MATH453M Homework Solutions Week4

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2.1.3 a False: if $a = 1 = c$ and $b = -2 = d$ then $ac = 1 < 4 = bd$. But the result is true if $a > b > 0$ and $c > d > 0$ since then $a > b$ and $c > 0$ imply $ac > bc$, and $c > d$ and $b > 0$ imply $bc > bd$ hence $ac > bd$.

2.1.3 b True. If $a = b$ then $ac = bc$ and if $a < b$ then $c > 0$ implies $ac < bc$. Therefore we must have $a > b$.

2.1.5 a $r^2 + s^2 - 2rs = (r - s)^2 \geqslant 0$ with equality when $r = s$.

2.1.5 b $r^2 + 2rs + 3s^2 = (r + s)^2 + 2s^2 \geqslant 0$ with equality when $r + s = s = 0$, in other words, when $r = s = 0$.

2.1.5 c $4(r^2 + rs + s^2) = (2r + s)^2 + 3s^2 \geqslant 0$ with equality when $2r + s = s = 0$, in other words, when $r = s = 0$.

2.1.8 False. In $\mathbb{Z}_5$ we have $\overline{1}^2 + \overline{2}^2 = \overline{5} = \overline{0}$.

2.1.10 Yes. Between rational numbers r and s we have the rational number $(r+ms)/(1+m)$ for any $m \in \mathbb{N}$.

2.1.16 The only such function is the identity map $f(x) = x$. *Proof:* First we note that $1 = f(1) = f(1 + 0) = f(1) + f(0) = 1 + f(0)$ so $f(0) = 0$. Therefore $0 = f(x + (-x)) = f(x) + f(-x)$ so $f(-x) = -f(x)$ for any $x \in \mathbb{Q}$. Now an induction proof shows that $f(nx) = nf(x)$ for any $n \in \mathbb{N}$ and $x \in \mathbb{Q}$ (true for $n = 1$ and if true for $n-1$ then $f(nx) = f((n-1)x) + f(x) = (n-1)f(x) + f(x) = nf(x)$). We therefore have $f(n) = n$ for all $n \in \mathbb{Z}$. Finally if $x = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ then $nf(x) = f(nx) = f(m) = m$ so $f(x) = m/n = x$.

2.2.3 Let $a = (x + y)/2$ and $b = \sqrt{xy}$ so $a, b \in \mathbb{R}^+$. Now

$$4(a^2 - b^2) = 4\left[\left(\frac{x+y}{2}\right)^2 - (\sqrt{xy})^2\right] = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2 \geqslant 0,$$

so $(a - b)(a + b) = a^2 - b^2 \geqslant 0$. But $a + b > 0$ so $a - b \geqslant 0$, hence $a \geqslant b$.

2.2.5 No. For example $\sqrt{2}$ and $-\sqrt{2}$ are both irrational, bur $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$ and $\sqrt{2}(-\sqrt{2}) = -4 \in \mathbb{Q}$.

2.2.6 a Suppose that $\sqrt{3} \in \mathbb{Q}$. Then $\sqrt{3} = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Now $3 = a^2/b^2$ so $3b^2 = a^2$ and hence $3 \mid a$, say $a = 3c$ with $c \in \mathbb{N}$. Therefore $3b^2 = 9c^2$ so $b^2 = 3c^2$, hence $3 \mid b$, which contradicts $\gcd(a, b) = 1$. Therefore $\sqrt{3} \notin \mathbb{Q}$.

2.2.6 b Suppose that $\sqrt[3]{2} \in \mathbb{Q}$. Then $\sqrt[3]{2} = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Now $2 = a^3/b^3$ so $2b^3 = a^3$ and hence a is even, say $a = 2c$ with $c \in \mathbb{N}$. Therefore $2b^3 = 8c^3$ so $b^3 = 4c^3$, hence b is even, which contradicts $\gcd(a, b) = 1$. Therefore $\sqrt[3]{2} \notin \mathbb{Q}$.

2.2.6 c Suppose that $x = \log_{10} 3 \in \mathbb{Q}$, so $10^x = 3$. Then $x = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Now $10^{a/b} = 3$ so $10^a = 3^b$, but this is impossible, since 10^a is even and 3^b is odd. Therefore $\log_{10} 3 \notin \mathbb{Q}$.