Solutions for HWs (§2.4)

#2.4.1 Proof Let $s_n = b_1 + \cdots + b_n$ be the partial sum sequence of $\sum b_n$.

$$s_{2^k} = b_1 + b_2 + \cdots + b_{2^k}$$
$$= b_1 + b_2 + (b_3 + b_4) + (b_5 + \cdots + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k})$$
$$\geq b_1 + b_2 + 2b_4 + 4b_8 + 2^{k-1} b_{2^k} \qquad \text{since } b_n \geq 0 \text{ decreasing.}$$
$$\geq \frac{1}{2}\left[b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right] = \frac{1}{2} t_k$$

where t_k is partial sum sequence of $\sum 2^n b_{2^n}$ that diverges.

$\Rightarrow \sum b_n$ diverges. #

#2.4.2 (a) Proof Using proof by induction to show that

$0 \leq x_n \leq 3$ and x_n is decreasing $\Longrightarrow \{x_n\}$ converges.

(b)

(c) $x = \frac{1}{4-x} \Rightarrow x = 2 - \sqrt{3}$ since $0 \leq x \leq 3$.

#2.4.5 (a) Proof $x_1^2 = 4 \geq 2$.

Assume that $x_n^2 \geq 2$

$$\Rightarrow x_{n+1}^2 = \frac{1}{4}\left(x_n + \frac{2}{x_n}\right)^2 = \frac{(x_n^2+2)^2}{4x_n^2} \overset{?}{\geq} 2$$

$$\Leftrightarrow (x_n^2+2)^2 \geq 8x_n^2 \Leftrightarrow (x_n^2-2)^2 \geq 0. \Longrightarrow x_{n+1}^2 \geq 2.$$

$$x_{n+1} \overset{?}{\leq} x_n \Leftrightarrow \frac{1}{2}\frac{x_n^2+2}{x_n} \leq x_n \Longleftrightarrow x_n^2 + 2 \leq 2x_n^2 \Leftrightarrow 2 \leq x_n^2.$$

$\Rightarrow \{x_n\}$ is decreasing and bounded below by $\sqrt{2}$.

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \Rightarrow x = \sqrt{2}$$

(c) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$.