MA 224 FORMULAS

THE SECOND DERIVATIVE TEST

Suppose f is a function of two variables x and y, and that all the second-order partial derivatives are continuous. Let

$$D = f_{xx}f_{yy} - (f_{xy})^2$$

and suppose (a, b) is a critical point of f.

1. If $D(a, b) < 0$, then f has a saddle point at (a, b),
2. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b).
3. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b).
4. If $D(a, b) = 0$, the test is inconclusive.

LEAST-SQUARES LINE

The equation of the least-squares line for the n points $(\mathrm{x}_1,\mathrm{y}_1)$, $(\mathrm{x}_2,\mathrm{y}_2)$, ... , $(\mathrm{x}_n,\mathrm{y}_n)$, is $y = mx + b$, where m and b are solutions to the system of equations

$$\begin{aligned}(x_1^2 + x_2^2 + \cdots + x_n^2)m + (x_1 + x_2 + \cdots + x_n)b &= x_1y_1 + x_2y_2 + \cdots + x_ny_n\\ (x_1 + x_2 + \cdots + x_n)m + nb &= y_1 + y_2 + \cdots + y_n\end{aligned}$$

TRAPEZOIDAL RULE

$$\int_a^b f(x)dx \equiv \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big],$$

where $a = x_0, x_1, x_2, \ldots, x_n = b$ subdivides $[a, b]$ into n equal subintervals of length $\Delta x = \dfrac{b-a}{n}$.

ERROR ESTIMATE FOR THE TRAPEZOIDAL RULE

If M is the maximum value of $|f''(x)|$ on the interval $a \le x \le b$, then

$$|E_n| \le \frac{M(b-a)^3}{12n^2}$$

GEOMETRIC SERIES

If $0 < |r| < 1$, then

$$\sum_{n=0}^{\infty} r^n \;=\; \frac{1}{1-r}.$$

TAYLOR SERIES

The Taylor series of $f(x)$ about $x = a$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots$$

Examples: (with $a = 0$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ for } -\infty < x < \infty; \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,x^n, \text{ for } -1 < x < 1.$$

Table of Integrals

Forms Involving $a + bu$

1. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}[a + bu - a\ln|a+bu|] + C$

2. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|] + C$

3. $\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$

4. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu - 2a)(a+bu)^{3/2} + C$

5. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a+bu} + C$

6. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C \qquad (\text{if } a > 0)$

Forms Involving $\sqrt{a^2+u^2}$

7. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln|u + \sqrt{a^2+u^2}| + C$

8. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln|u + \sqrt{a^2+u^2}| + C$

9. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln|u + \sqrt{a^2+u^2}| + C$

10. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

11. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

12. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Forms Involving $\sqrt{u^2-a^2}$

13. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u + \sqrt{u^2-a^2}| + C$

14. $\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u + \sqrt{u^2-a^2}| + C$

15. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln|u + \sqrt{u^2-a^2}| + C$

16. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u + \sqrt{u^2-a^2}| + C$

17. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

18. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Forms Involving $\sqrt{a^2-u^2}$

19. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

20. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

21. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

22. $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

Forms Involving e^{au} and $\ln u$

23. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$

24. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

25. $\int \frac{du}{1+be^{au}} = u - \frac{1}{a}\ln(1+be^{au}) + C$

26. $\int \ln u\,du = u\ln u - u + C$

27. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C \qquad (n \neq -1)$

28. $\int \frac{du}{u\ln u} = \ln|\ln u| + C$

29. $\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$