MATH453M Homework Solutions Week 6

Nick Inglis
ninglis@math.purdue.edu

2.5.7 a $f(z) - (2+3i) = e^{i\pi/3}\big(z - (2+3i)\big) = \left(\frac{1+i\sqrt{3}}{2}\right)\big(z-(2+3i)\big)$ so

$$f(z) = \left(\frac{1+i\sqrt{3}}{2}\right)z + \left(\frac{1-i\sqrt{3}}{2}\right)(2+3i) = \left(\frac{1+i\sqrt{3}}{2}\right)z + \frac{2+3\sqrt{3}+(3-2\sqrt{3})i}{2}.$$

2.5.7 c Reflection in the line $z + \overline{z} = 0$ is $z \mapsto -\overline{z}$. Therefore $f(z) - 1 = -\overline{(z-1)} + 3i$ and hence $f(z) = -\overline{z} + 2 + 3i$.

2.5.10 a This is a rotation by $3\pi/2$ about $2/(1+i) = 1-i$.

2.5.10 b This is a glide reflection in the real axis with translation 1.

2.5.10 c This is a glide reflection in the line through $z = 1/2$ at an angle of $\pi/4$ with the real axis and with translation $(1+i)/2$.

3.1.1 c

$$\begin{array}{r|rrrrrrr}
 & & & x^4 & +x^3 & +x^2 & +x & -2 \\ \hline
x^2+2x-3 & x^6 & +3x^5 & +0x^4 & +0x^3 & -3x^2 & -3x & +2 \\
 & x^6 & +2x^5 & -3x^4 & & & & \\ \cline{2-4}
 & & x^5 & +3x^4 & +0x^3 & & & \\
 & & x^5 & +2x^4 & -3x^3 & & & \\ \cline{3-5}
 & & & x^4 & +3x^3 & -3x^2 & & \\
 & & & x^4 & +2x^3 & -3x^2 & & \\ \cline{4-6}
 & & & & x^3 & +0x^2 & -3x & \\
 & & & & x^3 & +2x^2 & -3x & \\ \cline{5-7}
 & & & & & -2x^2 & +0x & +2 \\
 & & & & & -2x^2 & +3x & -1 \\ \cline{6-8}
 & & & & & & -3x & +3
\end{array}$$

3.1.2 a $x^4 + x^3 - x^2 - 2x - 2 = (x+1)(x^3-1) - (x^2+x+1)$ and $x^3 - 1 = (x-1)(x^2+x+1)$ so $\gcd(f,g) = x^2+x+1 = (x+1)f(x) - g(x)$.

3.1.2 e $x^2+1 = x^2+2x+2-(2x+1)$ and $x^2+2x+2 = (x+3/2)(x+1/2)+5/4$. Therefore

$$\begin{aligned}
\gcd(f,g) = 1 &= \frac{4}{5}f(x) - \frac{4}{5}\left(x+\frac{3}{2}\right)\left(x+\frac{1}{2}\right) \\
&= \frac{4}{5}f(x) - \frac{4}{5}\left(x+\frac{3}{2}\right)\frac{1}{2}[f(x)-g(x)] \\
&= \frac{2x+3}{5}g(x) - \frac{2x-1}{5}g(x).
\end{aligned}$$

3.1.3 b $x^4-1 = (x^2=1)(x-1)(x+1)$ and $1 = \frac{1}{2}(x^2+1) - \frac{1}{2}(x^2-1) = \frac{1}{2}(x=1) - \frac{1}{2}(x-1)$ so

$$\begin{aligned}
\frac{2x^3+x^2+2x-1}{x^4-1} &= \frac{1}{2}\frac{2x^3+x^2+2x-1}{x^2-1} - \frac{1}{2}\frac{2x^3+x^2+2x-1}{x^2=1} \\
&= x+\frac{1}{2}+\frac{2x}{x^2-1} - x - \frac{1}{2} + \frac{2}{x^2+1} = \frac{x}{x-1} - \frac{x}{x=1} - \frac{2}{x^2} \\
&= 1+\frac{1}{x-1} - 1 + \frac{1}{x+1} + \frac{2}{x^2+1} = \frac{1}{x-1} + \frac{1}{x+1} + \frac{2}{x^2+1}
\end{aligned}$$

3.1.3 c $x^3-x^2 = x^2(x-1)$ and $1 = x^2-(x+1)(x-1)$ so

$$\begin{aligned}
\frac{7x^2+x-3}{x^3-x^2} &= \frac{7x^2+x-3}{x-1} - \frac{(7x^2+x-3)(x+1)}{x^2} \\
&= 7x+8+\frac{5}{x-1} - 7x - 8 + \frac{2x+3}{x^2} = \frac{5}{x-1} + \frac{2}{x} + \frac{3}{x^2}
\end{aligned}$$

3.1.6 Proof by induction on n. if $n=0$ then f is a non-zero constant and has no root, so the result is true. Now assume that $n>0$ and that the result is true for polynomials of degree less than n. If f has no root, then the result holds, otherwise let a be a root of f, so that by the Remainder Theorem $f(x) = (x-a)g(x)$ for some polynomial $g(x)$. Now for any root $b \neq a$ of f, we have $0 = f(b) = (b-a)g(b)$ so $g(b) = 0$. By the inductive hypothesis g has at most $n-1$ roots, so f has at most $n-1$ roots different from a and hence f has at most n roots.

3.1.9 **Apologies!** The first two parts of this question don't really work: (a) doesn't really demonstrate the failure of unique factorisation and (b) is much too difficult and probably false. In (c), f has roots 2 and $5=-1$.

3.1.10 a $f(x) = (x-2)(x+2)$ is not irreducible.

3.1.10 b $f(x)$ is irreducible since -1 is not a square in $\mathbb{Z}_7$.

3.1.10 d $4^3 = 64 = -2 = 9$ so $f(4) = 0$ hence $f(x)$ is not irreducible.

3.1.10 f $f(x) = (x^2+x+1)^2$ is not irreducible.