Solution for HWs (§1.4b)

#1.4.7 Proof Let $f: N \to B$ be 1–1 and onto (assume that B is infinite)

$A \subset B$ and f is onto $\Rightarrow \{n \in N \mid f(n) \in A\}$ is nonempty $\Rightarrow n_1 = \min\{n \in N \mid f(n) \in A\}$

A is infinite $\Rightarrow \{n \in N \mid f(n) \in A \setminus \{f(n_1)\}\}$ is nonempty $\Rightarrow n_2 = \min\{n \in N \mid f(n) \in A \setminus \{f(n_1)\}\}$

similarly, let $n_k = \min\{n \in N \mid f(n) \in A \setminus \{f(n_1), \cdots, f(n_{k-1})\}\}$

define $n_{k+1} = \min\{n \in N \mid f(n) \in A \setminus \{f(n_1), \cdots, f(n_k)\}\}$

$\Rightarrow A = \{f(n_1), f(n_2), \cdots, f(n_k), \cdots\}$

$\Rightarrow g: N \to A$ is 1-1 and onto $\Rightarrow A$ is countable.

#1.4.8 Proof (a) Assume that A and B are countable sets. Then

$A \cup B = A \cup \tilde{B}$ where $\tilde{B} = B \setminus A$ and $A \cap \tilde{B} = \emptyset$.

Let $A = \{a_1, a_2, \cdots\}$ and $\tilde{B} = \{\tilde{b}_1, \tilde{b}_2, \cdots\}$, then

$$\begin{array}{ccccc} a_1, & \tilde{b}_1, & a_2, & \tilde{b}_2, & \cdots \\ 1, & 2, & 3, & 4, & \cdots \end{array}$$

defines a 1-1 and onto function $\Rightarrow A \cup B$ is countable

This implies $A_1 \cup A_2$ is countable. By induction, $\bigcup_{n=1}^{m} A_m$ is countable.

Assume that $\tilde{A} = \bigcup_{n=1}^{k} A_n$ is countable, then $\bigcup_{n=1}^{k+1} A_n = \left(\bigcup_{n=1}^{k} A_n\right) \cup A_{k+1} \equiv \tilde{A} \cup A_{k+1}$

is countable because $\tilde{A}$ and A_{k+1} are countable.

(c) $E = \bigcup_{n=1}^{\infty} A_n$ Assume that $A_i \cap A_j = \emptyset$ and A_i is countable.

$$\begin{array}{lllll} A_1: & a_{11} & a_{12} & a_{13} & \cdots \\ A_2: & a_{21} & a_{22} & a_{23} & \cdots \\ A_3: & a_{31} & a_{32} & a_{33} & \cdots \\ & \vdots & \vdots & \vdots & \end{array}
\qquad
\begin{array}{cccc} 1 & 3 & 6 & 10 \\ 2 & 5 & 9 & \\ 4 & 8 & & \\ 7 & & & \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

#1.4.9 (a) $A \sim B \Rightarrow \exists f : A \to B$ is 1-1 and onto

$\forall x \in B$, $\exists y \in A$, s.t., $x = f(y)$ (since f is onto)

define $f^{-1}(x) = y$ (since f is 1-1)

$\Rightarrow f^{-1} : B \to A$ is 1-1 and onto $\Rightarrow B \sim A$.

(b) $A \sim B \Rightarrow \exists f : A \to B$ is 1-1 and onto

$B \sim C \Rightarrow \exists g : B \to C$ is 1-1 and onto

Define $h : A \to C$ by $h(x) = g(f(x))$ $\forall x \in A$.

For $x_1, x_2 \in A$ and $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ (f is 1-1)

and $g(f(x_1)) \neq g(f(x_2))$ (g is 1-1) $\Longrightarrow h(x_1) \neq h(x_2)$

$\Longrightarrow h(x)$ is 1-1.

$\forall y \in C$, g is onto $\Rightarrow \exists z \in B$ s.t. $y = g(z)$

for this $z \in B$, f is onto $\Rightarrow \exists x \in A$ s.t. $z = f(x)$

$\Rightarrow y = g(f(x)) = h(x) \Longrightarrow h$ is onto.