Solution for HWs (§2.3)

#2.3.1 Proof $\forall \varepsilon>0$, $|a-a|=0<\varepsilon$ for all $n\in N$.

#2.3.2 Proof (a) $\forall \varepsilon>0$, since $x_n\to 0$, $\exists N$ s.t. $|x_n|=x_n<\varepsilon^2$ for all $n>N$

$\Rightarrow \sqrt{x_n}<\varepsilon$.

Proof (b) ~~Since $x_n\to x$, then $\{x_n\}$ is bounded, say $|x_n|<M$~~.

For any $\varepsilon>0$, $x_n\to x \Rightarrow \exists N$ s.t. $|x_n-x|<(\sqrt{x})\varepsilon$.

Now, $|\sqrt{x_n}-\sqrt{x}| = \dfrac{|x_n-x|}{\sqrt{x_n}+\sqrt{x}} \le \dfrac{|x_n-x|}{\sqrt{x}} < \varepsilon \Rightarrow \sqrt{x_n}\to\sqrt{x}$.

#2.3.3 $\forall \varepsilon>0$, $x_n\to l \Rightarrow \exists N_1$ s.t. $|x_n-l|<\varepsilon$ for $n\ge N_1$ $\Rightarrow x_n>l-\varepsilon$ for all $n\ge N_1$

$z_n\to l \Rightarrow \exists N_2$ s.t. $|z_n-l|<\varepsilon$ for all $n\ge N_2 \Rightarrow z_n<l+\varepsilon$, $\forall n\ge N_2$

for $n\ge \max\{N_1,N_2\}=N$, we have $l-\varepsilon < x_n \le y_n \le z_n < l+\varepsilon$

$\Rightarrow |y_n-l|<\varepsilon \quad \forall n\ge N$.

#2.3.4

$l_1-l_2=\lim a_n-\lim a_n=\lim(a_n-a_n)=0$.

or

$|l_1-l_2|=|l_1-a_n+a_n-l_2|$
$\le |l_1-a_n|+|a_n-l_2|$
$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$
for $n\ge N=\max\{N_1,N_2\}$

#2.3.6 Proof (a) $\big||b_n|-|b|\big| \le |b_n-b|<\varepsilon$.

(b) No. For example $\{1,-1,1,-1,\dots\}=\{(-1)^{n+1}\}$

$|b_n|=1\to 1$ but b_n does not converge.

#2.3.7 Proof $\{a_n\}$ is bounded $\Rightarrow |a_n|\le M \quad \forall n\in N$.

$\forall \varepsilon>0$, $\lim b_n=0 \Rightarrow \exists N$, s.t. $|b_n|<\varepsilon/M \Rightarrow |a_nb_n|\le M|b_n|<\varepsilon \quad \forall n\ge N$.

(b) a_n does not necessarily convergent.

#2.3.10 Proof $\forall\varepsilon>0$, $a_n\to 0 \Rightarrow \exists N$, s.t. $|a_n|<\varepsilon \ \forall n\ge N$

$\Rightarrow |b_n-b|\le a_n=|a_n|<\varepsilon \quad \forall n\ge N$.