text{ over } [-\frac{1}{2}, 4].$$

Step 1.

① $a = -\frac{1}{2}, b = 4$

② $f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(c) = 3c(c-2) = 0 \longrightarrow c = 0, 2$

Step 2.

$$\begin{cases} f\left(-\frac{1}{2}\right) = \frac{1}{8}, & f(4) = 17 \\ f(0) = 1, & f(2) = -3 \end{cases}$$

$\longrightarrow$

Absolute Max	17	at $x = 4$
Absolute Min	-3	at $x = 2$

Show the picture of the graph.

Example Problem 2: Find abs. max and abs. min of

$$y = f(x) = x^3 \text{ over } [-1, 2].$$

Step 1.

① $a = -1, b = 2$

② $f'(x) = 3x^2$

$f'(c) = 3c^2 = 0 \longrightarrow c = 0$

Step 2.

$$\begin{cases} f(-1) = -1, & f(2) = 8 \\ f(0) = 0, & \end{cases}$$

$\longrightarrow$

Absolute Max	8	at $x = 2$
Absolute Min	-1	at $x = -1$

Show the picture of the graph.

Note: 0 at $x = 0$ is NOT either abs. max or abs. min.
0 at $x = 0$ is NOT either local max or local min.

Example Problem 3: Find abs. max and abs. min of

$$y = f(t) = 2\cos t + \sin(2t) \text{ over } [0, \frac{\pi}{2}].$$

Step 1.

① $a = 0, b = \frac{\pi}{2}$

②

$$\begin{aligned} f'(t) &= -2\sin t + 2\cos(2t) \\ &= -2\sin t + 2(\cos^2 t - \sin^t) \\ &= -2\sin t + 2(1 - 2\sin^t) \\ &= -2(2\sin^2 t + \sin t - 1) \\ &= -2(2\sin t - 1)(\sin t + 1) \end{aligned}$$

$f'(c) = -2(2\sin c - 1)(\sin c + 1) = 0$

$\longrightarrow$

$\sin c = \frac{1}{2}$ or ~~-1~~

$\longrightarrow c = \frac{\pi}{6} \in [0, \frac{\pi}{2}]$

Step 2.

$$\longrightarrow \begin{cases} f(0) = 2, & f\left(\frac{\pi}{2}\right) = 0 \\ f\left(\frac{\pi}{6}\right) & = 2\cos\left(\frac{\pi}{6}\right) + \sin\left(2 \cdot \frac{\pi}{6}\right) \\ & = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \\ & = \frac{3\sqrt{3}}{2} \end{cases}$$

Absolute Max	$\frac{3\sqrt{3}}{2}$	at $t = \frac{\pi}{6}$
Absolute Min	0	at $t = \frac{\pi}{2}$

Note:

$$2 = \frac{4}{2} < \frac{3\sqrt{3}}{2}$$

since

$$2^2 = \left(\frac{4}{2}\right)^2 = \frac{16}{4} < \frac{9 \cdot 3}{4} = \left(\frac{3\sqrt{3}}{2}\right)^2$$

Show the picture of the graph.

Lesson 20

Topics: Mean Value Theorem & What derivatives tell us Part 1
(1st derivative test & 2nd derivative test)
Section Number: 4.2, 4.3
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 19. This should serve as a review of Lesson 19.
- Review of Lesson 19
- MyLabMath Homework for Lesson 19

(2) **Mean Value Theorem**
- Statement

  $y = f(x)$ a function continuous over $[a, b]$
  differentiable over (a, b)

  $\Longrightarrow$

  $\exists\ c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- Explanations
  - ◦ using the slopes **Draw the picture.**

    Let's call point $P(a, f(a))$ and $Q(b, f(b))$.
    Then $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the line PQ.
    On the other hand, $f'(c)$ is the slope of the tangent at the point $(c, f(c))$. Therefore, what Mean Value Theorem is telling us is that at some point $c \in (a, b)$ the slope of the tangent is equal to the slope of the line PQ.
  - ◦ using the velocity

    Say, I drive from West Lafayette to Chicago, total of 240 miles in 3 hours. In Chicago, I get pulled over by a police (who has been watching me from the air). I insist that I was always going under the speed limit of 70 miles, always looking at the speedometer. The policeman then smiles and says "I took Calculus at Purdue. The average speed is $\dfrac{240 - 0}{3 - 0}$. By the Mean Value Theorem at some point $c \in (0, 3)$, the reading $f'(c)$ of the speedometer is exactly the average speed, i.e., $f'(c) = \dfrac{240 - 0}{3 - 0} = 80$. Therefore, I have to give you a speeding ticket."
- Example

$f(x) = 2x^3 - 3x + 1$

Check continuous over $[-2, 2]$ ✓
Check differentiable over $(-2, 2)$ ✓

Compute

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{11 - (-9)}{4} = 5$$

Mean Value Theoirem says

$\exists\ c \in (-2, 2)$ s.t. $f'(c) = 5$

Examination of the statement:

$f'(x) = 6x^2 - 3$

$\longrightarrow$

$f'(c) = 6c^2 - 3 = 5$

$\longrightarrow$

$6c^2 = 8 \longrightarrow c^2 = \dfrac{4}{3}$

$\longrightarrow$

$c = \pm\sqrt{\dfrac{4}{3}}$ will do, i.e., $f'(c) = 5$.

(3) Rolle's Theorem (Special Case of Mean Value Theorem)

- Situation

  The same situation as in the Mean Value Theorem

  With the extra assumption $f(a) = f(b)$

  $\Longrightarrow$

  $$\exists\, c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

- Explanation **Draw the picture !**

(4) Corollary to Rolle's Theorem

- Statement

  $y = f(x)$ a function continuous over $[a, b]$

  differentiable over (a, b)

  &

  $f'(c) = 0$ for any $c \in (a, b)$

  $\Longrightarrow$

  f is a constant function.

- Proof: **Proof by contradiction**

  Suppose f is NOT a constant function

  $\longrightarrow$

  $\exists \alpha, \beta \in [a, b]$ s.t. $f(\alpha) \neq f(\beta)$ (say, $\alpha < \beta$)

  $y = f(x)$ a function continuous over $[\alpha, \beta]$

  differentiable over (α, β)

  $\overset{\text{M.V.Th.}}{\longrightarrow}$

  $$\exists\, c \in (\alpha, \beta)(\subset (a, b)) \text{ s.t. } f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \neq 0.$$

  This contradicts the assumption $f'(c) = 0$ for all $c \in (a, b)$.

  Therefore, f is a constant function.

- Application
  - Problem:

    $$\sin^{-1}\left(\frac{2}{5}\right) + \cos^{-1}\left(\frac{2}{5}\right) = ?$$

  - Solution:

    Consider $f(x) = \sin^{-1}(x) + \cos^{-1}(x)$, which is

    continous over $[-1, 1]$

    differentiable over $(-1, 1)$

  Compute

  $$f'(x) = \frac{1}{\sqrt{1 - x^2}} + \left(-\frac{1}{\sqrt{1 - x^2}}\right) = 0.$$

  $\overset{\text{Corollary}}{\longrightarrow}$

  f is a constant over $[-1, 1]$

  Now

  $$\begin{aligned} f\left(\frac{2}{5}\right) &= \sin^{-1}\left(\frac{2}{5}\right) + \cos^{-1}\left(\frac{2}{5}\right) \\ \| \\ f(0) &= \sin^{-1}(0) + \cos^{-1}(0) \\ &= 0 + \frac{\pi}{2} = \frac{\pi}{2} \end{aligned}$$

  That is to say,

  $$\sin^{-1}\left(\frac{2}{5}\right) + \cos^{-1}\left(\frac{2}{5}\right) = \frac{\pi}{2}.$$

Alternative Solution

Draw the picture of a right triangle with hypotenuse 5 and vertical side 2.

(5) Strange Example (?!)

Situation: $y = f(x) = x^{2/3}$ over $[-1, 1] = [a, b]$

Draw the picture.

Observe $f(a) = f(b) = 1$.

But there is no $c \in (a, b) = (-1, 1)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0!$$

Rolle's Theorem fails ?

No. Since f is NOT differentiable at 0, the assumption of f being differentiable over (a, b) is NOT satisfied.

Therefore, we can NOT apply Rolle's Theorem.

(6) 1st Derivative Test & 2nd Derivative Test

• 1st Derivative Test

Situation: f is differentiable near the point c

x		c	
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	local max	$\searrow$

x		c	
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	local min	$\nearrow$

• 2nd Derivative Test

Situation: f', f'' exist near the point c & f'' is continuous near c)

x		c	
$f'(x)$		0	
$f''(x)$		$-$	
$f(x)$		$\frown$ local max	

x		c	
$f'(x)$		0	
$f''(x)$		$+$	
$f(x)$		$\smile$ local min	

Warning: Consider the example

$$\begin{aligned} f(x) &= x^4 \\ f'(x) &= 4x^3 \\ f''(x) &= 12x^2 \end{aligned}$$

x		c	
$f'(x)$	$-$	0	$+$
$f''(x)$		0	
$f(x)$	$\searrow$	local min	$\nearrow$

1st Derivative Test $\longrightarrow$ local min

We cannot use 2nd Derivative Test

Lesson 21

Topics: What derivatives tell us Part II
Section Number: 4.3
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 20. This should serve as a review of Lesson 20.
- Review of Lesson 20
- MyLabMath Homework for Lesson 20

(2) How to figure out the shape of a graph

Example ①

$$\begin{aligned} y = f(x) &= 3x^4 - 4x^3 - 6x^2 + 12x + 1 \\ f'(x) &= 12x^3 - 12x^2 - 12x + 12 \\ &= 12(x^3 - x^2 - x + 1) \\ &= 12(x-1)(x^2-1) \\ &= 12(x-1)(x-1)(x+1) \\ &= 12(x+1)(x-1)^2 \\ f''(x) &= 36x^2 - 24x - 12 \\ &= 12(3x^2 - 2x - 1) \\ &= 12(3x+1)(x-1) \end{aligned}$$

$f'(x) = 0 \longrightarrow x = -1, 1$

$f''(x) = 0 \longrightarrow x = -\frac{1}{3}, 1$

x		-1		$-\frac{1}{3}$		1	
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$	0	$+$
$f(x)$		local min		inf. pt.		neither local min nor local max inf. pt.	

Desmos Picture

Note:

inf. point $\Longleftrightarrow$ where the concavity changes

$\not\Longleftrightarrow f'(c) = 0$

$\Longleftrightarrow f'(c) = 0$

Example related to the note:

$$\begin{aligned} y = f(x) &= x^4 + 3x \\ f'(x) &= 4x^3 + 3 \\ f''(x) &= 12x^2 \end{aligned}$$

x		0	
$f'(x)$	$+$	3	$+$
$f''(x)$	$+$	0	$+$

Then even though $f''(0) = 0$, the point $(0, 0)$ is NOT an inflection point.

Example ②

$$y = f(x) = x^x \text{ over } (0, \infty)$$

Computation of $f'(x)$

$$\left\{\begin{array}{rcl} y & = & x^x \\ \ln y & = & x \ln x \\ \dfrac{1}{y}\dfrac{dy}{dx} & = & 1 \cdot \ln x + x \cdot \dfrac{1}{x} \\ f'(x) & = & \dfrac{dy}{dx} \\ & = & y(\ln x + 1) \\ & = & x^x(\ln x + 1) \end{array}\right.$$

$$\begin{array}{rcl} y = f(x) & = & x^x \\ f'(x) & = & y(\ln x + 1) \\ & = & x^x(\ln x + 1) \\ f''(x) & = & (x^x)' \cdot \ln x + x^x \cdot \dfrac{1}{x} + (x^x)' \\ & = & x^x(\ln x + 1)\ln x + x^x \cdot \dfrac{1}{x} + x^x(\ln x + 1) \\ & = & x^x \left\{ (\ln x + 1)^2 + \dfrac{1}{x} \right\} > 0 \end{array}$$

x	0		$\frac{1}{e}$	
$f'(x)$	$\times$	$-$	0	$+$
$f''(x)$	$\times$	$+$	$+$	$+$
$f(x)$	$\searrow$ cancave up	local and absolute min	$\nearrow$ concave down	

Desmos Picture

Note: $\lim_{x \to 0^+} x^x = 1$. (We will see why this computation holds later.)

Example ③

$$\begin{array}{rcl} y = f(x) & = & \sin^2 x \text{ over } [0, 2\pi] \\ f'(x) & = & 2 \sin x \cos x \\ & = & 2\sin(2x) \\ f''(x) & = & 2\cos(2x) \cdot 2 \\ & = & 4\cos(2x) \end{array}$$

Picture of the unit circle to figure out in what anlges $f'(x)$ and/or $f''(x)$ becomes 0.

$$f'(x) = 0 \iff x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

$$f''(x) = 0 \iff x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

x	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$		π		$\frac{5\pi}{4}$		$\frac{3\pi}{2}$		$\frac{7\pi}{4}$		2π
$f'(x)$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0
$f''(x)$	4	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	4
$f(x)$																	

Desmos Picture !

Lesson 22

Topics: Graphing functions
Section Number: 4.4
Note: In the original syllabus the topic of graphing functions was to be discussed in Part 1 and Part 2 during the two seperate lessons. We find it better to discuss the subject in one single lesson.
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 21. This should serve as a review of Lesson 21.
- Review of Lesson 21
- MyLabMath Homework for Lesson 21

(2) How to sketch the graph of a function
Example ①

$$\begin{aligned} f(x) &= \frac{x^3}{3} - 400x = x\left(\frac{x^2}{3} - 400\right) = x\left(x+\sqrt{1200}\right)\left(x-\sqrt{1200}\right) \\ & \qquad \text{over the domain } (-\infty, \infty) \\ f'(x) &= x^2 - 400 = (x+20)(x-20) \\ f''(x) &= 2x \end{aligned}$$

$f(x) = 0 \Longleftrightarrow x = \pm\sqrt{1200} = \pm 20\sqrt{3}$
$f'(x) = 0 \Longleftrightarrow x = \pm 20$
$f''(x) = 0 \Longleftrightarrow x = 0$

x		-20	0		20		
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f(x)$		$\frac{16000}{3}$					
	$\nearrow$		$\searrow$	0	concave up		concave up
	concave up		concave down		$\searrow$		$\nearrow$
				inf. pt.		$-\frac{1600}{3}$	

Desmos Picture !

Example ②

$$\begin{aligned} f(x) &= \frac{x^3}{x^2-1} \\ & \text{over the domain } x \neq \pm 1 \text{ i.e., } (-\infty,-1)\cup(-1,1)\cup(1,\infty) \\ f'(x) &= \frac{3x^2(x^2-1) - x^3\cdot 2x}{(x^2-1)^2} \\ &= \frac{x^2(x^2-3)}{(x^2-1)^2} \\ f''(x) &= \frac{(4x^3-6x)(x^2-1)^2 - (x^4-3x^2)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} \\ &= \frac{2x(x^2+3)}{(x^2-1)^3} \end{aligned}$$

$f(x) = 0 \Longleftrightarrow x = 0$
$f'(x) = 0 \Longleftrightarrow x = 0, \pm\sqrt{3}$
$f''(x) = 0 \Longleftrightarrow x = 0$

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	$+$	0	$-$	$\times$	$-$	0	$-$	$\times$	$-$	0	
$f''(x)$	$-$	$-$	$-$	$\times$	$+$	0	$-$	$\times$	$+$	$+$	$+$
$f(x)$	c. d.		c. d.	$\times$	$\searrow$		c. d.	$\times$	$\searrow$		$\nearrow$
	$\nearrow$		$\searrow$	$\times$	c. u.		$\searrow$	$\times$	c. u.		c. u.

(i) **Behavior around where the function is not defined**

$$\lim_{x\to -1^-} \frac{x^3}{x^2-1} = -\infty, \lim_{x\to -1^+} \frac{x^3}{x^2-1} = \infty$$

$$\lim_{x\to 1^-} \frac{x^3}{x^2-1} = -\infty, \lim_{x\to 1^+} \frac{x^3}{x^2-1} = \infty$$

(ii) **Behavior around** $\pm\infty$

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} \text{ with } \lim_{x\to +\infty} \frac{x}{x^2-1} = \lim_{x\to -\infty} \frac{x}{x^2-1} = 0$$

That is to say, $y = x$ is the slant asymptote fpr $y = f(x) = \dfrac{x^3}{x^2-1}$ at $\pm\infty$

Desmos Picture !

Example ③

$$\begin{aligned} f(x) &= e^{-x^2} \\ &\quad \text{over the domain } (-\infty, \infty) \\ f'(x) &= e^{-x^2}(-2x) \\ &= -2xe^{-x^2} \\ f''(x) &= -2e^{-x^2} + (-2x)(-2xe^{-x^2}) = 2e^{-x^2}(2x^2-1) \end{aligned}$$

$f(x) > 0$

$f'(x) = 0 \iff x = 0$

$f''(x) = 0 \iff x = \pm\sqrt{\dfrac{1}{2}}$

x		$-\sqrt{\frac{1}{2}}$		0		$\sqrt{\frac{1}{2}}$	
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$	concave up		$\nearrow$	1	$\searrow$	concave up	
	$\nearrow$		concave down		concave down	$\searrow$	

(i) **Behavior around** $\pm\infty$

$$\lim_{x\to -\infty} e^{-x^2} = \lim_{x\to +\infty} e^{-x^2} = 0.$$

Desmos Picture !

Example ④

$$
\begin{aligned}
f(x) &= \frac{1}{8}x^{\frac{2}{3}}(9x^2 - 8x - 16) \\
&\quad \text{over the domain } (-\infty, \infty) \\
f'(x) &= \left(9x^{\frac{8}{3}} - x^{\frac{5}{3}} - 2x^{\frac{2}{3}}\right) \\
&= 3x^{\frac{5}{3}} - \tfrac{5}{3}x^{\frac{2}{3}} - \tfrac{4}{3}x^{-\frac{1}{3}} \\
&= \frac{(x-1)(9x+4)}{3x^{\frac{1}{3}}} \\
f''(x) &= \frac{45x^2 - 10x + 4}{9x^{\frac{4}{3}}}
\end{aligned}
$$

$f(x) = 0 \iff x = 0, \dfrac{4 \pm 4\sqrt{10}}{9}$

$f'(x) = 0 \iff x = -\dfrac{4}{9}, 1$

Note: $f'(x)$ DNE at $x = 0$

$f''(x) > 0$ (except for $x = 0$, where $f''(x)$ DNE)

since $45x^2 - 10x + 4 = 45\left(x - \dfrac{1}{9}\right)^2 + \dfrac{279}{81} > 0$ and since $x^{\frac{4}{3}} > 0$

x		$-\frac{4}{9}$		0		0	
$f'(x)$	$-$	0	$+$	$\times$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$	$\times$	$+$	$+$	$+$
$f(x)$	concave up		concave up	0	concave up	concave up	
	$\searrow$		$\nearrow$		$\searrow$	$\nearrow$	

Desmos Picture !

Lesson 23

Topics: Optimization Problem, Part 1
Section Number: 4.5
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 22. This should serve as a review of Lesson 22.
- Review of Lesson 22
- MyLabMath Homework for Lesson 22

(2) Typical Pattern of an Optimization Problem and its Solution
- Explanation of the pattern using an example

Example Problem ①

Picture with variables: Picture of a rancher constructing a recutangular corral to raise pigs. Set up some appropriate varibles.

Condition: $4x + y = 400$

Objective: Specification of the optimization problem

Maximize the area

$$\begin{aligned} A &= xy \\ A(x) &= x(400 - 4x) = -4x^2 + 400x \\ &\qquad 0 \le x \le 100 (= 400/4) \end{aligned}$$

Solution:

$$\begin{aligned} A'(x) &= -8x + 400 \\ &= -8(x - 50) \end{aligned}$$

x	0		50		400
$A'(x)$		+	0	−	
$A(x)$		↗	absolute max	↘	

Note:

(i) From 1st derivative test, the function takes the LOCAL max at $x = 50$. But looking at the global behavior of the function that it is increasing when $0 \le x \le 50$ and that it iss decreasing when $50 \le x \le 100$, we conclude that the function takes the GLOBAL max at $x = 50$.

(ii) We do NOT have to compute $A''(x)$, since we are only interested in finding the abs. max (i.e., the incease and decrease of the function).

<u>Grand Conclusion</u>

When

$$\begin{cases} x = 50 \\ y = 400 - 4 \cdot 50 = 200 \end{cases}$$

the area A takes its maximum

$$A = xy = 50 \cdot 200 = 1000 \quad (= A(50)).$$

(3) Example Problems

Example Problem ②

Picture with variables: Picture of a recutangular box with square bottom (side length $= w$ and height $= h$)

Condition: $2w + h = 64$

Objective: Specification of the optimization problem

Maximize the volume

$$\begin{aligned} V &= w^2 h \\ V(w) &= w^2(64 - 2w) = -2w^3 + 64w^2 \\ &\quad 0 \le w \le 32(= 64/2) \end{aligned}$$

Solution:

$$\begin{aligned} V'(w) &= -6w^2 + 128w \\ &= -6w\left(w - \frac{64}{3}\right) \quad \frac{64}{3} = \frac{128}{6} \end{aligned}$$

w	0		$\frac{64}{3}$		32
$V'(w)$		$+$	0	$-$	
$V(w)$		↗	absolute max	↘	

Note:

(i) From 1st derivative test, the function takes the LOCAL max at $w = \frac{64}{3}$. But looking at the global behavior of the function that it is increasing when $0 \le w \le \frac{64}{3}$ and that it is decreasing when $\frac{64}{3} \le w \le 100$, we conclude that the function takes the GLOBAL max at $w = 50$.

(ii) We do NOT have to compute $V''(w)$, since we are only interested in finding the abs. max (i.e., the incease and decrease of the function).

<u>Grand Conclusion</u>

When

$$\begin{cases} w = \frac{64}{3} \\ h = 64 - 2w = \frac{64}{3} \end{cases}$$

the volume V takes its maximum

$$V = w^2 h = \left(\frac{64}{3}\right)^2 \cdot \frac{64}{3}.$$

Example Problem ③ (Difficult ! and Optional !)

Picture with variables: Picture of going from the starting point S on a circular pond of radius 1 mile to the finishing point F on the opposite side, first by swimming from the starting point S straight to the point P (central angle $\angle SOP = \theta$) at the speed of 2 mi/hr and then walk along the pond from the point P to the finishing point F (central angle $\angle POF = \tau$) at the spped of 3 mi/hr.

Condition: $\theta + \tau = \pi$

Objective: Specification of the optimization problem

Minimize the time

$$\begin{aligned} T &= \frac{2\sin\left(\frac{\theta}{2}\right)}{2} + \frac{\tau}{3} \\ T(\theta) &= \sin\left(\frac{\theta}{2}\right) + \frac{\pi - \theta}{3} \end{aligned}$$

$$0 \le \theta \le \pi$$

Note: Explantion for $\overline{SP} = 2\sin\left(\frac{\theta}{2}\right)$.

Solution:

$$\begin{aligned} T'(\theta) &= \cos\left(\frac{\theta}{2}\right) \cdot \frac{1}{2} \\ &= \frac{1}{2}\left\{\cos\left(\frac{\theta}{2}\right) - \frac{2}{3}\right\} \end{aligned}$$

θ	0		$2\cos^{-1}\left(\frac{2}{3}\right)$		π
$T'(\theta)$		$+$	0	$-$	
$T(\theta)$	$\frac{\pi}{3}$	$\nearrow$		$\searrow$	1

Note:

(i) $\cos\left(\frac{\theta}{2}\right) = \frac{2}{3} \longrightarrow \frac{\theta}{2} = \cos^{-1}\left(\frac{2}{3}\right) \longrightarrow \theta = 2\cos^{-1}\left(\frac{2}{3}\right)$

(ii)

$$\begin{aligned} T(0) &= \sin\left(\frac{0}{2}\right) + \frac{\pi - 0}{3} &= \frac{\pi}{3} \\ & & \vee \\ T(\pi) &= \sin\left(\frac{\pi}{2}\right) + \frac{\pi - \pi}{3} &= 1 \end{aligned}$$

<u>Grand Conclusion</u>

When $\theta = \pi$, the time T takes its minimum $T(\pi) = 1$ over $[0, \pi]$.

That is to say, swimming all along to the finish point (and no running) will achieve the minimum time.

Lesson 24

Topics: Optimization Problem, Part 2
Section Number: 4.5
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 23. This should serve as a review of Lesson 23.
- Review of Lesson 23
- MyLabMath Homework for Lesson 23

(2) Example Problems

Example Problem ④

Picture with variables: Picture of a ladder over an 8-foot-tall fence, which is 3 feet away from the wall of a house

Condition: $b : (3+x) = 8 : x$ i.e., $\frac{b}{3+x} = \frac{8}{x} \longrightarrow b = 8 \cdot \frac{3+x}{x}$

Objective: Specification of the optimization problem

Maximize the length of the ladder

$$L = \sqrt{(3+x)^2 + b^2}.$$

This is equivalent to maximizing

$$\begin{array}{ccl} L^2 & = & (3+x)^2 + b^2 \\ \| & & \\ f(x) & = & (3+x)^2 + \left(8 \cdot \frac{3+x}{x}\right)^2 \\ & = & (3+x)^2 \left(1 + \frac{64}{x^2}\right) \quad 0 < x < \infty \quad \text{i.e.,} \quad (0, \infty) \end{array}$$

Solution:

$$\begin{array}{ccl} f'(x) & = & 2(3+x)\left(1 + \frac{64}{x^2}\right) + (3+x)^2 \left(-\frac{2 \cdot 64}{x^3}\right) \\ & = & 2(3+x)\left\{\left(1 + \frac{64}{x^2}\right) - (3+x)\frac{64}{x^3}\right\} \\ & = & 2(3+x)\left(1 - \frac{3 \cdot 64}{x^3}\right) = 2(3+x)\frac{x^3 - 192}{x^3} \end{array}$$

x	0		$\sqrt[3]{192}$	
$f'(x)$		$-$	0	$+$
$f(x)$		$\searrow$	absolute min	$\nearrow$

Note:

(i) From 1st derivative test, the function takes the LOCAL min at $x = \sqrt[3]{192}$. But looking at the global behavior of the function that it is decreasing when $0 < x \leq \sqrt[3]{192}$ and that it is increasing when $\sqrt[3]{192} \leq x$, we conclude that the function takes the GLOBAL min at $x = \sqrt[3]{192}$.

(ii) We do NOT have to compute $f''(x)$, since we are only interested in finding the abs. min (i.e., the incease and decrease of the function).

<u>Grand Conclusion</u>

When $x = \sqrt[3]{192}$, the length $L = \sqrt{(3 + \sqrt[3]{192})^2 \left\{1 + \frac{64}{(x = \sqrt[3]{192})^2}\right\}}$ is the (absolute) minimum.

Example Problem ⑤

Picture with variables: Picture of a water tank in the shape of a right circular cylinder with radius r and height h

Condition: $\pi r^2 \cdot h = 32000$

Objective: Specification of the optimization problem

Minimize the cost of cleaning (twice as much to clean the wall as to clean the floor, no claening the top)

$$\begin{aligned} C &= 2 \cdot 2\pi r h + \pi r^2 \\ C(r) &= 4\pi \cdot \frac{32000}{\pi r^2} + \pi r^2 \\ &= \frac{128000}{r} + \pi r^2 \quad 0 \leq r \end{aligned}$$

Solution:

$$\begin{aligned} C'(r) &= -\frac{12800}{r^2} + 2\pi r \\ &= \frac{2\pi r^3 - 128000}{r^2} \\ &= \frac{2\pi \left(r^2 - \frac{64000}{\pi} \right)}{r^2} \quad (\frac{64000}{\pi} = \frac{128000}{2pi} \end{aligned}$$

r	0		$\sqrt{\frac{64000}{\pi}} = \frac{80\sqrt{10}}{\pi}$	
$C'(r)$		$-$	0	$+$
$C(r)$		$\searrow$	absolute min	$\nearrow$

Note:

(i) From 1st derivative test, the function takes the LOCAL min at $r = \frac{80\sqrt{10}}{\pi}$. But looking at the global behavior of the function that it is decreasing when $0 < r \leq \frac{80\sqrt{10}}{\pi}$ and that it is increasing when $\frac{80\sqrt{10}}{\pi} \leq r$, we conclude that the function takes the GLOBAL min at $x = \sqrt[3]{192}$.

(ii) We do NOT have to compute $C''(r)$, since we are only interested in finding the abs. min (i.e., the incease and decrease of the function).

<u>Grand Conclusion</u>

When $r = \frac{80\sqrt{10}}{\pi}$, the cost $C = C(\frac{80\sqrt{10}}{\pi})$ is the (absolute) minimum.

Lesson 25

Topics: Linear Approximaation and Differentials
Section Number: 4.6
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 24. This should serve as a review of Lesson 24.
- Review of Lesson 24
- MyLabMath Homework for Lesson 24

(2) Review of the Slope-Point Formula
- **Picture !**
- Slope-Point Formula

Line
with slope: m
passing the point (a, b)

$$y - b = m(x - a)$$

$\longrightarrow$

$$\boxed{y = b + m(x - a)}$$

(3) Linear Approximation of $f(x)$ at $x = a$
- **Picture !**
  Graph of a function $y = f(x)$ with the tangent at the point $(a, f(a))$
- Equation of the line representing Linear Approximation of $f(x)$ at $x = a$

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

- **Main Idea**
  $L(x)$ approximates $f(x)$, i.e., $L(x) \approx f(x)$ when x is close to a
- Examples

Example Problem ①:

$$\sqrt{1.01} \approx ?$$

Answer:
We use the linear approximation of $f(x) = \sqrt{x}$ at $a = 1$ (which is close to 1.01).
Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= \sqrt{1} + \frac{1}{2}\sqrt{1}](x - 1) \\ &= 1 + \frac{1}{2}(x - 1) \end{aligned}$$

$$L(1.01) = 1 + \frac{1}{2}(1.01 - 1) = 1.005.$$

$$L(1.01) \approx f(1.01)$$

<u>Conclusion</u>

The value $\sqrt{1.01} = f(1.01)$ can be approximated by $1.005 = L(1.01)$.

Remark:

(i) We can also use the linear approximation of $g(x) = \sqrt{1+x}$ at $a = 0$ (which is close to 0.01). Then $\sqrt{1.01} = g(0.1) \approx L(0.1)$.

(ii) We can use any linear approximation, as long as the given value can be described as $f(x)$ where x is "close" to a.

Example Problem ②:

$$\sqrt{3.98} \approx ?$$

Answer:

We use the linear approximation of $f(x) = \sqrt{x}$ at $a = 4$ (which is close to 3.98). Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt{4} + \frac{1}{2}\sqrt{4}](x-4) \\ &= 2 + \frac{1}{4}(x-4) \end{aligned}$$

$$L(3.98) = 2 + \frac{1}{4}(3.98 - 4) = 1.995.$$

$$L(3.98) \approx f(3.98)$$

Conclusion

The value $\sqrt{3.98} = f(3.98)$ can be approximated by $1.995 = L(3.98)$.

(4) Differentials

• Picture of the graph of a function f and its tangent at the point $(x, f(x))$

• Notations

$$\begin{cases} \Delta x = (x + \Delta x) - x & : \text{increase of the value } x \\ \Delta y = f(x+\Delta x) - f(x) & : \text{increase of the value of the function} \\ dy & : \text{increase of the value on the tangent line} \\ dx = \Delta x & : \text{increase of the value } x \end{cases}$$

• Observation

Since the tangent line is a straight line (of course !), we have

$$\text{the slope} = \frac{\text{increase of the value on the line}}{\text{increase of the value } x}$$

Therefore, we conclude

$$f'(x) = \text{the slope of the tangent} = \frac{\text{increase of the value on the tangent line}}{\text{increase of the value } x} = \frac{dy}{dx}$$

Note: So far, we just understood that equation

$$f'(x) = \frac{dy}{dx}$$

is a matter of symbolic notation, where neither dy nor dx had its meaning by itself. Now the equation has the meaning, where dy (as well as dx) has its meaning by itself.

• Application to approximation

$\Delta y = f(x + \Delta x) - f(x)$ can be approximated by $dy = f'(x)dx$ when $\Delta x = dx$ is closed to 0.

• Example

$$\begin{cases} f(x) &= 3\cos^2 x \\ f'(x) &= 3\cdot 2\cos x\cdot(-\sin x) \\ &= -3\cdot 2\cos x\sin x \\ &= -3\sin(2x) \\ dy &= f'(x)dx \\ &= -3\sin(2x)dx \end{cases}$$

We consider the particular case of

$$x = \frac{\pi}{4} \text{ and } dx = \Delta x = 0.1.$$

$\longrightarrow$

$$x + \Delta x = \frac{\pi}{4} + 0.1$$

$\longrightarrow$

$$\begin{aligned} \Delta y &= f(x+\Delta x) - f(x) \\ &= f\left(\frac{\pi}{4} + 0.1\right) - f\left(\frac{\pi}{4}\right) \\ &= 3\cos^2\left(\frac{\pi}{4} + 0.1\right) - 3\cos^2\left(\frac{\pi}{4}\right) \end{aligned}$$

can be approximated by (i.e., $\Delta y \approx dy$)

$$\begin{aligned} dy &= f'(x)dx \\ &= f'\left(\frac{\pi}{4}\right)\cdot 0.1 \\ &= -3\cdot 0.1 = -0.3 \end{aligned}$$

Note: $f'(x) = f'\left(\frac{\pi}{4}\right) = -3\sin\left(2\cdot\frac{\pi}{4}\right) = -3.$

Lesson 26

Topics: L'Hospital's Rule
Section Number: 4.7
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 25. This should serve as a review of Lesson 25.
- Review of Lesson 25
- MyLabMath Homework for Lesson 25

(2) L'Hospital's Rule
- Statement

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \overset{\text{formally}}{=} \frac{0}{0}, \frac{\pm\infty}{\pm\infty}.$$

Note: We call $\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ "the provisional form" below.

(3) Examples

Example Problem ①:

$\lim_{x\to 0}\dfrac{\sin x}{x}$; provisional form $= \left(\dfrac{0}{0}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to 0}\dfrac{\cos x}{1} = 1.$

Example Problem ②:

$\lim_{x\to 0}\dfrac{e^x - x - 1}{x}$; provisional form $= \left(\dfrac{0}{0}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to 0}\dfrac{e^x - 1}{2x}$; provisional form $= \left(\dfrac{0}{0}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to 0}\dfrac{e^x}{2} = \dfrac{1}{2}$

Example Problem ③:

$\lim_{x\to\infty}\dfrac{x^2}{e^x}$; provisional form $= \left(\dfrac{\infty}{\infty}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\dfrac{2x}{e^x}$; provisional form $= \left(\dfrac{\infty}{\infty}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\dfrac{2}{e^x} = 0$

(4) Calculating the limits of indeterminate forms
- Indeterminate Form: $\infty \times 0$

Example Problem ④

$$\lim_{x\to\infty} x^2 \cdot \sin\left(\frac{1}{4x^2}\right)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \infty & \times & 0 \end{array}$$

$= \lim_{x\to\infty}\dfrac{\sin\left(\dfrac{1}{4x^2}\right)}{1/x^2}$; provisional form $= \left(\dfrac{0}{0}\right)$

$\overset{\text{L.H.}}{=} \lim_{x\to\infty}\dfrac{\cos\left(\dfrac{1}{4x^2}\right)\left\{\dfrac{1}{4}\cancel{(-2)\dfrac{1}{x^3}}\right\}}{\cancel{(-2)1/x^3}}$

$= \lim_{x\to\infty}\cos\left(\dfrac{1}{4x^2}\right)\cdot\dfrac{1}{4} = \dfrac{1}{4}.$

Remarks:

(i) (**Warning**) Some student might think that, since whenever we multiply 0, the limit of the form $\infty \times 0$ must be equal to 0. As observed above, this is NOT the case. One can also see the failre of "the limit of the form $\infty \times 0$ must be equal to 0" in the following easy example.

$$\begin{array}{cccccc} \lim_{x\to\infty} & x & \cdot & \dfrac{3}{x} & = & 3 \\ & \downarrow & & \downarrow & & \\ & \infty & \times & 0 & \neq & 0. \end{array}$$

(ii) (**FAQ**) If we see the limit of the form $\infty \times 0$, which factor should one bring it to the denominator, after taking its reciprocal ?

Answer: There is no universal rule. If the factor you happen to choose does not work, just take the other one !

Look at the following example.

$$\begin{array}{lccc} \lim_{x\to\infty} & x^2 & \cdot & e^x \\ & \downarrow & & \downarrow \\ & \infty & \times & 0 \end{array}$$

$$= \lim_{x\to\infty} \frac{e^x}{1/x^2}; \quad \text{provisional form} = \left(\frac{0}{0}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{e^x}{(-2)1/x^3}$$

$$= \lim_{x\to\infty} -\frac{1}{2}x^3e^x$$

With this choice of the factor, the resulting limit $\lim_{x\to\infty} -\frac{1}{2}x^3e^x$ is more complicated and haarder to compute than the original $\lim_{x\to\infty} x^2 \cdot e^x$.

We try the other one.

$$\begin{array}{lccc} \lim_{x\to\infty} & x^2 & \cdot & e^x \\ & \downarrow & & \downarrow \\ & \infty & \times & 0 \end{array}$$

$$= \lim_{x\to\infty} \frac{x^2}{1/e^x}$$

$$= \lim_{x\to\infty} \frac{x^2}{e^{-x}}; \quad \text{provisional form} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{2x}{-e^{-x}}; \quad \text{provisional form} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty} -\frac{2}{-e^{-x}} = 0$$

• Indeterminate Form: $\infty - \infty$

Example Problem ⑤

$$\lim_{x\to\infty} \underset{\infty}{\underset{\downarrow}{x}} - \underset{\infty}{\underset{\downarrow}{\sqrt{x^2-3x}}}$$

$$= \lim_{x\to\infty} \underset{\infty}{\underset{\downarrow}{x}} \cdot \underset{0}{\underset{\downarrow}{\left(1-\sqrt{1-\frac{3}{x}}\right)}}$$

$$= \lim_{x\to\infty} \frac{1-\sqrt{1-\frac{3}{x}}}{1/x} \quad \text{provisional form } \left(\frac{0}{0}\right)$$

$$\overset{\text{L.H.}}{=} \lim_{x\to\infty} -\frac{(-3)\left(-\frac{1}{x^2}\right)}{2\sqrt{1-\frac{3}{x}}} \Big/ \left(-\frac{1}{x^2}\right)$$

$$= -\frac{(-3)}{2} = \frac{3}{2}$$

Remark: We can compute the above limit in a different way by multtiplying the “conjugate”:

$$\lim_{x\to\infty}\left\{x-\sqrt{x^2-3x}\right\}$$
$$= \lim_{x\to\infty}\frac{\left\{x-\sqrt{x^2-3x}\right\}\left\{x+\sqrt{x^2-3x}\right\}}{\left\{x+\sqrt{x^2-3x}\right\}}$$
$$= \lim_{x\to\infty}\frac{x^2-\left(\sqrt{x^2-3x}\right)^2}{\left\{x+\sqrt{x^2-3x}\right\}}$$
$$= \lim_{x\to\infty}\frac{3x}{\left\{x+\sqrt{x^2-3x}\right\}}$$
$$= \lim_{x\to\infty}\frac{3x/x}{\left\{x+\sqrt{x^2-3x}\right\}/x}$$
$$= \lim_{x\to\infty}\frac{3}{1+\sqrt{1-\frac{3}{x}}}$$
$$= \frac{3}{2}$$

• Indeterminate Forms: $1^\infty, 0^0, \infty^0$

Example Problem ⑥: Compute the following limit

$$\lim_{x\to 0^+} x^x; \text{of form } 0^0$$

Solution:

Set

$$y = x^x.$$

Want to compute

$$\lim_{x\to 0^+} y.$$

Instead we compute first

$$\lim_{x\to 0^+} \ln y.$$

Observe

$$\ln y = \ln(x^x) = x \ln x.$$

Now we compute

$$\begin{array}{llll} \lim_{x\to 0^+} \ln y & = & \lim_{x\to 0^+} & \begin{array}{ccc} x & \cdot & \ln x \\ \downarrow & & \downarrow \\ 0 & \times & (-\infty) \end{array} \\ & = & \lim_{x\to 0^+} & \dfrac{\ln x}{1/x} \quad \text{provisional form } \left(\dfrac{0}{0}\right) \\ & \stackrel{\text{L.H.}}{=} & \lim_{x\to 0^+} & \dfrac{1/x}{-1/x^2} \\ & = & \lim_{x\to 0^+}(-x) = 0 & \end{array}$$

That is to say, we conclude

$$\lim_{x\to 0^+} \ln y = 0.$$

Now we go back to the computation of $\lim_{x\to 0^+} y$.

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

Example Problem ⑦: Compute the following limit

$$\lim_{x\to\infty} \left(\frac{2x+1}{2x-1}\right)^{4x+5} \text{; of form } 1^\infty$$

Solution:

Set

$$y = \left(\frac{2x+1}{2x-1}\right)^{4x+5}.$$

Want to compute

$$\lim_{x\to\infty} y.$$

Instead we compute first

$$\lim_{x\to\infty} \ln y.$$

Observe

$$\ln y = \ln\left(\frac{2x+1}{2x-1}\right)^{4x+5} = (4x+5)\ln\left(\frac{2x+1}{2x-1}\right)$$

Now we compute

$$\begin{array}{llllll}
\lim_{x\to\infty} \ln y & = & \lim_{x\to\infty} & (4x+5) & \cdot & \ln\left(\frac{2x+1}{2x-1}\right) \\
 & & & \downarrow & & \downarrow \\
 & & & \infty & \times & 0 \\
 & = & \lim_{x\to\infty} & \dfrac{\ln\left(\frac{2x+1}{2x-1}\right)}{1/(4x+5)} & & \text{provisional form } \left(\frac{0}{0}\right) \\
 & \overset{\text{L.H.}}{=} & \lim_{x\to\infty} & \dfrac{\{\ln(2x+1)-\ln(2x-1)\}'}{\{1/(4x+5)\}'} & & \\
 & = & \lim_{x\to\infty} & \left\{\dfrac{2}{2x+1} - \dfrac{2}{2x-1}\right\} & / & \left\{-\dfrac{4}{(4x+5)^2}\right\} \\
 & = & \lim_{x\to\infty} & \left\{\dfrac{2(2x-1)-2(2x+1)}{(2x+1)(2x-1)}\right\} & / & \left\{-\dfrac{4}{(4x+5)^2}\right\} \\
 & = & \lim_{x\to\infty} & \dfrac{\cancel{-4}(4x+5)^2}{\cancel{-4}(2x+1)(2x-1)} = 4 & &
\end{array}$$

That is to say, we conclude

$$\lim_{x\to\infty} \ln y = 4.$$

Now we go back to the computation of $\lim_{x\to\infty} y$.

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^4.$$

Lesson 27

Topics: Antiderivatives
Section Number: 4.9
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 26. This should serve as a review of Lesson 26.
- Review of Lesson 26
- MyLabMath Homework for Lesson 26

(2) Antiderivative
- Definition

$F(x)$ such that $F'(x) = f(x)$ is called an **antiderivative** of $f(x)$.
- Example

$F(x) = x^3$ and $F'(x) = (x^3)' = 3x^2$.
Therefore, we say $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.
- **Important Observation**

Observe
$$\begin{cases} (x^3+2)' &= 3x^2 \\ (x^2-5)' &= 3x^2 \end{cases}$$
and hence both x^3+2 and x^3-5 are antiderivatives of $3x^2$.

In general,

an antiderivative of $f(x)$ is determined only up to a constant

in the following sense:
(i) If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+C$ for any constant C.
(ii) If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then they only differ by a constant. That is to say, there exists a constant C such that $G(x) = F(x)+C$.
We write
$$\int f(x)\ dx = F(x) + C.$$

(3) Examples
$$\begin{aligned}
\int e^x\ dx &= e^x + C \\
\int \cos x\ dx &= \sin x + C \\
\int \sin^x\ dx &= -\cos x + C \\
\int x^n\ dx \quad (n \neq -1) &= \frac{1}{n+1}x^{n+1} + C \\
\int x^n\ dx \quad (n = -1) &= \int \frac{1}{x}\ dx = \ln|x| + C \\
\int b^x\ dx \quad (b > 0) &= \frac{b^x}{\ln b} + C \\
\left(\text{e.g.} \int 2^x\ dx \right. &= \left. \frac{2^x}{\ln 2} + C \right) \\
\int \sec^2 x\ dx &= \tan x + C \\
\int \sec x \tan x\ dx &= \sec x + C
\end{aligned}$$

Note: We will deal with more difficult questions such as

$$\begin{aligned}\int \tan x \; dx &= ?\\ \int \ln x \; dx &= ?\end{aligned}$$

later.

(4) Example Problems

- Purely computational

Example Problem ①:

$$\begin{aligned}\int \frac{x^2-3}{x^2+1}\,dx &= \int \frac{1\cdot(x^2+1)-4}{x^2+1}\,dx\\ &= \int\left(1-\frac{4}{x^2+1}\right)dx\\ &= x-4\tan^{-1}x+C\end{aligned}$$

Example Problem ②:

$$\begin{aligned}\int\left(3x^5+2-5\sqrt{x}\right)dx &= \int 3\cdot\frac{1}{6}x^6+2\cdot x-5\cdot\frac{1}{1+\frac{1}{2}}x^{1+\frac{1}{2}}+C\\ &= \frac{1}{2}x^6+2x-\frac{10}{3}x^{\frac{3}{2}}+C.\end{aligned}$$

- Easy Differential Equation

Example Problem ③: We have a function satsifying

$$f'(x)=x^2-2x, f(1)=\frac{1}{3}.$$

Determine $f(x)$.

Solution.

$$\begin{aligned}f(x) &= \int f'(x)\;dx\\ &= \int(x^2-2x)\;dx\\ &= \frac{1}{3}x^3-x^2+C.\end{aligned}$$

$$f(1)=\frac{1}{3}1^3-1^2+C=\frac{1}{3}.$$

$\longrightarrow$

$C=1$

<u>Final Answer</u>: $f(x)=\frac{1}{3}x^3-x^2+1.$

Example Problem ④: We have $\begin{cases} a(t) = -9.8\\ v(0) = 40\\ s(0) = 100.\end{cases}$ That is to say, in terms of the function $s(t)$, we have $\begin{cases} s''(t) = -9.8\\ s'(0) = 40\\ s(0) = 100.\end{cases}$ Determine $s(t)$.

Picture of a guy standing on top pf a cliff 100 m high, throwing a rock upward at the initial speed of 40 m/s, where the gravitational acceleration is -9.8 m/s^2. Note that the negative sign indicates the acceleration is downward (of course !).

Solution.

$$
\begin{aligned}
& a(t) = v'(t) = -9.8 \\
\longrightarrow \\
& v(t) = -9.8t + C \\
\& \\
& v(0) = -9.8 \cdot 0 + C = 40 \quad \longrightarrow \quad C = 40 \\
\longrightarrow \\
& \begin{array}{ccc} v(t) & = & -9.8t + 40 \\ \| & & \\ s'(t) & & \end{array} \\
\longrightarrow \\
& s(t) = -9.8 \cdot \frac{1}{2}t^2 + 40t + D \\
\& \\
& s(0) = -9.8 \cdot \frac{1}{2}0^2 + 40 \cdot 0 + D = 100 \quad \longrightarrow \quad D = 100
\end{aligned}
$$

Final Answer: $s(t) = -9.8 \cdot \frac{1}{2}t^2 + 40t + 100 = -4.9t^2 + 40t + 100.$

Lesson 28

Topics: Approximating the area under curve (**Riemann Sum**)
Section Number: 5.1
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 27. This should serve as a review of Lesson 27.
- Review of Lesson 27
- MyLabMath Homework for Lesson 27

(2) Explanation by example
- Situation

We consider the area enclosed by the line $y = 2x, x = 3$ and the x-axis. Of course, the area is easily computed to be that of the right trangle

$$A = \frac{1}{2} 3 \cdot 6 = 9.$$

We are going to compute the area in the following very complicated way:
- ○ Draw the picture of the line $y = 2x$ over the interval $[0, 3]$.
- ○ We devide the interval into n equal subintervals, each of which, therefore, has the length $\frac{3}{n}$.
- ○ We construct the k-th column over the k-th subinterval of height $2 \cdot k \cdot \frac{3}{n}$, which is the value on the line $y = 2x$ of the right end point $x = k \cdot \frac{3}{n}$ $(k = 1, \ldots, n)$. We compute the area of the k-th column to be

$$\text{height} \times \text{width} = 2 \cdot k \cdot \frac{3}{n} \times \frac{3}{n}.$$

- ○ We compute the sum A_n of the areas of all the columns.

$$\begin{aligned}
A_n &= 2 \cdot \left(1 \cdot \frac{3}{n}\right) \cdot \frac{3}{n} \\
&\quad +2 \cdot \left(2 \cdot \frac{3}{n}\right) \cdot \frac{3}{n} \\
&\quad \ldots \\
&\quad +2 \cdot \left(k \cdot \frac{3}{n}\right) \cdot \frac{3}{n} \\
&\quad \ldots \\
&\quad +2 \cdot \left(n \cdot \frac{3}{n}\right) \cdot \frac{3}{n} \\
&= \sum_{k=1}^{n} 2 \cdot \left(k \cdot \frac{3}{n}\right) \cdot \frac{3}{n} \\
&= 2 \cdot \frac{3}{n} \cdot \frac{3}{n} \cdot (1 + 2 + \cdots + (n-1) + n) \\
&= 2 \cdot \frac{3}{n} \cdot \frac{3}{n} \cdot \sum_{k=1}^{n} k \\
&= 2 \cdot \frac{3}{n} \cdot \frac{3}{n} \cdot \frac{n(n+1)}{2} \\
&= 9 \cdot \frac{n(n+1)}{n^2}.
\end{aligned}$$

- **Main Idea**
  The sum A_n gives an approximation of the genuine area, and as n gets bigger and bigger, the estimation becomes better and better.
  When $n \to \infty$, we expect $A_n \to A$.
  Let's check.

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} 9 \cdot \frac{n(n+1)}{n^2} = 9 = A!$$

  BINGO !

(3) Distraction: How to compute

$$S_n = 1 + 2 + \cdots + n = \sum_{k=1}^{n} k.$$

Solution (Gauss).

$$\begin{array}{rcccccccccccc} & S_n & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ +) & S_n & = & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline & 2S_n & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

$\longrightarrow$

$$2S_n = n \cdot (n+1)$$

$\longrightarrow$

$$S_n = \frac{n \cdot (n+1)}{2}$$

<u>Grand Conclusion</u>

$$S_n = \frac{n(n+1)}{2}$$

(4) General setting for the **Riemann Sum**

• Situation
Want to compute the area between
  the graph of a function (continuous and ≥ 0) over the interval $[a, b]$
and
  the x-axis.

We are going to compute the area in the following way:
- Draw the picture of the graph of the function $y = f(x)$ over the interval $[a, b]$.
- We devide the interval into n equal subintervals, each of which, therefore, has the length $\Delta x = \dfrac{b-a}{n}$
- We construct the k-th column over the k-th subinterval of height $f(x_k^*)$, which is the value of the function $y = f(x)$ at some point $x = x_k^*$ in the k-th interval $(k = 1, \ldots, n)$. We compute the area of the k-th column to be

$$\text{height} \times \text{width} = f(x_k^*) \cdot \Delta x.$$

○ We compute the sum A_n of the areas of all the columns.

$$\begin{aligned} A_n &= f(x_1^*)\cdot\Delta x \\ &\quad +f(x_2^*)\cdot\Delta x \\ &\quad \ldots \\ &\quad +f(x_k^*)\cdot\Delta x \\ &\quad \ldots \\ &\quad +f(x_n^*)\cdot\Delta x \\ &= \sum_{k=1}^{n} f(x_k^*)\cdot\Delta x \end{aligned}$$

○ **Main Idea**
The sum A_n gives an approximation of the genuine area, and as n gets bigger and bigger, the estimation becomes better and better.
When $n\to\infty$, we expect $A_n\to A$.

$$\boxed{\lim_{n\to\infty} A_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\cdot\Delta x = A.}$$

Warning: When $y=f(x)$ is continuous over $[a,b]$, the limit actually exists, and this is the definition of the "genuine" area A. That s to say, we define the geneuine area by the Riemman sum.

(5) One more example for Riemann Sum

• Situation
We consider the area enclosed by the parabola $y=x^2, x=1$ and the x-axis.
We are going to compute the area in the following way:

○ Draw the picture of the line $y=x^2$ over the interval $[0,1]$.

○ We devide the interval into n equal subintervals, each of which, therefore, has the length $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$.

○ We construct the k-th column over the k-th subinterval of height $\left(\dfrac{k}{n}\right)^2$, which is the value on the parabola $y=x^2$ of the right end point $x = k\cdot\dfrac{1}{n}\dfrac{k}{n}$ $(k=1,\ldots,n)$. We compute the area of the k-th column to be

$$\text{height}\times\text{width} = \left(\frac{k}{n}\right)^2\times\frac{1}{n}.$$

- We compute the sum A_n of the areas of all the columns.

$$
\begin{aligned}
A_n &= \left(\frac{1}{n}\right)^2 \times \frac{1}{n} \\
&\quad + \left(\frac{2}{n}\right)^2 \times \frac{1}{n} \\
&\quad \dots \\
&\quad + \left(\frac{k}{n}\right)^2 \times \frac{1}{n} \\
&\quad \dots \\
&\quad + \left(\frac{n}{n}\right)^2 \times \frac{1}{n} \\
&= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \times \frac{1}{n} \\
&= \frac{1}{n^3}(1^2 + 2^2 + \cdot + n^2) \\
&= \frac{1}{n^3} \cdot \sum_{k=1}^{n} k^2 \\
&= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{n(n+1)(2n+1)}{6n^3}.
\end{aligned}
$$

- **Main Idea**
  The sum A_n gives an approximation of the genuine area, and as n gets bigger and bigger, the estimation becomes better and better.
  When $n \to \infty$, we observe $A_n \to A$.

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{2}.$$

(6) Distraction (Optional !): How to compute

$$S_n = 1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2.$$

Solution (Gauss).

$$
\begin{array}{rcccccccccc}
 & T_n & = & 1^2 & + & 2^2 & + & \cdots & + & (n-1)^2 & + & n^2 \\
+) & T_n & = & n^2 & + & (n-1)^2 & + & \cdots & + & 2^2 & + & 1^2 \\
\hline
 & & & \not\parallel & & \not\parallel & & \cdots & & \not\parallel & & \not\parallel \\
 & 2T_n & \neq & (n+1)^2 & + & (n+1)^2 & + & \cdots & + & (n+1)^2 & + & (n+1)^2
\end{array}
$$

We can NOT conclude $2T_n = n \cdot (n+1)^2$.
So our previous method to compute S_n does NOT work with T_n.

We need a different idea !

Step 1. We consider

$$\begin{array}{ccccc} & \cancel{1\cdot 2\cdot 3} & - & 0\cdot 1\cdot 2 \\ + & \cancel{2\cdot 3\cdot 4} & - & \cancel{1\cdot 2\cdot 3} \\ & & \cdots & \\ + & \cancel{k(k+1)(k+2)} & - & \cancel{(k-1)k(k+1)} \\ & & \cdots & \\ + & n(n+1)(n+2) & - & \cancel{(n-1)n(n+1)} \end{array}$$

$$= n(n+1)(n+2).$$

Step 2.
What we computed in Step 1 can be written

$$\begin{aligned} &\textstyle\sum_{k=1}^{n} \{k(k+1)(k+2) - (k-1)k(k+1)\} \\ &= \textstyle\sum_{k=1}^{n} 3k(k+1) \\ &= 3\textstyle\sum_{k=1}^{n} k(k+1) \\ &= 3\textstyle\sum_{k=1}^{n} (k^2+k) \\ &= 3\left\{\textstyle\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right\} \\ &= 3(T_n + S_n) \end{aligned}$$

Step 3.
From Step 1 and Ste 2, we conclude

$$n(n+1)(n+2) = 3(T_n + S_n)$$

$$\longrightarrow$$

$$\frac{n(n+1)(n+2)}{3} = T_n + S_n$$

$$\longrightarrow$$

$$\begin{aligned} T_n &= \frac{n(n+1)(n+2)}{3} - S_n \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ &= n(n+1)\left\{\frac{n+2}{3} - \frac{1}{2}\right\} \\ &= n(n+1)\left\{\frac{2(n+2)-3}{6}\right\} \\ &= n(n+1)\left\{\frac{2n+1}{6}\right\} \\ &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

<u>Grand Conclusion</u>

$$T_n = \frac{n(n+1)(2n+1)}{6}.$$

Lesson 29

Topics: Definite Integrals
Section Number: 5.2
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 28. This should serve as a review of Lesson 28.
- Review of Lesson 28
- MyLabMath Homework for Lesson 28

(2) Definite integral
- Picture of the graph of a function $y = f(x)$ (continuous and ≥ 0) over the interval $[a, b]$
- Definition

$\int_a^b f(x)\ dx$: the area between the graph andd the x-axis over the interval $[a, b]$

Remark: **Relation between** $\int_a^b f(x)\ dx$ **and** $\int f(x)\ dx$

Note that we use the symbol $\int f(x)\ dx$ to indicate a (general) antiderivartive of $f(x)$. Here the symbol $\int_a^b f(x)\ dx$ indicates the area specified as above.

The relation between them will be clear when we discuss

Fundamental Theorem of Calculus.

- the case when the value of $f(x)$ is not necessarily positive.

○ Picture of the graph of the function where the values over $[a, b]$ are not necessarily positive

○ In this case,

$\int_a^b f(x)\ dx$: the sum of the areas ABOVE the x-axis MINUS the sum of the areas BELOW the x-axis

Remark: In any case (stay positive or not), the Riemann Sum formula holds. That is to say,

$$\boxed{\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x}$$

(3) Example Problems

Example Problem ①: $\int_3^4 \sqrt{1-(x-3)^2}\ dx = ?$

Solution.

Set $y = \sqrt{1-(x-3)^2}$.
Then $y^2 = 1-(x-3)^2$.

$\longrightarrow$

$(x-3)^+ y^2 = 1$ the circle with center $(3, 0)$ and of radius 1

When $3 \leq x \leq 4$, and when $y = \sqrt{1-(x-3)^2}$, it sweeps out the upper right quarter of this circle.

The area between this curve and the x-axis is, therefore, $\frac{1}{4}\pi \cdot 1^2 = \frac{\pi}{4}$.

$\longrightarrow$

$$\int_3^4 \sqrt{1-(x-3)^2}\ dx = \frac{\pi}{4}.$$

Example Problem ②: Compute $\int_0^2 (x^3+1)\,dx$ using the Riemann Sum.

Solution.

$$
\begin{aligned}
\int_0^2 (x^3+1)\,d &= \lim_{n\to\infty} \textstyle\sum_{k=1}^n f(x_k^*)\cdot \Delta x \\
&\left\{\begin{array}{rcl}
\Delta x &=& \dfrac{b-a}{n} = \dfrac{2-0}{n} \\
f(x) &=& x^3+1 \\
x_k^* &;& \text{we choose it to be the right end point} \\
&& \text{of the } k\text{-th interval} \\
&=& k\cdot \dfrac{2}{n}
\end{array}\right. \\
&= \lim_{n\to\infty} \textstyle\sum_{k=1}^n \left\{\left(k\cdot\dfrac{2}{n}\right)^3+1\right\}\cdot\dfrac{2}{n} \\
&= \lim_{n\to\infty} \dfrac{2}{n}\left\{\left(\dfrac{2}{n}\right)^3 \textstyle\sum_{k=1}^n k^3 + \sum_{k=1}^n 1\right\} \\
&= \lim_{n\to\infty} \dfrac{2}{n}\left\{\left(\dfrac{2}{n}\right)^3 \cdot \dfrac{n^2(n+1)^2}{4} + n\right\} \\
&= \lim_{n\to\infty} \left\{\dfrac{16}{4}\cdot\dfrac{n^2(n+1)^2}{n^4}+2\right\} = 4+2 = 6.
\end{aligned}
$$

Lesson 30

Topics: Fundamental Theorem of Calculus
Section Number: 5.3
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 29. This should serve as a review of Lesson 29.
- Review of Lesson 29
- MyLabMath Homework for Lesson 29

(2) Fundamental Theorem of Calculus
- Picture

The graph of a function f continuous over the interval $[a, b]$.
We use the letter x for the value in $[a, b]$, i.e., $c \in [a, b]$.
- Statement

Set $A(x) = \int_a^x f(t)\ dt$.

Part 1

$A(x)$ is an antiderivative of $f(x)$,

i.e.,

$A'(x) = f(x)$,

i.e.,

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\ dt\right) = f(x).}$$

Part 2

Let $F(x)$ be any (your favorite) antiderivative of $f(x)$.

$\longrightarrow$

$A(x) = F(x) + C$

$\longrightarrow$

$$\begin{aligned} \int_a^b f(t)\ dt &= A(b) \\ &= A(b) - A(a) \\ &= \{F(b) + C\} - \{F(a) + C\} \\ &= F(b) - F(c) \end{aligned}$$

i.e.,

$$\boxed{\int_a^b f(t)\ dt = F(b) - F(a).}$$

- Reasons

Part 1

Why $A'(x) = f(x)$?

Picture: Again draw the picture of the graph of a function f continuous over the interval $[a, b]$.

Emphasis is on the region over the interval $[x, x+h]$, whose area represents the difference $A(x+h) - A(x)$.

When h is small ($h \sim 0$), this region almost looks like the column with height $f(x)$ and width $(x+h) - x = h$.

$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} \\ &= \lim_{h\to 0} \frac{\displaystyle\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt}{h} \\ &= \lim_{h\to 0} \frac{\text{the area of the region} \approx f(x)\cdot h}{h} \\ &= f(x). \end{aligned}$$

Part 2

Part 2 follows immediately from Part 1, saying that $A(x)$ is an antiderivative of $f(x)$ and hence that $A(x) = F(x) + C$ where C is some constant.

(3) Example Problems for Part 1

Example Problem ①: $\dfrac{d}{dx}\left(\displaystyle\int_1^x \sin^2 t\ dt\right) = ?$

Solution.

$$\frac{d}{dx}\left(\int_1^x \sin^2 t\ dt\right) = \sin^2 x$$

Example Problem ②: $\dfrac{d}{dx}\left(\displaystyle\int_x^5 \sqrt{t^2+1}\ dt\right) = ?$

Solution.

$$\begin{aligned} \frac{d}{dx}\left(\int_x^5 \sqrt{t^2+1}\ dt\right) &= \frac{d}{dx}\left(-\int_5^x \sqrt{t^2+1}\ dt\right) \\ &= -\sqrt{x^2+1} \end{aligned}$$

Example Problem ③: $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^4} \sec t\ dt\right) = ?$

Warning (**Wrong Solution !**): $\dfrac{d}{dx}\left(\displaystyle\int_1^{x^4} \sec t\ dt\right) \neq \sec(x^4)$

Solution.

Set $x^4 = u$.

$$\begin{aligned} \frac{d}{dx}\left(\int_1^{x^4} \sec t\ dt\right) &= \frac{d}{dx}\left(\int_1^{u} \sec t\ dt\right) \\ &= \frac{du}{dx}\cdot\frac{d}{du}\left(\int_1^{u} \sec t\ dt\right) \\ &= 4x^3 \cdot \sec u \\ &= 4x^3 \cdot \sec(x^4) \end{aligned}$$

In the above solution, we are actually using the Chain Rule as follows:

$$\begin{aligned} \frac{d}{dx}\left(\int_1^{x^4} \sec t\ dt\right) &= \frac{dy}{dx} \\ &= \frac{du}{dx}\cdot\frac{dy}{du} \\ &= \frac{du}{dx}\cdot\frac{d}{du}\left(\int_1^{u} \sec t\ dt\right) \\ &= 4x^3 \cdot \sec u \\ &= 4x^3 \cdot \sec(x^4) \end{aligned}$$

(4) Example Problems Part 2

- Formula

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Example Problem ④: $\int_1^3 e^x\ dx = ?$

Solution.

Choose $F(x) = e^x$.
Then we have

$$\int_1^3 e^x\ dx = e^3 - e^1$$

Example Problem ⑤: $\int_{-1}^3 \frac{1}{x^2}\ dx = ?$

Solution (????)

Choose $F(x) = -\frac{1}{x}$.
Then we have

$$\int_{-1}^3 \frac{1}{x^2}\ dx = \left(-\frac{1}{3}\right) - \left(-\frac{1}{-1}\right) = -\frac{1}{3}.$$

But WAIT A MINUTE !

Since we always have $\frac{1}{x^2} \geq 0$, wouldn't that imply $\int_{-1}^3 \frac{1}{x^2}\ dx \geq 0$?????
But the above computation says it is negative ????

WHAT IS HAPPENING HERE ?

Explanation: In order to use the Fundamental Theorem of Calculus, the function $f(x)$ has to be defined and continuous for the whole interval $[a, b]$. Here it is NOT defined at $0 \in [-1, 3]$ (and hence not continuous). Setting $F(x) = -\frac{1}{x}$, we have $F'(x) = f(x)$ when $x \neq 0$, but not over the whole interval $[-1, 3]$. Trouble happened because we used the F.T.C. when we cannot.

(5) Difficult Problem

Example Problem ⑥: What value of b (> -1) maximizes the integral

$$\int_{-1}^b x^2(5 - x)\ dx\ ?$$

Solution.

Step 1. Change of letters so that it will fit into our statement of F.T.C.: What value of x (> -1) maximizes the integral

$$F(x) = \int_{-1}^x t^2(5 - t)\ dt\ ?$$

Step 2. Compute $F'(x)$ using the F.T.C.

$$\begin{aligned} F'(x) &= \frac{d}{dx}F(x) \\ &= \frac{d}{dx}\left(\int_{-1}^x t^2(5 - t)\ dt\right) \\ &= x^2(5 - x). \end{aligned}$$

Step 3. Construction of the table as in the optimization problem

x	-1		0		5	
$F'(x)$		$+$	0	$+$	0	$-$
$F(x)$	0	$\nearrow$		$\nearrow$	max	$\searrow$

Step 4. Grand Conclusion

$F(x)$ takes its (absolute) maximum when $x = 5$.

Lesson 31

Topics: Working with Integrals
Section Number: 5.4
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 30. This should serve as a review of Lesson 30.
- Review of Lesson 30
- MyLabMath Homework for Lesson 30

(2) Even function
- Picture
- Definition

$$\begin{aligned} f \text{ even} \quad &\iff \quad f(-x) = f(x) \\ &\iff \quad \text{symmetric w.r.t. } y\text{-axis} \end{aligned}$$

- Property about the integral
  - Picture
  - $\displaystyle\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$
  - Example ①:

$$\int_{-2}^{2} x^4\ dx = 2\int_{0}^{2} x^4\ dx = 2\left[\frac{x^5}{5}\right]_0^2 = \frac{64}{5}$$

  - Example ②:

$$\int_{-\pi/2}^{\pi/2} \cos x\ dx = 2\int_{0}^{\pi/2} \cos x\ dx = 2\left[\sin x\right]_0^{\pi/2} = 2$$

(3) Odd function
- Picture
- Definition

$$\begin{aligned} f \text{ odd} \quad &\iff \quad f(-x) = -f(x) \\ &\iff \quad \text{symmetric w.r.t. the origin} \end{aligned}$$

- Property about the integral
  - Picture
  - $\displaystyle\int_{-a}^{a} f(x)\ dx = 0$
  - Example ③:

$$\int_{-2}^{2} 3x^3\ dx = 0$$

  - Example ④:

$$\int_{-\pi/2}^{\pi/2} \sin x\ dx = 0$$

  - Example ⑤:

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\ dx = 0$$

(4) Average of a function

- Picture

Draw the graph of a function $y = f(x)$ (continuous) over the interval $[a, b]$.
Show the picture of a "wave" machine.

- Definition and formula

$$\boxed{f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\ dx}$$

(5) Mean Value Theorem for Integrals

- Picture

Draw the graph of a function $y = f(x)$ (continuous) over the interval $[a, b]$.

- Statement

$f(x)$ a continuous function over the interval $[a, b]$

$f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\ dx$

$\Longrightarrow$

$\exists\ c \in (a, b)$ s.t. $f(c) = f_{\text{ave}}$

- Proofs

Proof I: Obvious from the picture !

Proof II: Reduction to the usual Mean Value Theorem

Set $A(x) = \displaystyle\int_a^x f(t)\ dt.$

$\longrightarrow$

$A(x)$ continuous over $[a, b]$, and differentiable over (a, b)

Actually we have $A'(x) = f(x)$ for $x \in (a, b)$.

$\longrightarrow$ (via the usual Mean Value Theorem)

$\exists c \in (a, b)$ s.t. $A'(c) = \dfrac{A(b) - A(a)}{b-a}.$

Observing

$$\begin{aligned} A'(c) &= f(c) \\ \frac{A(b) - A(a)}{b-a} &= \frac{1}{b-a}\int_a^b f(t)\ dt = f_{\text{ave}}, \end{aligned}$$

we finally conclude

$$\exists\ c \in (a, b) \text{ s.t. } f(c) = f_{\text{ave}}.$$

• Example

Consider $f(x) = 2x(1-x)$ over the interval $[0,1]$.

Picture !

We compute

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{1-0}\int_0^1 2x(1-x)\ dx \\ &= \int_0^1 (2x-2x^2)\ dx \\ &= \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}. \end{aligned}$$

Now the Mean Value Theirem for Integrals says

$$\exists\ c \in (0,1) \text{ s.t. } f(c) = f_{\text{ave}}.$$

Let's check if such $c \in (0,1)$ really exsists.

We should have the equation

$f(c) = 2c(1-c) = \dfrac{1}{3}$

$\longrightarrow$

$-2c^2 + 2c = \dfrac{1}{3}$

$\longrightarrow$

$2c^2 - 2c + \dfrac{1}{3} = 0$

$\longrightarrow$

$$\begin{aligned} c &= \frac{2 \pm \sqrt{(-2)^2 - 4\cdot\frac{1}{3}}}{2\cdot 2} \\ &= \frac{2 \pm \sqrt{(-2)^2 - 4\cdot\frac{1}{3}}}{2\cdot 2} \\ &= \frac{2 \pm 2\sqrt{\frac{1}{3}}}{2\cdot 2} = \frac{1 \pm \sqrt{\frac{1}{3}}}{2} \end{aligned}$$

Therefore, we cnclude that, for $c = \dfrac{1 \pm \sqrt{\frac{1}{3}}}{2}$, we have $f(c) = f_{\text{ave}}$, as stated in the M.V. Th. for Integrals.

Lesson 32

Topics: Substitution Rules
Section Number: 5.5
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 31. This should serve as a review of Lesson 31.
  - Review of Lesson 31
  - MyLabMath Homework for Lesson 31

(2) Substitution Rule
  - Formula

$$\boxed{\int f(u) \cdot \frac{du}{dx} \cdot dx = \int f(u)\ du}$$

  - Examples

Example ①:

$$\begin{aligned}
&\int 2x\sqrt{1+x^2}\ dx \\
&\qquad \left\{ \begin{array}{rcl} u &=& 1+x^2 \\ \dfrac{du}{dx} &=& 2x \end{array} \right. \\
&= \int \sqrt{u} \cdot \frac{du}{dx} \cdot dx \\
&= \int \sqrt{u}\ du = u^{\frac{1}{2}}\ du \\
&= \frac{1}{1+\frac{1}{2}} u^{1+\frac{1}{2}} + C = \frac{2}{3} u^{\frac{3}{2}} + C \\
&= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C
\end{aligned}$$

Example ②:

$$\begin{aligned}
&\int x^3 \cos(x^4+2)\ dx \\
&\qquad \left\{ \begin{array}{rcl} u &=& x^4+2 \\ \dfrac{du}{dx} &=& 4x^3 \end{array} \right. \\
&= \int \cos u \cdot \frac{1}{4} \frac{du}{dx} \cdot dx \\
&= \int \cos u \cdot \frac{1}{4} du = \frac{1}{4} \int \cos u\ du \\
&= \frac{1}{4}(\sin u + C) = \frac{1}{4} \sin u + C
\end{aligned}$$

(We replace $\frac{1}{4}C$ with a new C, which by abuse of notation, is denoted by the same C.)

$$= \frac{1}{4} \sin(x^4+2) + C.$$

Example ③:

$$\int \sqrt{2x+1}\ dx$$
$$= \int \sqrt{2x+1}\cdot 1 \cdot dx$$
$$\begin{cases} u &= 2x+1 \\ \dfrac{du}{dx} &= 2 \end{cases}$$
$$= \int \sqrt{u}\cdot \frac{1}{2}\frac{du}{dx}\cdot dx$$
$$= \int \sqrt{u}\cdot \frac{1}{2}du = \frac{1}{2}\int \sqrt{u}\ du = \frac{1}{2}\int u^{\frac{1}{2}}\ du$$
$$= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} + C\right) = \frac{1}{3}u^{\frac{3}{2}} + C$$

(We replace $\frac{1}{2}C$ with a new C, which by abuse of notation, is denoted by the same C.)

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)\sqrt{2x+1} + C$$

Example ④:

$$\int \sqrt{1+x^2}\cdot x^5 \cdot dx$$
$$\begin{cases} u &= 1+x^2 \longrightarrow x^2 = u-1 \\ \dfrac{du}{dx} &= 2x \end{cases}$$
$$\int \sqrt{1+x^2}\cdot x^4 \cdot x \cdot dx$$
$$\int \sqrt{1+x^2}\cdot (x^2)^2 \cdot x \cdot dx$$
$$= \int \sqrt{u}\cdot (u-1)^2 \frac{1}{2}\frac{du}{dx}\cdot dx$$
$$= \int \sqrt{u}(u-1)^2 \cdot \frac{1}{2}du = \frac{1}{2}\int \sqrt{u}(u-1)^2\ du = \frac{1}{2}\int \sqrt{u}(u^2-2u+1)\ du$$
$$= \frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)\ du$$
$$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C\right) = \frac{1}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C$$
$$= \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

(3) Substitution Rule for Definite Integrals

- Formula

$$\boxed{\int_a^b f(u)\cdot\frac{du}{dx}\cdot dx = \int_{u(a)}^{u(b)} f(u)\ du}$$

- Important Remark

Substitution Rule for Indefinite Integrals: We have to come back from the u-world to the x-world.

Substitution Rule for Definite Integrals: Once we go to the u-world, we carry out all the computation there, without ever coming back to the x-world.

- Examples

Example ⑤:

$$\int_0^4 \sqrt{2x+1}\ dx$$

$$\left\{\begin{array}{llll} x & u & = & 2x+1 \\ 4 & 9 & & \\ 0 & 1 & & \\ & du & = & 2\ dx \end{array}\right.$$

$$= \int_1^9 \sqrt{u} \cdot \frac{1}{2}\ du = \frac{1}{2} \int_1^9 \sqrt{u}\ du$$

$$= \frac{1}{2}\left[\frac{2}{3} u^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[\left\{(9)^{\frac{1}{2}}\right\}^3 - 1\right] = \frac{1}{3}\left[27-1\right] = \frac{1}{26}.$$

Example ⑥:

$$\int_1^e \frac{\ln x}{x}\ dx$$

$$\left\{\begin{array}{llll} x & u & = & \ln x \\ e & 1 & & \\ 1 & 0 & & \\ & du & = & \dfrac{1}{x}\ dx \end{array}\right.$$

$$= \int_0^1 u\ du$$

$$= \left[\frac{1}{2} u^2\right]_0^1 = \frac{1}{2}\left[u^2\right]_0^1$$

$$= \frac{1}{2}\left[1^1 - 0^2\right] = \frac{1}{2}.$$

Lesson 33

Topics: Exponential Models (Growth & Decay)
Section Number: 7.2
Lecture Plan:

(1) Use the first 10 minutes to discuss some difficult problems from MyLabMath HW for Lesson 32. This should serve as a review of Lesson 32.
- Review of Lesson 32
- MyLabMath Homework for Lesson 32

(2) Population Growth
- Common Sense Observation

Population P	Birth Rate $\frac{dP}{dt}$ (babies per year)
1000	20
2000	40
$\cdots$	$\cdots$
5000	100

$$\frac{dP}{dt}/P = \frac{20}{1000} = \frac{40}{2000} = \frac{100}{5000} = k = \text{constant}$$

∘ Basic Differential Equation $\boxed{\frac{dP}{dt} = kP}$

∘ How to solve the differential equation

$$\frac{dP}{dt} = kP$$

$$\longrightarrow \quad \frac{1}{P}\frac{dP}{dt} = kP$$

$$\longrightarrow \quad \int \frac{1}{P}\frac{dP}{dt}\,dt = \int k\,dt$$

$$\int \frac{1}{P}\frac{dP}{dt}\,dt = \int \frac{1}{P}\,dP = \ln P \qquad \int k\,dt = kt + C$$

(actually $\ln|P|$ but $|P| = P$ since $P > 0$.)

$$\longrightarrow \quad \ln P = kt + C$$

$$\longrightarrow \quad e^{\ln P} = e^{kt+C}$$

$$\longrightarrow \quad \begin{aligned} P &= e^{kt+C} = e^{kt}e^{C} \\ &= Ae^{kt} \quad (A = e^{C}) \end{aligned}$$

That is to say, we have

$$P(t) = Ae^{kt}.$$

What is the constant A ?

$$P(0) = ae^{k\cdot 0} = A \cdot 1 = A$$

<u>Grand Final Conclusion</u>

$$\boxed{P(t) = P(0)e^{kt} = P_0e^{kt}}$$

where

$$P(0) = P_0 : \text{ initial population}$$

• Example Problem for the population growth:
Example Problem ①:
Stats

t	year	Population (billion)
0	1999	6.0
18	2017	7.4

Find the formula for $P(t)$.
Solution.
From the general formula, we have $P(t) = P(0)e^{kt} = 6.0e^{kt}$
What is k ?

$$\begin{aligned} P(18) &= P(0)e^{k\cdot 18} \\ 7.4 &= 6.0e^{k\cdot 18} \\ \frac{7.4}{6.0} &= e^{18k} \\ \ln\left(\frac{7.4}{6.0}\right) &= 18k \end{aligned}$$

$\longrightarrow$

$$k = \frac{1}{18}\ln\left(\frac{7.4}{6.0}\right) = 0.01165 \text{ billion/year}$$

<u>Conclusion</u>

$P(t) = 6.0e^{kt}$ with $k = 0.01165$ billion/year

(3) Radioactive Decay

<u>Situation</u>

m: mass
$\frac{dm}{dt}$: rate of decay

◦ Basic Differential Equation $\boxed{\frac{dm}{dt} = km}$
◦ the same process as before to solve the differential equation

<u>Grand Final Conclusion</u>

$$\boxed{m(t) = m(0)e^{kt} = m_0e^{kt}}$$

where

$$m(0) = m_0 : \text{ initial mass}$$

Note: The only difference between the population growth and the radioactive decay is:
Population Growth: $k > 0$
Radioactive Decay: $k < 0$
• Example Problem for the radioactive decay
Example problem ②: Radium 226 has the half-life of 1590 years.
That is to say,

yaer t	mass m
0	1
1590	$\frac{1}{2}$

Find the formula for $m(t)$, when the initial mass is $m(0) = 100$ mg.

Solution.

From the general formula, we have $m(t) = m(0)e^{kt} = 100e^{kt}$
What is k ?

$$\begin{cases} t = 0 & : \quad 100 \qquad = m(0) = 100e^{k\cdot 0} \\ t = 1590 & : \quad 100 \cdot \frac{1}{2} \quad = m(1590) = 100e^{k\cdot 1590} \end{cases}$$

$\longrightarrow$

$$\frac{1}{2} = e^{k\cdot 1590}$$

$\longrightarrow$

$$\begin{array}{ccc} \ln\left(\frac{1}{2}\right) & = & \ln\left(e^{k\cdot 1590}\right) \\ \| & & \| \\ -\ln 2 & & k\cdot 1590 \end{array}$$

$\longrightarrow$

$$k = -\frac{\ln 2}{1590}$$

Conclusion

We obtain the mathematician's formula

$$m(t) = 100 \cdot e^{\left(-\frac{\ln 2}{1590}\right)t}$$

Half-Life: Physicist's formula

- Simple Observation

t	m
0	100
$1590 \cdot 1$	$100 \cdot \left(\frac{1}{2}\right)^1$
$1590 \cdot 2$	$100 \cdot \left(\frac{1}{2}\right)^2$
$1590 \cdot 3$	$100 \cdot \left(\frac{1}{2}\right)^3$
$\cdots$	$\cdots$

$$\begin{aligned} m(t) &= 100 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1590}} \\ &= 100 \cdot \left(2^{-1}\right)^{\frac{t}{1590}} \\ &= 100 \cdot 2^{-\frac{t}{1590}} \end{aligned}$$

Grand General Formula in tems of Half-Life h

$$\boxed{m(t) = 100 \cdot 2^{-\frac{t}{1590}} = m(0) \cdot 2^{-\frac{t}{h}}}$$

Remark:
Mathematician's formula and Physicist's formula are the same !

$$\begin{aligned} m(t) &= m(0) \cdot e^{\left(-\frac{\ln 2}{1590}\right)t} \\ &\qquad \text{Mathematician's formula} \\ &= m(0) \cdot \left(e^{\ln 2}\right)^{-\frac{t}{1590}} \\ &= m(0) \cdot 2^{-\frac{t}{1590}} \\ &\qquad \text{Physicist's formula} \end{aligned}$$