Solution for HWs (§1.3)

#1.3.2 (a) $s \in R$ is the infimum for $A \subset R$ if

(1) s is a lower bound for A; (2) if b is any lower bound for A, then $s \geq b$.

(b) Assume $s \in R$ is a lower bound for $A \subset R$. Then

$$s = \inf A \iff \forall \varepsilon > 0, \exists a \in A \text{ s.t. } s > a - \varepsilon.$$

Proof "$\Rightarrow$" Let $s = \inf A$. Assume that $\exists \varepsilon_0 > 0$, s.t. $s + \varepsilon_0 \leq a \quad \forall a \in A$

$\Rightarrow s + \varepsilon_0$ is a lower bound for $A \Rightarrow s$ is not the infimum $\Rightarrow$ contradiction

"$\Leftarrow$" Let s be a lower bound, if $b > s \Rightarrow b = s + (b - s) \equiv s + \varepsilon > a$

for some $a \in A \Rightarrow b$ is not a lower bound $\Rightarrow s = \inf A$.

#1.3.3 (a) Let b be any lower bound for $A \Rightarrow b \in B$

Since $\sup B \geq b$ for any $b \in B$, it's then sufficient to prove $\sup B$ is a lower bound for A.

Assume that it is not true, i.e., $\sup B$ is not a l.b. for $A \Rightarrow \exists a \in A$ s.t. $a > \sup B$.

$\Rightarrow \exists \varepsilon_0 > 0$ s.t. $a > \sup B - \varepsilon_0 \Rightarrow \sup B - \varepsilon_0$ is not a l.b. for A

On the other hand, $\exists b \in B$ s.t. $b > \sup B - \varepsilon_0 \Rightarrow \sup B - \varepsilon_0$ is a l.b. $\Rightarrow$ contradiction.

(b) A is bounded below.

Assume that $\inf A$ does not exist $\iff \sup B$ d.n.e.

But B is bounded above $\Rightarrow \sup B$ exists $\Rightarrow$ contradiction.

(c) A is bounded below ($\exists\ b \in R$ s.t. $b \leq a \quad \forall a \in A$)

Let $D = \{-a \mid \forall a \in A\}$, then $-b \geq -a \quad \forall a \in A \Rightarrow -b$ is an u.b. for D

$\Rightarrow D$ is bounded above $\xRightarrow{AoC} s = \sup D$ exists $\Rightarrow -s$ is a l.b. for A.

Now, use proof by contradiction to show that $-s$ is the largest l.b. for A.

Assume it is not, then $\exists\ t \in R$, s.t., t is a l.b. for A and $t > -s$.

$\Rightarrow -t$ is an u.b. for D and $-t < s$

This contradicts with $s = \sup D$.