MATH 162 – FALL 2006 – THIRD EXAM, NOVEMBER 16, 2006
SOLUTIONS

1) (10 points) Which of the following series converge?

$$S_1 = \sum_{n=1}^{\infty} \frac{n^3 + 9n^2}{300n^4 + 3n}, \quad S_2 = \sum_{n=1}^{\infty} \frac{8n^2 + 7n}{n^4 + 9n^3}, \quad S_3 = \sum_{n=1}^{\infty} \frac{8n^6 + 7n}{600n^5 + 200n^3}$$

A) Only S_1

B) S_1, S_2 and S_3

C) S_1 and S_2

D) S_2 and S_3

E) Only S_2

Solution: It is easy to see that the third series diverges because

$$\lim_{n\to\infty} \frac{8n^6 + 7n}{600n^5 + 200n^3} = \infty.$$

To analyze the convergence of the other two, we use the limit comparison theorem. Notice that

$$\lim_{n\to\infty} \frac{\frac{n^3+9n^2}{300n^4+3n}}{\frac{1}{n}} = \frac{1}{300} \text{ and } \lim_{n\to\infty} \frac{\frac{8n^2+7n}{n^4+9n^3}}{\frac{1}{n^2}} = 8$$

Since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, we find that S_1 diverges and S_2 converges. So S_2 is the only convergent series. Correct answer: E

2) (9 points) Which of the following is true about the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$?

I) It converges by the integral test

II) It converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

III) It diverges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

A) I, II and III are true

B) Only I and II are true

C) Only II is true

D) Only I is true

E) Only III is true

Solution: Notice that, substituting $u = \ln x$,

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^\infty \frac{du}{u^2} = \frac{1}{\ln 2}.$$

So the integral test says that $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges. Although it is true that $\frac{1}{n(\ln n)^2} < \frac{1}{n}$, the comparison test cannot be used to decide the convergence of this series. Correct answer: D

3) (9 points) Let $f(x)$ be a function defined for $x \geq 1$, such that $0 \leq f(x) \leq 1$, for all $x \geq 1$. What can be said about the series

$$S_1 = \sum_{n=1}^\infty \frac{f(n)}{n}, \quad S_2 = \sum_{n=1}^\infty \frac{f(n)}{n^2} \quad ?$$

A) S_1 and S_2 converge

B) S_1 diverges and S_2 converges

C) S_1 converges and S_2 diverges

D) S_1 and S_2 diverge

E) S_2 converges, but S_1 might converge or diverge.

Solution: Since $f(n) \leq 1$, it follows that

$$\frac{f(n)}{n^2} \leq \frac{1}{n^2}.$$

So S_2 converges. However S_1 may converge or not. For example if $f(n) = 1$, S_1 diverges. On the other hand, if $f(n) = \frac{1}{n}$, S_1 converges. Correct answer: E

4) (9 points) Using that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and the limit comparison theorem, the following is true about the series

$$S_1 = \sum_{n=1}^\infty \sin\left(\frac{1}{n}\right), \quad S_2 = \sum_{n=1}^\infty \sin\left(\frac{1}{n^2}\right):$$

A) S_1 and S_2 converge

B) S_1 diverges and S_2 converges

C) S_1 converges and S_2 diverges

D) S_1 and S_2 diverge

E) S_2 converges, but S_1 might converge or diverge.

Solution: Using the limit above we find that

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1, \quad \text{and} \quad \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1.$$

Since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, we deduce from the limit comparison theorem that S_1 diverges and S_2 converges. Correct answer: B

5)(9 points) Find the smallest number of terms which one needs to add to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 n!}$ with an error strictly less than 10^{-3}.

A) 2 terms

B) 3 terms

C) 4 terms

D) 5 terms

E) 11 terms

Solution: This is an alternating series. We know that if $S = \sum_{n=1}^{\infty}(-1)^n b_n$, and $S_N = \sum_{n=1}^{N}(-1)^n b_n$, where $b_n \geq 0$, $b_{n+1} \leq b_n$, and $\lim_{n\to\infty} b_n = 0$, then

$$|S - S_N| \leq b_{N+1}.$$

In our case $b_n = \frac{1}{n^3 n!}$ so it satisfies the three assumptions about b_n stated above. Since we want the error to be strictly less than 10^{-3}, we impose that $b_{N+1} < 10^{-3}$. In this case,

$$\frac{1}{(N+1)^3(N+1)!} < 10^{-3}.$$

So we must have

$$(N+1)^3(N+1)! > 1000.$$

The first value of N for which this is true is $N = 3$. So we need at least 3 terms. Correct answer B.

6)(9 points) Each of the following series converge

$$S_1 = \sum_{n=1}^{\infty}(-1)^n \frac{2}{3n+1}, \quad S_2 = \sum_{n=1}^{\infty}(-1)^n \frac{n}{2^n}, \quad S_3 = \sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n\sqrt{n}}.$$

Which ones converge absolutely?

A) S_1, S_2 and S_3

B) S_1 and S_2

C) S_1 and S_3

D) S_2 and S_3

E) Only S_2

Solution: The first series does not converge absolutely by comparison with $\sum \frac{1}{n}$ because

$$\left|(-1)^n \frac{2}{3n+1}\right| = \frac{2}{3n+1} > \frac{2}{4n} > \frac{1}{2n}.$$

The third series converges absolutely by comparison with $\sum \frac{1}{n^{\frac{3}{2}}}$ because

$$\left|(-1)^n \frac{\sin n}{n\sqrt{n}}\right| = \frac{\sin n}{n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}.$$

The second series converges absolutely by the ratio test,

$$\lim_{n\to\infty} \frac{n+1}{2^{n+1}} \; \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1.$$

Correct answer: D

7)(9 points) Which of the following is the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2(x-2)^n}{3^n(n^3+2)} \; ?$$

A) $(0,6)$

B) $[0,6)$

C) $(-1,5]$

D) $[-1,5)$

E) $(0, 6]$

Solution: First we use the ratio test to find the radius of convergence

$$\lim_{n\to\infty}\left|\frac{(n+1)^2(x-2)^{n+1}}{3^{n+1}((n+1)^3+2)}\,\frac{3^n(n^3+2)}{n^2(x-2)^n}\right| = \lim_{n\to\infty}\frac{(n+1)^2(n^3+2)}{n^2((n+1)^3+2)}\,\frac{|x-2|}{3} = \frac{|x-2|}{3}.$$

So the series converges when $\frac{|x-2|}{3} < 1$, which is the same as $-1 < x < 5$. Next we test the convergence of the series at the end points $x = -1$ and $x = 5$. When $x = -1$, the series is

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2(-3)^n}{3^n(n^3+2)} = \sum_{n=1}^{\infty}\frac{n^2}{n^3+2},$$

which diverges by limit comparison with $\sum\frac{1}{n}$. When $x = 5$, the series is

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^2(3)^n}{3^n(n^3+2)} = \sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+2}.$$

This is an alternating series. If $b_n = \frac{n^2}{n^3+2}$, we see that b_n satisfies: $b_n \geq 0$, $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. So it converges conditionally by the alternating series test. Correct answer: C.

8) (9 points) Let $f(x)$ be the function which is represented by the power series

$$f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{n^3}.$$

The fifth derivative of the function f at $x = 1$ is equal to

A) $\frac{1}{2}$

B) $-\frac{37}{81}$

C) $-\frac{24}{25}$

D) $\frac{25}{96}$

E) $\frac{1}{4}$

Solution: We know that if a function $f(x)$ is represented by a power series

$$f(x) = \sum_{n=0}^{\infty}c_n(x-a)^n, \quad \text{if} \quad |x-a| < R,$$

then $c_n = \frac{f^{(n)}(a)}{n!}$. So in this case

$$\frac{f^{(5)}(1)}{5!} = (-1)^5 \frac{1}{5^3}.$$

Therefore $f^{(5)}(1) = \frac{-5!}{5^3} = \frac{-24}{25}$. Correct asnwer: C

9) (9 points) The coefficient of the x^4 term of the binomial series of $f(x) = \sqrt{1+x}$ is

A) $\frac{1}{57}$

B) $-\frac{75}{128}$

C) $-\frac{5}{128}$

D) $\frac{8}{57}$

E) $\frac{9}{77}$

Solution: The binomial series is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + ...$$

In this case $k = \frac{1}{2}$. So the coefficient of x^4 is

$$c_4 = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!} = \frac{-5}{128}.$$

Correct answer: C

10)(9 points) The Taylor series of $f(x) = \frac{1}{5-x}$ centered $a = 1$ is

A) $\sum_{n=0}^{\infty}(x-1)^n$

B) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$

C) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{5^n}$

D) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^{n+1}}$

E) $\sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^n}{4^{n+1}}$

Solution: First write

$$\frac{1}{5-x} = \frac{1}{4-(x-1)} = \frac{1}{4} \frac{1}{1-\frac{(x-1)}{4}}.$$

By the formula for the geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

we get that

$$\frac{1}{1-\frac{(x-1)}{4}} = \sum_{n=0}^{\infty} \left(\frac{x-1}{4}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$$

So

$$\frac{1}{5-x} = \frac{1}{4-(x-1)} = \frac{1}{4} \frac{1}{1-\frac{(x-1)}{4}} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{4^{n+1}}.$$

Correct answer: D.

11)(9 points) Recall that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty.$$

Using this, we find that Maclaurin series of

$$\int x^2 \sin x \, dx \quad \text{is}$$

A) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

B) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}$

C) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3)(2n+1)!}$

D) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)!}$

E) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)!(2n+1)!}$

Solution: From the given formula we have:

$$x^2 \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}, \quad -\infty < x < \infty.$$

To find $\int x^2 \sin x \, dx$ we only need to integrate the series term by term. We obtain

$$\int x^2 \sin x \, dx = \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n+3}}{(2n+1)!} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)!}.$$

Correct answer: D.