***************** Disclaimer *****************

This represents a very brief outline of most of the topics covered MA261

****

I. Vectors, Lines and Planes

1. Vector arithmetic; directed vector $\overline{P_0P_1}$ from P_0 to P_1; dot product of vectors $(a_1\vec{\mathbf{i}}+a_2\vec{\mathbf{j}}+a_3\vec{\mathbf{k}})\cdot(b_1\vec{\mathbf{i}}+b_2\vec{\mathbf{j}}+b_3\vec{\mathbf{k}}) = a_1b_1+a_2b_2+a_3b_3$; angle between two vectors, $\cos\theta = \dfrac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{||\vec{\mathbf{a}}||\,||\vec{\mathbf{b}}||}$; cross product $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and their properties: $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, $\frac{1}{2}||\vec{\mathbf{a}}\times\vec{\mathbf{b}}|| =$ area of triangle spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$; projections $\boldsymbol{pr}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \left(\dfrac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{||\vec{\mathbf{a}}||^2}\right)\vec{\mathbf{a}}$; $\vec{\mathbf{v}} = ||\vec{\mathbf{v}}||(\cos\theta\,\vec{\mathbf{i}}+\sin\theta\,\vec{\mathbf{j}})$.

2. Equation of line containing (x_0, y_0, z_0), direction vector $\vec{\mathbf{L}} = a\vec{\mathbf{i}}+b\vec{\mathbf{j}}+c\vec{\mathbf{k}}$:

   (a) Vector Form: $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\,\vec{\mathbf{L}}$, where $\vec{\mathbf{r}}_0 = x_0\vec{\mathbf{i}}+y_0\vec{\mathbf{j}}+z_0\vec{\mathbf{k}}$

   (b) Parametric Form: $\begin{cases} x = x_0 + a\,t \\ y = y_0 + b\,t \\ z = z_0 + c\,t \end{cases}$

   (c) Symmetric Form: $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$
   (if say $b = 0$, then $\frac{x-x_0}{a} = \frac{z-z_0}{c}$; $y = y_0$)

3. Equation of plane containing (x_0, y_0, z_0), normal vector $\vec{\mathbf{N}} = a\vec{\mathbf{i}}+b\vec{\mathbf{j}}+c\vec{\mathbf{k}}$:

$$\vec{\mathbf{N}}\cdot\overline{\mathbf{P_0P}} = 0 \quad \text{or} \quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0\,.$$

4. Sketching planes (look at intercepts : $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$).

II. Vector-Valued Functions

1. Differentiating and integrating vector-valued functions and sketching the corresponding curves.

2. Parameterizing curves of the form say $y = f(x)$, $a \le x \le b$
   ($C:\ \vec{\mathbf{r}}(t) = t\,\vec{\mathbf{i}} + f(t)\,\vec{\mathbf{j}}$, $a \le t \le b$).

3. Unit tangent vector $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{||\vec{\mathbf{r}}\,'(t)||}$; length of a curve $\displaystyle\int_a^b ||\vec{\mathbf{r}}\,'(t)||\,dt$.

III. Partial Derivatives

1. Domains of functions of several variables; level curves $f(x,y)=C$, level surfaces $f(x,y,z)=C$; sketching surfaces using level curves.

2. Quadric surfaces.

3. Computing limits, determining when limits exist.

4. Partial derivatives; CHAIN RULE (consider tree diagrams).

5. Implicit Differentiation, for example :

   (a) If $y=y(x)$ is defined implicitly by $F(x,y)=0$, then $\dfrac{dy}{dx}=-\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

   (b) If $z=z(x,y)$ is defined implicitly by $F(x,y,z)=0$, then

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\,.$$

6. Gradients: $\nabla f(x,y,z)=f_x\,\vec{\mathbf{i}}+f_y\,\vec{\mathbf{j}}+f_z\,\vec{\mathbf{k}}$; the gradient $\nabla f(x,y)$ is perpendicular to level curve $f(x,y)=C$ and $\nabla f(x,y,z)$ is perpendicular to level surface $f(x,y,z)=C$.

7. Directional derivative : $D_{\vec{\mathbf{u}}}f(x,y,z)=\nabla f(x,y,z)\cdot\vec{\mathbf{u}}$, where $\vec{\mathbf{u}}$ is a UNIT vector; $-||\nabla f||\le D_{\vec{\mathbf{u}}}f\le||\nabla f||$;
   $f(x,y,z)$ increases fastest in the direction ∇f.

8. Normal vector $\vec{\mathbf{n}}$ to surfaces $\sum$:

   (a) $\sum$ is a level surface, $F(x,y,z)=C$, then a normal is $\vec{\mathbf{n}}=\nabla F(x,y,z)$.

   (b) $\sum$ is the graph of $z=f(x,y)$, then a normal is $\vec{\mathbf{n}}=-f_x\,\vec{\mathbf{i}}-f_y\,\vec{\mathbf{j}}+\vec{\mathbf{k}}$

9. Tangent planes to surfaces; Tangent Plane Approximation Formula:

$$f(x+h,y+k)\approx f(x,y)+f_x(x,y)\,h+f_y(x,y)\,k\,.$$

10. Critical points of $f(x,y,z)$: points where $\nabla f(x,y,z)=\vec{\mathbf{0}}$ or $\nabla f(x,y,z)$ does not exist.

11. Finding relative extrema of $f(x,y)$ at those particular critical points (x_0,y_0) where $\nabla f(x_0,y_0)=\vec{\mathbf{0}}$ using 2^{nd} Partials Test: let $D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

    (a) If $D(x_0,y_0)>0$ and $f_{xx}(x_0,y_0)>0 \Rightarrow f$ has rel minimum value at (x_0,y_0)

    (b) If $D(x_0,y_0)>0$ and $f_{xx}(x_0,y_0)<0 \Rightarrow f$ has rel maximum value at (x_0,y_0)

    (c) If $D(x_0,y_0)<0 \Rightarrow f$ has a saddle point at (x_0,y_0).

12. Finding absolute extrema over closed, bounded regions: find interior critical points, find points on the boundary where extrema may occur, make a table of values of f at all these points.

13. Constrained extremal problems: Maximize and/or minimize $f(x,y)$ subject to the condition $g(x,y)=C$; Lagrange Multipliers: $\begin{cases} \nabla f=\lambda \nabla g \\ g(x,y)=C \end{cases}$

IV. <u>Multiple Integrals</u>

1. Double integrals; vertically and horizontally simple regions, iterated integrals; double integrals in polar coordinates $(dA = r\,dr\,d\theta)$

2. Applications of double integrals: areas between curves, volumes, surface area $S=\int\int_R \sqrt{f_x^2+f_y^2+1}\,dA$.

3. Changing the order of integration in double integrals.

4. Triple integrals; iterated triple integrals; applications of triple integrals: volumes, mass $m=\int\int\int_D \delta(x,y,z)\,dV$.

5. Triple integrals in Rectangular, Cylindrical, and Spherical Coordinates:

    (a) Rectangular Coordinates: $dV = dz\,dy\,dx$ or $dV = dz\,dx\,dy$, etc

    (b) Cylindrical Coordinates: $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$ $dV = r\,dz\,dr\,d\theta$

    (c) Spherical Coordinates: $\begin{cases} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{cases}$ $dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

V. Vector Fields

1. Vector fields $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$; divergence and curl of a vector field $\vec{\mathbf{F}}$:

$$\text{div}\,\vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = M_x + N_y + P_z$$

$$\text{curl}\,\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix};$$

   Laplacian of $f = \text{div}\nabla f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$.

2. Conservative vector fields $\vec{\mathbf{F}} = \nabla f$; how to determine if $\vec{\mathbf{F}}$ is conservative : check that $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$ (if region has no "holes"); given that $\vec{\mathbf{F}} = \nabla f$, know how to determine the potential function $f(x, y, z)$.

3. Line integrals of functions $\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\,||\vec{\mathbf{r}}\,'(t)||\,dt$; line integrals of vector fields $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt$$

   or equivalently $\int_C M\,dx + N\,dy + P\,dz = \int_a^b Mx'dt + Ny'dt + Pz'dt$, where C : $\vec{\mathbf{r}}(t) = x(t)\,\vec{\mathbf{i}} + y(t)\,\vec{\mathbf{j}} + z(t)\,\vec{\mathbf{k}}$, $a \le t \le b$.

4. Fundamental Theorem of Line Integrals: $\int_C \nabla f \cdot \vec{\mathbf{dr}} = f(P_1) - f(P_0)$; independence of path (check if $\vec{\mathbf{F}} = \nabla f$ or $\text{curl}\,\vec{\mathbf{F}} = \vec{\mathbf{0}}$) ; applications to work $W = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}}$.

5. Green's Theorem : If C is a closed curve traversed counterclockwise, then

$$\int_C M(x, y)\,dx + N(x, y)\,dy = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\,.$$

6. Surface integrals : if $\sum$ is the graph of $z = f(x, y)$ with $(x, y) \in R$, then $\int\int_{\sum} g(x, y, z)\,dS = \int\int_R g(x, y, f(x, y))\sqrt{f_x^2 + f_y^2 + 1}\;dA\,.$

7. Flux integral of $\vec{\mathbf{F}} = M\,\vec{\mathbf{i}} + N\,\vec{\mathbf{j}} + P\,\vec{\mathbf{k}}$ over the surface $\sum$, the graph of $z = f(x, y)$ with $(x, y) \in R$, and $\vec{\mathbf{n}} =$ upper unit normal vector to $\sum$:

$$\int\int_{\sum} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\; dS = \int\int_{R} \{-M\, f_x - N\, f_y + P\}\; dA\,.$$

8. <u>Divergence Theorem (Gauss' Theorem)</u> : If D is a solid region and $\sum$ is its closed boundary surface, $\vec{\mathbf{n}} =$ outer unit normal to $\sum$, then

$$\int\int_{\sum} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\; dS = \int\int\int_{D} \operatorname{div} \vec{\mathbf{F}}\; dV\,.$$