$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x}=\frac{\lim\limits_{x\to -2}(x^3+2x^2-1)}{\lim\limits_{x\to -2}(5-3x)} \qquad \text{(by Law 5)}$$

$$=\frac{\lim\limits_{x\to -2}x^3+2\lim\limits_{x\to -2}x^2-\lim\limits_{x\to -2}1}{\lim\limits_{x\to -2}5-3\lim\limits_{x\to -2}x} \qquad \text{(by 1, 2, and 3)}$$

$$=\frac{(-2)^3+2(-2)^2-1}{5-3(-2)} \qquad \text{(by 9, 8, and 7)}$$

$$=-\frac{1}{11}$$

NOTE ◦ If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 53 and 54). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x-1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$=\lim_{x\to 1}(x+1)$$

$$=1+1=2$$

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola $y = x^2$ at the point (1, 1).

NEWTON AND LIMITS

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague in 1665 and 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

NOTE In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, if $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

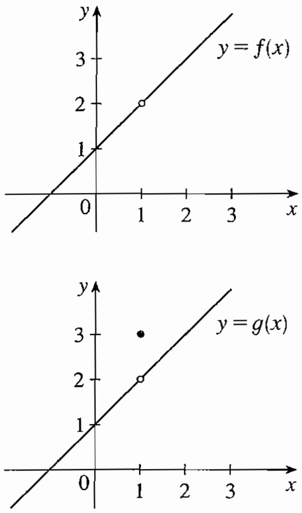


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

EXAMPLE 5 Evaluate $\displaystyle\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$

Explore a limit like this one interactively.
Resources / Module 2
/ The Essential Examples
/ Example C

EXAMPLE 6 Find $\displaystyle\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2 + 9} + 3\right)} = \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2 + 9} + 3\right)} \\ &= \lim_{t \to 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2.

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

$\boxed{1}$ **Theorem** $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

The result of Example 7 looks plausible from Figure 3.

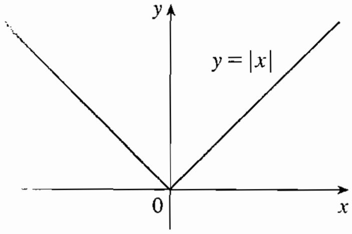


FIGURE 3

EXAMPLE 8 Prove that $\lim_{x \to 0} \dfrac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

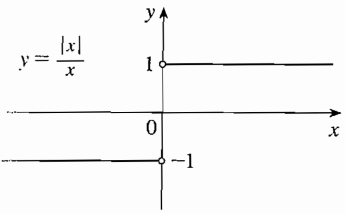


FIGURE 4

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x - 4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

SOLUTION Since $f(x) = \sqrt{x - 4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0$$

It is shown in Example 3 in Section 2.4 that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus, the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 5.

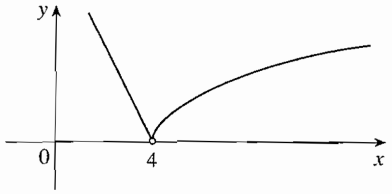


FIGURE 5

EXAMPLE 10 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

|||| Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1.

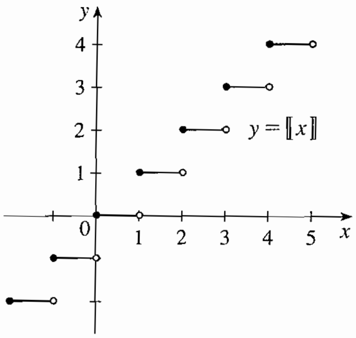


FIGURE 6
Greatest integer function

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

$\boxed{2}$ **Theorem** If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

$\boxed{3}$ **The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then
$$\lim_{x \to a} g(x) = L$$

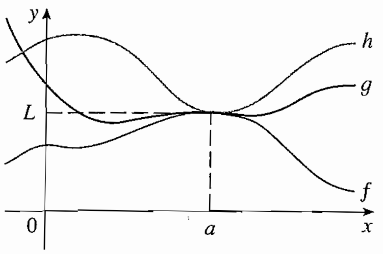


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 11 Show that $\lim_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

because $\lim_{x\to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin\frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin\frac{1}{x} \leqslant x^2$$

Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

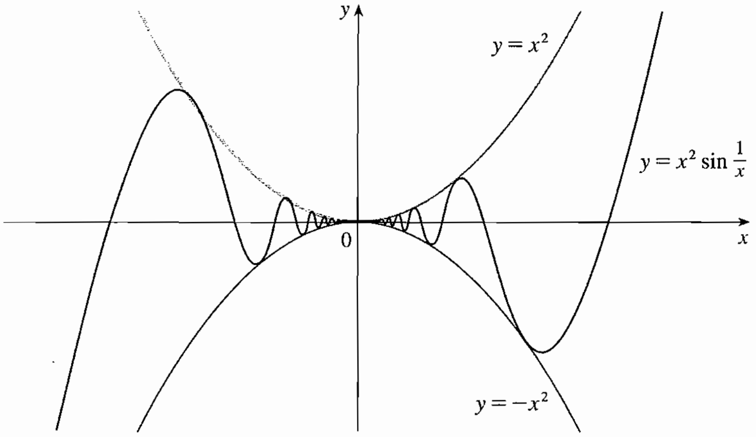


FIGURE 8

We know that

$$\lim_{x\to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x\to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x\to a} f(x) = -3 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to a} [f(x) + h(x)]$

(b) $\lim_{x\to a} [f(x)]^2$

(c) $\lim_{x\to a} \sqrt[3]{h(x)}$

(d) $\lim_{x\to a} \dfrac{1}{f(x)}$

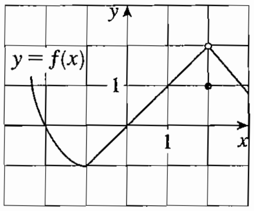
(e) $\lim_{x\to a} \dfrac{f(x)}{h(x)}$

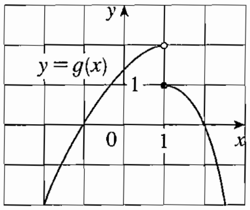
(f) $\lim_{x\to a} \dfrac{g(x)}{f(x)}$

(g) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$

(h) $\lim_{x\to a} \dfrac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2}[f(x)+g(x)]$ (b) $\lim_{x\to 1}[f(x)+g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ (d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2}x^3f(x)$ (f) $\lim_{x\to 1}\sqrt{3+f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to -2}(3x^4+2x^2-x+1)$

4. $\lim_{x\to 2}\dfrac{2x^2+1}{x^2+6x-4}$

5. $\lim_{x\to 3}(x^2-4)(x^3+5x-1)$

6. $\lim_{t\to -1}(t^2+1)^3(t+3)^5$

7. $\lim_{x\to 1}\left(\dfrac{1+3x}{1+4x^2+3x^4}\right)^3$

8. $\lim_{u\to -2}\sqrt{u^4+3u+6}$

9. $\lim_{x\to 4^-}\sqrt{16-x^2}$

* * *

10. (a) What is wrong with the following equation?

$$\frac{x^2+x-6}{x-2}=x+3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}(x+3)$$

is correct.

11–30 Evaluate the limit, if it exists.

11. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}$

12. $\lim_{x\to -4}\dfrac{x^2+5x+4}{x^2+3x-4}$

13. $\lim_{x\to 2}\dfrac{x^2-x+6}{x-2}$

14. $\lim_{x\to 4}\dfrac{x^2-4x}{x^2-3x-4}$

15. $\lim_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3}$

16. $\lim_{x\to -1}\dfrac{x^2-4x}{x^2-3x-4}$

17. $\lim_{h\to 0}\dfrac{(4+h)^2-16}{h}$

18. $\lim_{x\to 1}\dfrac{x^3-1}{x^2-1}$

19. $\lim_{h\to 0}\dfrac{(1+h)^4-1}{h}$

20. $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}$

21. $\lim_{t\to 9}\dfrac{9-t}{3-\sqrt{t}}$

22. $\lim_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}$

23. $\lim_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}$

24. $\lim_{x\to 2}\dfrac{x^4-16}{x-2}$

25. $\lim_{x\to -4}\dfrac{\dfrac{1}{4}+\dfrac{1}{x}}{4+x}$

26. $\lim_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)$

27. $\lim_{x\to 9}\dfrac{x^2-81}{\sqrt{x}-3}$

28. $\lim_{h\to 0}\dfrac{(3+h)^{-1}-3^{-1}}{h}$

29. $\lim_{t\to 0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)$

30. $\lim_{x\to 1}\dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$

* * *

31. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}$$

by graphing the function $f(x)=x/(\sqrt{1+3x}-1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

32. (a) Use a graph of

$$f(x)=\frac{\sqrt{3+x}-\sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0}f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

33. Use the Squeeze Theorem to show that $\lim_{x\to 0}x^2\cos 20\pi x=0$. Illustrate by graphing the functions $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$, and $h(x)=x^2$ on the same screen.

34. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x}=0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

35. If $1\leqslant f(x)\leqslant x^2+2x+2$ for all x, find $\lim_{x\to -1}f(x)$.

36. If $3x\leqslant f(x)\leqslant x^3+2$ for $0\leqslant x\leqslant 2$, evaluate $\lim_{x\to 1}f(x)$.

37. Prove that $\lim_{x\to 0}x^4\cos\dfrac{2}{x}=0$.

38. Prove that $\lim_{x\to 0^+}\sqrt{x}\,e^{\sin(\pi/x)}=0$.

39–44 Find the limit, if it exists. If the limit does not exist, explain why.

39. $\lim_{x\to -4}|x+4|$

40. $\lim_{x\to -4^-}\dfrac{|x+4|}{x+4}$

41. $\lim \dfrac{|x-2|}{x-2}$

42. $\lim\limits_{x \to 1.5} \dfrac{2x^2 - 3x}{|2x-3|}$

43. $\lim \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

44. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

45. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim\limits_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim\limits_{x \to 0^-} \operatorname{sgn} x$

(iii) $\lim\limits_{x \to 0} \operatorname{sgn} x$ (iv) $\lim\limits_{x \to 0} |\operatorname{sgn} x|$

46. Let

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \leqslant 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

(a) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

(b) Does $\lim_{x \to 2} f(x)$ exist?

(c) Sketch the graph of f.

47. Let $F(x) = \dfrac{x^2 - 1}{|x - 1|}$.

(a) Find

(i) $\lim\limits_{x \to 1^+} F(x)$ (ii) $\lim\limits_{x \to 1^-} F(x)$

(b) Does $\lim_{x \to 1} F(x)$ exist?

(c) Sketch the graph of F.

48. Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leqslant 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.

(i) $\lim\limits_{x \to 0^+} h(x)$ (ii) $\lim\limits_{x \to 0} h(x)$ (iii) $\lim\limits_{x \to 1} h(x)$

(iv) $\lim\limits_{x \to 2^-} h(x)$ (v) $\lim\limits_{x \to 2^+} h(x)$ (vi) $\lim\limits_{x \to 2} h(x)$

(b) Sketch the graph of h.

49. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim\limits_{x \to -2^+} [\![x]\!]$ (ii) $\lim\limits_{x \to -2} [\![x]\!]$ (iii) $\lim\limits_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim\limits_{x \to n^-} [\![x]\!]$ (ii) $\lim\limits_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

50. Let $f(x) = x - [\![x]\!]$.

(a) Sketch the graph of f.

(b) If n is an integer, evaluate

(i) $\lim\limits_{x \to n^-} f(x)$ (ii) $\lim\limits_{x \to n^+} f(x)$

(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

51. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

52. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

53. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

54. If r is a rational function, use Exercise 53 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

55. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

56. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

57. Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

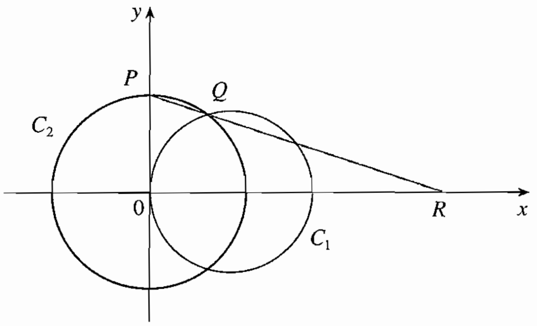
58. Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

59. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

60. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



44. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Prove each statement.

(a) $\lim_{x\to a} [f(x) + g(x)] = \infty$

(b) $\lim_{x\to a} [f(x)g(x)] = \infty \quad \text{if } c > 0$

(c) $\lim_{x\to a} [f(x)g(x)] = -\infty \quad \text{if } c < 0$

2.5 Continuity

Explore continuous functions interactively.
Resources / Module 2
/ Continuity
/ Start of Continuity

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

$\boxed{1}$ **Definition** A function f is **continuous at a number a** if

$$\lim_{x\to a} f(x) = f(a)$$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

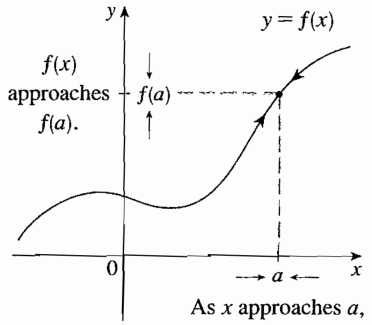


FIGURE 1

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus, a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at a**, or f has a **discontinuity** at a, if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [See Example 6 in Section 2.2, where the Heaviside function is discontinuous at 0 because $\lim_{t\to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

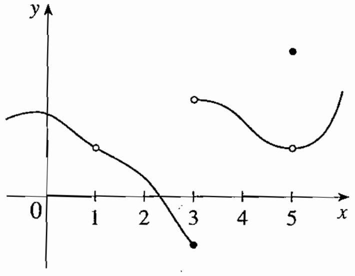


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that *f* is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So *f* is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x\to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Now let's see how to detect discontinuities when a function is defined by a formula.

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EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. (See Example 8 in Section 2.2.) So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$$

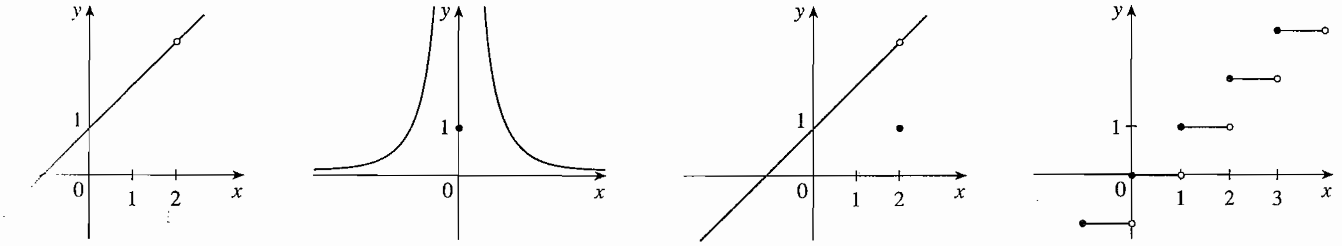
exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x \to n} [\![x]\!]$ does not exist if n is an integer. (See Example 10 and Exercise 49 in Section 2.3.)

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ (d) $f(x) = [\![x]\!]$

FIGURE 3 Graphs of the functions in Example 2

$\boxed{2}$ **Definition** A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n, the function $f(x) = [\![x]\!]$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [\![x]\!] = n = f(n)$$

but
$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [\![x]\!] = n - 1 \neq f(n)$$

$\boxed{3}$ **Definition** A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 7)} \\ &= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} && \text{(by 11)} \\ &= 1 - \sqrt{1 - a^2} && \text{(by 2, 7, and 9)} \\ &= f(a) \end{aligned}$$

Thus, by Definition 1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$

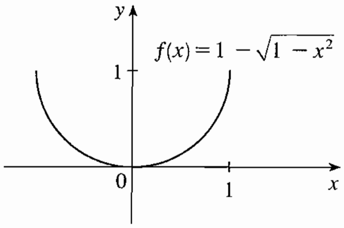


FIGURE 4

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\frac{f}{g}$ if $g(a) \neq 0$

Proof Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a) + g(a) \\ &= (f + g)(a) \end{aligned}$$

This shows that $f + g$ is continuous at a.

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Proof

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad \text{(by Law 7)}$$

and

$$\lim_{x \to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 9)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D.

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet after t seconds is given by the formula $h = 50t - 16t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2(b) in Section 2.3.

EXAMPLE 5 Find $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\begin{aligned} \lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \lim_{x \to -2} f(x) = f(-2) \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11} \end{aligned}$$

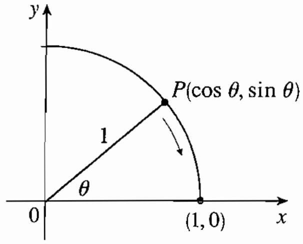


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 10 (page 106) implies that root functions are continuous. [Example 3 in Section 2.4 shows that $f(x) = \sqrt{x}$ is continuous from the right at 0.]

From the appearance of the graphs of the sine and cosine functions (Figure 18 in Section 1.2), we would certainly guess that they are continuous. We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$. Thus

$$\boxed{6} \qquad \lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

|||| Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ (for $\theta > 0$), which is proved in Section 3.4.

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 56 and 57).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

EXAMPLE 7 Evaluate $\lim\limits_{x \to 1} \arcsin\left(\dfrac{1 - \sqrt{x}}{1 - x}\right)$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned} \lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\ &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\ &= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\ &= \arcsin \frac{1}{2} = \frac{\pi}{6} \end{aligned}$$

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and
$$f\left(\lim_{x \to a} g(x)\right) = \sqrt[n]{\lim_{x \to a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$$

and so Limit Law 11 has now been proved. (We assume that the roots exist.)

$\boxed{9}$ **Theorem** If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

Proof Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a.

EXAMPLE 8 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \qquad \text{and} \qquad f(x) = \sin x$$

Now g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus, $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) We know from Theorem 7 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F(x) = f(g(x))$ is continuous wherever it is defined. Now $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$. So it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm 3\pi, \ldots$. Thus, F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure 7).

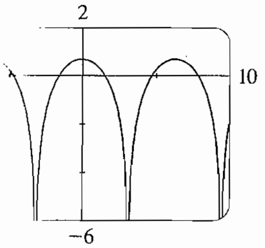


FIGURE 7

$y = \ln(1 + \cos x)$

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

> $\boxed{10}$ **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

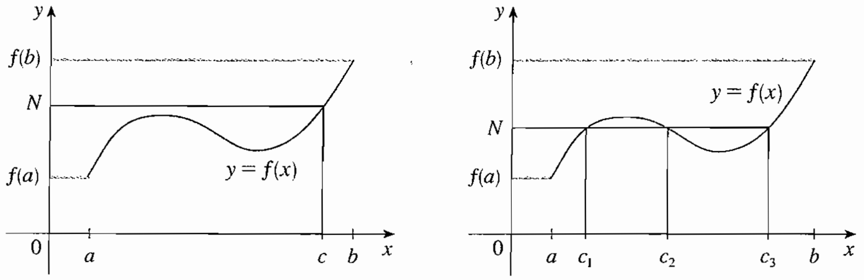


FIGURE 8

(a) (b)

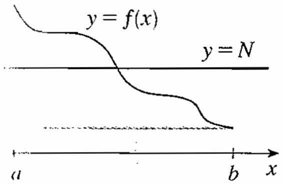


FIGURE 9

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 44).

One use of the Intermediate Value Theorem is in locating roots of equations as in the following example.