Solutions for HWs (§2.5)

#2.5.1 Proof Assume that $\lim_{j\to\infty} a_j = a$. Let $\{a_{n_j}\}_{j=1}^{\infty}$ be an arbitrary subsequence. $\forall \varepsilon > 0$,

$$\lim_{j\to\infty} a_j = a \Rightarrow \exists J \text{ s.t. } |a_j - a| < \varepsilon \ \forall j \geq J.$$

Since $n_j \geq j \geq J$, we have $|a_{n_j} - a| < \varepsilon$. #

#2.5.2

(a) Proof The partial sum sequence of

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots$$

is

$$t_j = (a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + \cdots + (a_{n_{j-1}+1} + \cdots + a_{n_j})$$
$$= s_{n_j}$$

where $s_n = a_1 + \cdots + a_n$ is the partial sum of $\sum_{n=1}^{\infty} a_n$. This implies that the p.s. sequence of the new series is a subsequence of $\{s_n\}$. By #2.5.1, $s_{n_j} \to L$ as $j \to \infty$ and hence the new series converges to L.

(b)

#2.5.3

(a) $\left\{\frac{1}{2}, \cdots, \frac{1}{n}, \cdots\right\} \to 0$ $\quad\Longrightarrow\quad$ $\left\{\frac{1}{2}, \frac{2}{3}, \cdots, \frac{1}{n}, \frac{n}{n+1}, \cdots\right\}$

$\left\{\frac{2}{3}, \cdots, \frac{n}{n+1}, \cdots\right\} \to 0$

(b) Impossible. A divergent monotone sequence is bounded, but its convergent subsequence is also monotone, and hence, bounded.

(c) $\left\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$

(d) $\{n\}$ is unbounded $\quad\Longrightarrow\quad \{1, 1, 1, 2, 1, 3, 1, 4, \cdots, 1, n, \cdots\}$

$\{1\} \to 1$

(e) Impossible. A subseq. is bounded $\Rightarrow$ containing a convergent subseq.

#2.5.4 Proof Assume that $\{a_n\}$ does not converges to a.

$\Leftrightarrow \exists\, \varepsilon_0 > 0,\ \forall N,\ \exists\, n \geq N$ s.t. $|a_n - a| \geq \varepsilon_0$

In particular, for $N=1$, $\exists\, n_1 \geq 1$ s.t. $|a_{n_1} - a| \geq \varepsilon_0$

for $N=2$, $\exists\, n_2 \geq 2$ and $n_2 \geq n_1$ s.t. $|a_{n_2} - a| \geq \varepsilon_0$

$\vdots$

for $N=j$, $\exists\, n_j \geq j$ and $n_j \geq n_{j-1} \geq \cdots \geq n_1$ s.t. $|a_{n_j} - a| \geq \varepsilon_0$

$\Rightarrow$ we construct a subsequence $\{a_{n_j}\}$ s.t. $|a_{n_j} - a| \geq \varepsilon_0$ (*)

but $\{a_{n_j}\}$ is bounded $\Rightarrow$ it has a convergent subseq. converging to a that contradict to (*)

#2.5.5 Proof Case $b=0$ $\quad \{b^n\} = \{0\} \to 0$

Case $0<b<1$ Example 2.5.3 $\Leftrightarrow |b^n| = |b|^n < \varepsilon$

Case $-1<b<0$ $\lim\limits_{n\to\infty} b^n = 0 \Leftrightarrow \varepsilon > |b^n - 0| = |b|^n$ $< \varepsilon$ by Example 2.5.3.