Supplementary Problems

A For what value(s), if any, of A will $y = Axe^{-x}$ be a solution of the differential equation $2y' + 2y = e^{-x}$? For what value(s), if any, of B will $y = Be^{-x}$ be a solution ?

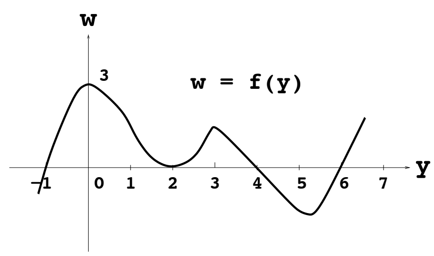
B Using the substitution $u(x) = y + x$, solve the differential equation $\frac{dy}{dx} = (y+x)^2$.

C Using the substitution $u(x) = y^3$, solve the differential equation $y^2 \frac{dy}{dx} + \frac{y^3}{x} = \frac{1}{x}$ $(x > 0)$.

D Use **dfield6** to plot the slope field for the differential equation $y' = 2y - 3e^{-t}$.
Plot the solution satisfying $y(0) = 1.001$. What happens to the solution as $t \longrightarrow \infty$?
Plot the solution satisfying $y(0) = 0.999$. What happens to this solution as $t \longrightarrow \infty$?

E Find the explicit solution of the Separable Equation $\frac{dy}{dt} = 4y - y^2$, $y(0) = 8$. What is the largest open interval containing $t = 0$ for which the solution is defined ?

F The graph of $f(y)$ vs y is as shown:



(a) Find the equilibrium solutions of the autonomous differential equation $\frac{dy}{dt} = f(y)$.

(b) Determine the stability of each equilibrium solution.

G Solve the differential equation $\frac{d\theta}{dr} = \frac{2r\theta}{\theta^2 - r^2}$.

H (a) If $y' = -2y + e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (**eul**) with step sizes of the form $h = \frac{1}{n}$ to find the smallest value of n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05 .

I (a) If $y' = 2y - 3\,e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (**eul**) with step sizes of the form $h = \frac{1}{n}$ to find the smallest value of n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05 .

J Consider the initial value problem $\begin{cases} y' = 2ty - y^2 \\ y(1) = 2.5 \end{cases}$. Using the the Euler, Improved Euler and Runge-Kutta Methods (**eul, rk2, rk4** respectively) with $h = 0.1$ to complete this table:

t_n	Euler y_n	Improved Euler y_n	Runge-Kutta y_n
1.0			
1.1			
1.2			
1.3			
1.4			
1.5			
1.6			

K Choosing smaller and smaller step sizes h does not guarantee better and better approximations even for a simple initial value problem like $\begin{cases} \dfrac{dy}{dt} = t\,(y-1) \\ y(-10) = 0 \end{cases}$.

(a) Verify that $y(t) = 1 - e^{\frac{(t^2-100)}{2}}$ is a solution of the above initial value problem.

(b) Approximate the actual solution at $t = 10$ (note that $y(10) = 0$) using the Runge-Kutta Method (**rk4**) with $h = 0.2, h = 0.1$ and $h = 0.05$ and fill in the table:

h	Runge-Kutta Approximation at $t = 10$	Actual Solution at $t = 10$
0.20		0.0000
0.10		0.0000
0.05		0.0000

L Approximation methods for differential equations can be used to estimate definite integrals:

(a) Show that $y(x) = \int_0^x e^{-t^2}\,dt$ satisfies the initial value problem $\dfrac{dy}{dx} = e^{-x^2}, \quad y(0) = 0$.

(b) Use the Runge-Kutta Method (**rk4**) with $h = 0.1$ to approximate $y(1.5)$, i.e., $\int_0^{1.5} e^{-t^2}\,dt$.

M To transform any 2^{nd} order linear differential equation $P(t)\,y'' + Q(t)\,y' + R(t)\,y = G(t)$ into an equivalent 1^{st} order linear system of equations

$$\begin{cases} x_1'(t) = a_{11}(t)\,x_1(t) + a_{12}(t)\,x_2(t) + g_1(t) \\ \\ x_2'(t) = a_{21}(t)\,x_1(t) + a_{22}(t)\,x_2(t) + g_2(t) \end{cases}$$

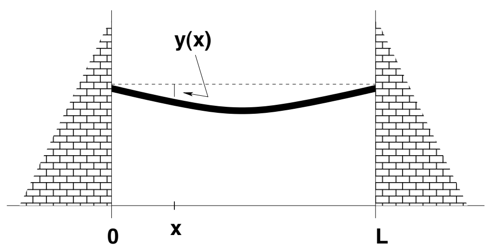
one can use the substitution $\boldsymbol{x_1(t) = y}$ and $\boldsymbol{x_2(t) = y'}$. Transform the initial value problem

$$2y'' + 3y' - t\,y = 3\,e^t,\; y(0) = 1,\; y'(0) = -4$$

into an equivalent system of 1^{st} order equations with initial conditions.

N If $y' = xy^2 - y^3$ and $y(1) = 2$, find $y''(1)$ and $y'''(1)$.

O From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement $y(x)$ satisfies the *boundary value problem* $\begin{cases} y'''' = -P \\ y(0) = y(L) = 0 \\ y'(0) = y'(L) = 0 \end{cases}$, where $P > 0$ is a constant depending on the beam's density and rigidity and L is the distance between supporting walls:



(a) Solve the above boundary value problem when $L = 4$ and $P = 24$.

(b) Show that the maximum displacement occurs at the center of the beam $x = \dfrac{4}{2} = 2$.

P Using Laplace Transforms, solve this *boundary value problem* : $\begin{cases} y'' + 4y = 16\,t \\ y(0) = 0 \\ y\left(\frac{\pi}{4}\right) = 0 \end{cases}$ $(*)$.

Hint : Solve the initial value problem $\begin{cases} y'' + 4y = 16\,t \\ y(0) = 0 \\ y'(0) = A \end{cases}$ and then determine A from $(*)$.

Q You can use Laplace transforms to find particular solutions to some nonhomogeneous differential equations. Use Laplace fransforms to find a particular solution, $y_p(t)$, of $y'' + 4\,y = 10\,e^t$.

Hint : Solve the initial value problem $\begin{cases} y'' + 4\,y = 10\,e^t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$.

(You will get a different particular solution if you use Undetermined Coefficients or Variation of Parameters.)

R Tank # 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank # 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank # 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank # 1 into Tank # 2 at the same rate of 5 gal/min. The solution in Tank # 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank # 1 and Tank # 2, respectively, SET UP BUT DO NOT SOLVE an initial value problem describing this system.

S If $\vec{\mathbf{x}}^{(1)}(t)$ and $\vec{\mathbf{x}}^{(2)}(t)$ are linearly independent solutions to the 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, then the matrix $\Phi(t) = \left(\vec{\mathbf{x}}^{(1)}(t),\, \vec{\mathbf{x}}^{(2)}(t)\right)$ is called a **Fundamental Matrix** for the system. Find a Fundamental Matrix $\Phi(t)$ of the system $\vec{\mathbf{x}}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \vec{\mathbf{x}}$.

T Find a particular solution $\vec{\mathbf{x}}_p(t)$ of these nonhomogeneous systems:

(a) $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 5\,e^{2t} \\ 3 \end{pmatrix}$

(b) $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ 4\,e^{t} \end{pmatrix}$

(c) $\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 10\,\cos t \\ 0 \end{pmatrix}$