EXAMPLE 9 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore, we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus, $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \qquad \text{and} \qquad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \qquad \text{and} \qquad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$.

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 10 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.
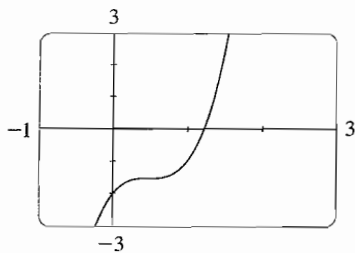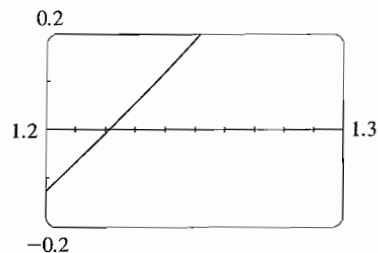

FIGURE 10

FIGURE 11

In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

2.5 Exercises

1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.

4. From the graph of g, state the intervals on which g is continuous.

5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.
(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City

(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x \to 3} [2f(x) - g(x)] = 4$, find $g(3)$.

10-12 IIII Use the definition of continuity and the properties of limits to show that the function is continuous at the given number.

10. $f(x) = x^2 + \sqrt{7 - x}$, $\quad a = 4$

11. $f(x) = (x + 2x^3)^4$, $\quad a = -1$

12. $g(x) = \dfrac{x + 1}{2x^2 - 1}$, $\quad a = 4$

13-14 IIII Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f(x) = \dfrac{2x + 3}{x - 2}$, $\quad (2, \infty)$ **14.** $g(x) = 2\sqrt{3 - x}$, $\quad (-\infty, 3]$

15-20 IIII Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

15. $f(x) = \ln|x - 2|$ $\qquad a = 2$

16. $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ $\qquad a = 1$

17. $f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ $\qquad a = 0$

18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$ $\qquad a = 1$

19. $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x + 3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$ $\qquad a = -3$

20. $f(x) = \begin{cases} 1 + x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$ $\qquad a = 1$

21-28 IIII Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

21. $F(x) = \dfrac{x}{x^2 + 5x + 6}$ **22.** $G(x) = \sqrt[3]{x}(1 + x^3)$

23. $R(x) = x^2 + \sqrt{2x - 1}$

24. $h(x) = \dfrac{\sin x}{x + 1}$

25. $f(x) = e^x \sin 5x$

26. $F(x) = \sin^{-1}(x^2 - 1)$

27. $G(t) = \ln(t^4 - 1)$

28. $H(x) = \cos(e^{\sqrt{x}})$

29–30 ▌▌▌▌ Locate the discontinuities of the function and illustrate by graphing.

29. $y = \dfrac{1}{1 + e^{1/x}}$

30. $y = \ln(\tan^2 x)$

31–34 ▌▌▌▌ Use continuity to evaluate the limit.

31. $\displaystyle\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

32. $\displaystyle\lim_{x \to \pi} \sin(x + \sin x)$

33. $\displaystyle\lim_{x \to 1} e^{x^2 - x}$

34. $\displaystyle\lim_{x \to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

35–36 ▌▌▌▌ Show that f is continuous on $(-\infty, \infty)$.

35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

36. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

37–39 ▌▌▌▌ Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geq 3 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

40. The gravitational force exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\[2mm] \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

41. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

42. Find the constant c that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

43. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous on $\mathbb{R}$.

(a) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}, \quad a = -2$

(b) $f(x) = \dfrac{x - 7}{|x - 7|}, \quad a = 7$

(c) $f(x) = \dfrac{x^3 + 64}{x + 4}, \quad a = -4$

(d) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}, \quad a = 9$

44. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

45. If $f(x) = x^3 - x^2 + x$, show that there is a number c such that $f(c) = 10$.

46. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$. (This proves the existence of the number $\sqrt{2}$.)

47–50 ▌▌▌▌ Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

47. $x^4 + x - 3 = 0, \quad (1, 2)$

48. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

49. $\cos x = x, \quad (0, 1)$

50. $\ln x = e^{-x}, \quad (1, 2)$

51–52 ▌▌▌▌ (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

51. $e^x = 2 - x$

52. $x^5 - x^2 + 2x + 3 = 0$

53–54 ▌▌▌▌ (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

53. $x^5 - x^2 - 4 = 0$

54. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

11. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

56. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 55 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

57. Prove that cosine is a continuous function.

58. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

59. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

60. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

61. Is there a number that is exactly 1 more than its cube?

62. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is $|f|$.
(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

63. A Tibetan monk leaves the monastery at 7:00 A.M. and takes his usual path to the top of the mountain, arriving at 7:00 P.M. The following morning, he starts at 7:00 A.M. at the top and takes the same path back, arriving at the monastery at 7:00 P.M. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

	$f(x)$
0	-1
1	0
2	0.600000
3	0.800000
4	0.882353
5	0.923077
10	0.980198
50	0.999200
100	0.999800
1000	0.999998

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

FIGURE 1

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ become closer and closer to L as x becomes larger and larger.

> **1 Definition** Let f be a function defined on some interval (a, ∞). Then
> $$\lim_{x \to \infty} f(x) = L$$
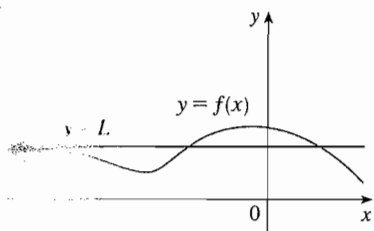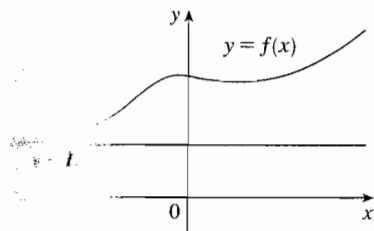> means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x \to \infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.
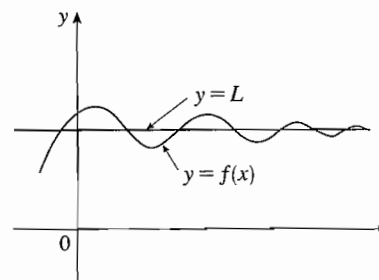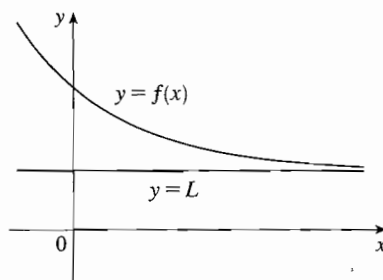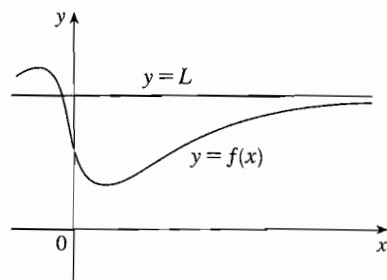
FIGURE 2
Examples illustrating $\lim_{x \to \infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close as we like to 1. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

> **2 Definition** Let f be a function defined on some interval $(-\infty, a)$. Then
> $$\lim_{x \to -\infty} f(x) = L$$
> means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.
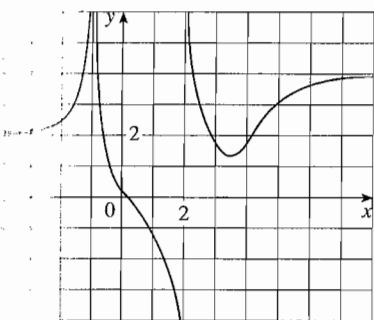
Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x \to -\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

> **3** **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either
>
> $$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

FIGURE 3
Examples illustrating $\lim_{x \to -\infty} f(x) = L$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 4.) In fact,

> **4** $$\lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

FIGURE 4
$y = \tan^{-1}x$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.)

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus, both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

FIGURE 5

EXAMPLE 2 Find $\lim\limits_{x \to \infty} \dfrac{1}{x}$ and $\lim\limits_{x \to -\infty} \dfrac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10,000} = 0.0001 \qquad \frac{1}{1,000,000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have
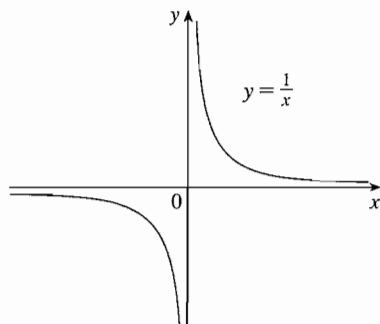
$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.)

FIGURE 6

$\lim\limits_{x \to \infty} \dfrac{1}{x} = 0, \quad \lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Laws 6 and 11 with the results of Example 2, we obtain the following important rule for calculating limits.

---

5 **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

EXAMPLE 3 Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have
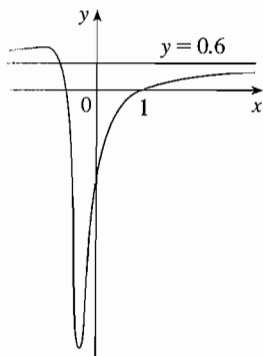
$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x \to \infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\displaystyle\lim_{x \to \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2} \right)}{\displaystyle\lim_{x \to \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2} \right)} \qquad \text{(by Limit Law 5)}$$

$$= \frac{\displaystyle\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}}{\displaystyle\lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{(by 7 and Theorem 5)}$$

$$= \frac{3}{5}$$

FIGURE 7
$$\frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} \qquad \left(\text{since } \sqrt{x^2} = x \text{ for } x > 0 \right)$$

$$= \frac{\displaystyle\lim_{x \to \infty} \sqrt{2 + \frac{1}{x^2}}}{\displaystyle\lim_{x \to \infty} \left(3 - \frac{5}{x} \right)} = \frac{\sqrt{\displaystyle\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x^2}}}{\displaystyle\lim_{x \to \infty} 3 - 5 \lim_{x \to \infty} \frac{1}{x}}$$

$$= \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore, the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{1}{x} \sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}} \sqrt{2x^2 + 1} = -\sqrt{2 + \frac{1}{x^2}}$$
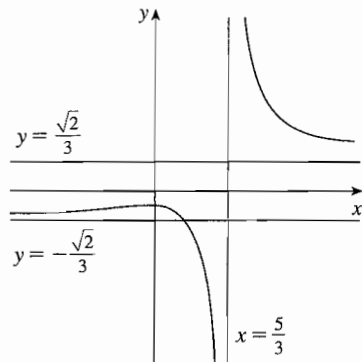
Therefore
$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}}$$

$$= \frac{-\sqrt{2 + \lim\limits_{x \to -\infty} \dfrac{1}{x^2}}}{3 - 5 \lim\limits_{x \to -\infty} \dfrac{1}{x}} = -\frac{\sqrt{2}}{3}$$

Thus, the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive. Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x - 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8.

FIGURE 8
$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

EXAMPLE 5 Compute $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right)$.

SOLUTION Because both $\sqrt{x^2 + 1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

IIII We can think of the given function as having a denominator of 1.

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right) = \lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x}$$

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by x. Doing this and using the Limit Laws, we obtain

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + x} = \lim_{x \to \infty} \frac{\dfrac{1}{x}}{\dfrac{\sqrt{x^2 + 1} + x}{x}}$$

$$= \lim_{x \to \infty} \frac{\dfrac{1}{x}}{\sqrt{1 + \dfrac{1}{x^2}} + 1} = \frac{0}{\sqrt{1 + 0} + 1} = 0$$

$$y = \sqrt{x^2 + 1} - x$$

FIGURE 9

Figure 9 illustrates this result.

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base $a > 1$.) In

fact, from the graph in Figure 10 and the corresponding table of values, we see that

$$\boxed{6} \qquad \lim_{x \to -\infty} e^x = 0$$

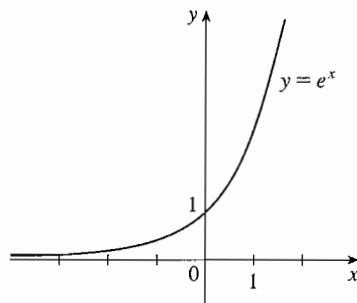Notice that the values of e^x approach 0 very rapidly.

x	e^x
0	1.00000
-1	0.36788
-2	0.13534
-3	0.04979
-5	0.00674
-8	0.00034
-10	0.00005

FIGURE 10

EXAMPLE 6 Evaluate $\lim_{x \to 0^-} e^{1/x}$.

The problem-solving strategy for Example 6 is *introducing something extra* (see page 80). Here the something extra, the auxiliary aid, is the new variable t.

SOLUTION If we let $t = 1/x$, we know that $t \to -\infty$ as $x \to 0^-$. Therefore, by (6),

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

(See Exercise 67.)

EXAMPLE 7 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus, $\lim_{x \to \infty} \sin x$ does not exist.

|||| Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

EXAMPLE 8 Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1,000,000 \qquad 1000^3 = 1,000,000,000$$

In fact, we can make x^3 as big as we like by taking x large enough. Therefore, we can write

$$\lim_{x \to \infty} x^3 = \infty$$

FIGURE 11

$\lim_{x \to \infty} x^3 = \infty$, $\lim_{x \to -\infty} x^3 = -\infty$

FIGURE 12

e^x is much larger than x^3
when x is large.

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11.

Looking at Figure 10 we see that

$$\lim_{x \to \infty} e^x = \infty$$

but, as Figure 12 demonstrates, $y = e^x$ becomes large as $x \to \infty$ at a much faster rate than $y = x^3$.

EXAMPLE 9 Find $\lim_{x \to \infty} (x^2 - x)$.

⊘ **SOLUTION** Note that we *cannot* write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x$$

$$= \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we can write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too.

EXAMPLE 10 Find $\lim_{x \to \infty} \dfrac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is just x:

$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x} = \lim_{x \to \infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to -1$ as $x \to \infty$.

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without having to plot a large number of points.

EXAMPLE 11 Sketch the graph of $y = (x - 2)^4(x + 1)^3(x - 1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The y-intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x-intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x - 2)^4$ is positive, the function doesn't change sign at 2; thus, the graph doesn't cross the x-axis at 2. The graph crosses the axis at -1 and 1.

When x is large positive, all three factors are large, so

$$\lim_{x \to \infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x \to -\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

Combining this information, we give a rough sketch of the graph in Figure 13.
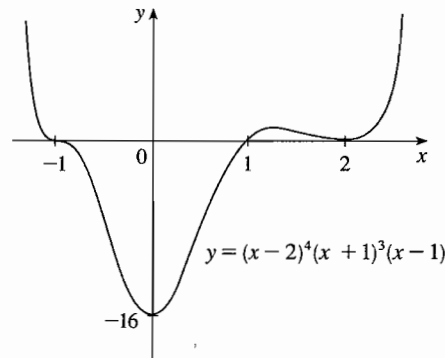
FIGURE 13

|||| Precise Definitions

Definition 1 can be stated precisely as follows.

> **7 Definition** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x \to \infty} f(x) = L$$
>
> means that for every $\varepsilon > 0$ there is a corresponding number N such that
>
> $$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > N$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where N depends on ε). Graphically it says that by choosing x large enough (larger than

some number N) we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14. This must be true no matter how small we choose ε. Figure 15 shows that if a smaller value of ε is chosen, then a larger value of N may be required.
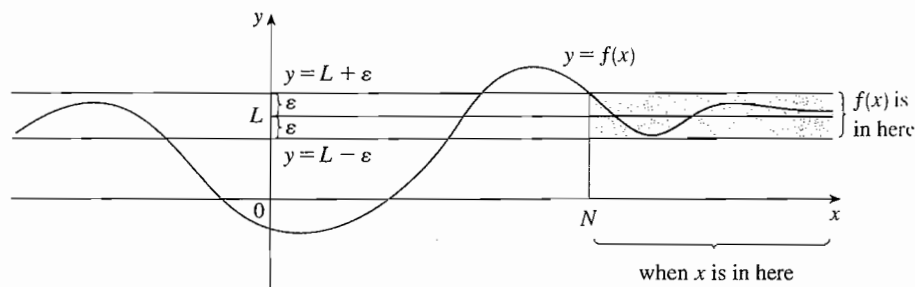
FIGURE 14
$\lim\limits_{x \to \infty} f(x) = L$
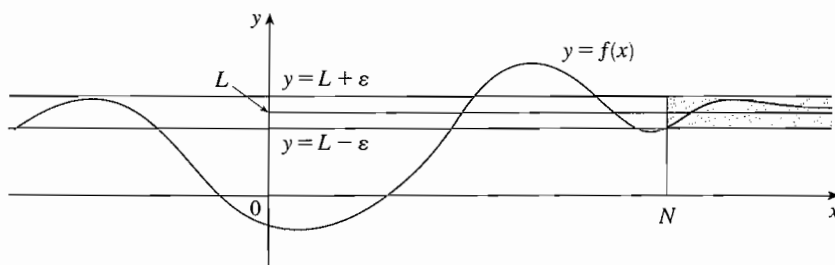
FIGURE 15
$\lim\limits_{x \to \infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 8, which is illustrated in Figure 16.

---

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that
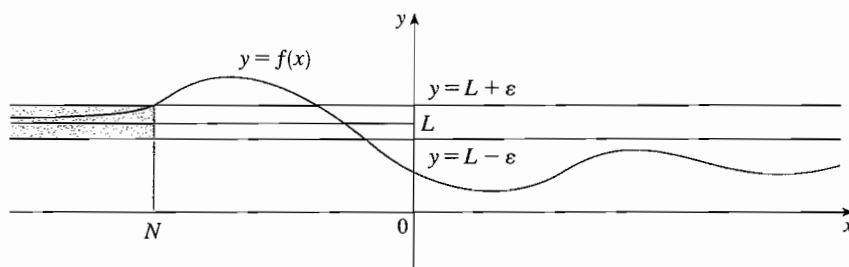
$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x < N$$

FIGURE 16
$\lim\limits_{x \to -\infty} f(x) = L$

In Example 3 we calculated that

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 7 with $L = \frac{3}{5}$ and $\varepsilon = 0.1$.

EXAMPLE 12 Use a graph to find a number N such that

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \qquad \text{whenever} \qquad x > N$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 17. Then we use the cursor to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that

$$\left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1 \qquad \text{whenever} \qquad x > 7$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 7.
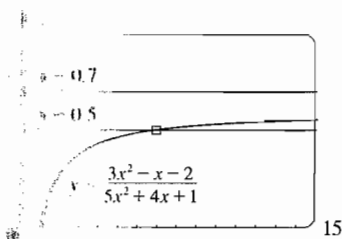
FIGURE 17

EXAMPLE 13 Use Definition 7 to prove that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$.

SOLUTION

1. *Preliminary analysis of the problem* (*guessing a value for N*). Given $\varepsilon > 0$, we want to find N such that

$$\left| \frac{1}{x} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad x > N$$

In computing the limit we may assume $x > 0$, in which case

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| = \frac{1}{x}$$

Therefore, we want

$$\frac{1}{x} < \varepsilon \qquad \text{whenever} \qquad x > N$$

that is,

$$x > \frac{1}{\varepsilon} \qquad \text{whenever} \qquad x > N$$

This suggests that we should take $N = 1/\varepsilon$.

2. *Proof* (*showing that this N works*). Given $\varepsilon > 0$, we choose $N = 1/\varepsilon$. Let $x > N$. Then

$$\left| \frac{1}{x} - 0 \right| = \frac{1}{|x|} = \frac{1}{x} < \frac{1}{N} = \varepsilon$$

Thus

$$\left| \frac{1}{x} - 0 \right| < \varepsilon \qquad \text{whenever} \qquad x > N$$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.
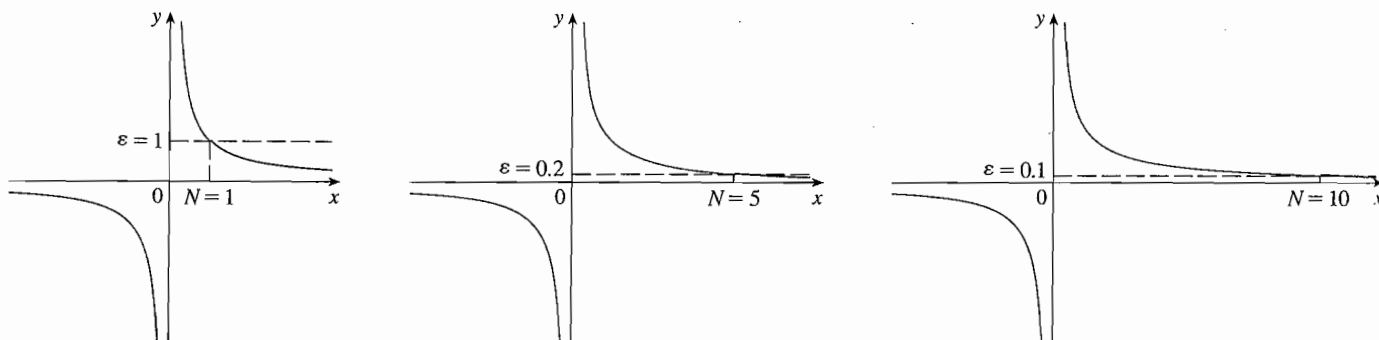
FIGURE 18
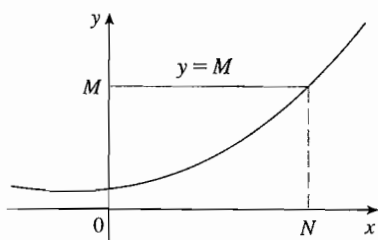
Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

> **9** **Definition** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x \to \infty} f(x) = \infty$$
>
> means that for every positive number M there is a corresponding positive number N such that
>
> $$f(x) > M \qquad \text{whenever} \qquad x > N$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 66.)

|||| **2.6** Exercises

1. Explain in your own words the meaning of each of the following.
   (a) $\lim_{x \to \infty} f(x) = 5$
   (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
   (b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
   (a) $\lim_{x \to 2} f(x)$
   (b) $\lim_{x \to -1^-} f(x)$
   (c) $\lim_{x \to -1^+} f(x)$
   (d) $\lim_{x \to \infty} f(x)$
   (e) $\lim_{x \to -\infty} f(x)$

(f) The equations of the asymptotes
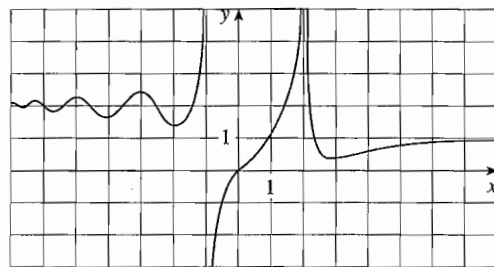
4. For the function g whose graph is given, state the following.
   (a) $\lim_{x \to \infty} g(x)$
   (b) $\lim_{x \to -\infty} g(x)$

(c) $\lim\limits_{x \to \cdot} g(x)$ (d) $\lim\limits_{x \to 0} g(x)$

(e) $\lim\limits_{x \to 2} g(x)$ (f) The equations of the asymptotes
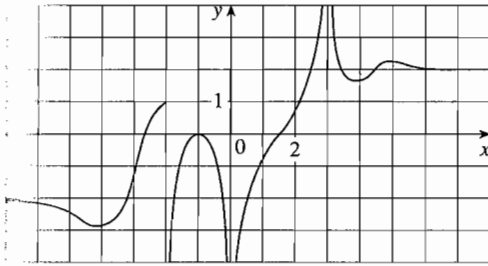
3. Sketch the graph of an example of a function f that satisfies all of the given conditions.

4. $f(0) = 0$, $f(1) = 1$, $\lim\limits_{x \to \infty} f(x) = 0$, f is odd

5. $\lim\limits_{x \to 0^+} f(x) = \infty$, $\lim\limits_{x \to 0^-} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = 1$,
$\lim\limits_{x \to -\infty} f(x) = 1$

6. $\lim\limits_{x \to 2} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = 0$,
$\lim\limits_{x \to 0^+} f(x) = \infty$, $\lim\limits_{x \to 0^-} f(x) = -\infty$

7. $\lim\limits_{x \to 2} f(x) = \infty$, $\lim\limits_{x \to -\infty} f(x) = 3$, $\lim\limits_{x \to \infty} f(x) = -3$

8. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5,$ 6, 7, 8, 9, 10, 20, 50, and 100. Then use a graph of f to support your guess.

9. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

10–12 |||| Evaluate the limit and justify each step by indicating the appropriate properties of limits.

11. $\lim\limits_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$ **12.** $\lim\limits_{x \to \infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

13–34 |||| Find the limit.

13. $\lim\limits_{x \to \infty} \dfrac{1}{2x + 3}$ **14.** $\lim\limits_{x \to \infty} \dfrac{3x + 5}{x - 4}$

15. $\lim\limits_{x \to -\infty} \dfrac{1 - x - x^2}{2x^2 - 7}$ **16.** $\lim\limits_{y \to \infty} \dfrac{2 - 3y^2}{5y^2 + 4y}$

17. $\lim\limits_{x \to \infty} \dfrac{x^3 + 5x}{2x^3 - x^2 + 4}$ **18.** $\lim\limits_{t \to -\infty} \dfrac{t^2 + 2}{t^3 + t^2 - 1}$

19. $\lim\limits_{u \to \infty} \dfrac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)}$ **20.** $\lim\limits_{x \to \infty} \dfrac{x + 2}{\sqrt{9x^2 + 1}}$

21. $\lim\limits_{x \to \infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$ **22.** $\lim\limits_{x \to -\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

23. $\lim\limits_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$ **24.** $\lim\limits_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

25. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$ **26.** $\lim\limits_{x \to \infty} \cos x$

27. $\lim\limits_{x \to \infty} \sqrt{x}$ **28.** $\lim\limits_{x \to -\infty} \sqrt[3]{x}$

29. $\lim\limits_{x \to \infty} \left(x - \sqrt{x}\right)$ **30.** $\lim\limits_{x \to \infty} \dfrac{x^3 - 2x + 3}{5 - 2x^2}$

31. $\lim\limits_{x \to -\infty} (x^4 + x^5)$ **32.** $\lim\limits_{x \to \infty} \tan^{-1}(x^2 - x^4)$

33. $\lim\limits_{x \to \infty} \dfrac{x + x^3 + x^5}{1 - x^2 + x^4}$ **34.** $\lim\limits_{x \to (\pi/2)^+} e^{\tan x}$

35. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.
(b) Use a table of values of $f(x)$ to guess the value of the limit.
(c) Prove that your guess is correct.

36. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
(c) Find the exact value of the limit.

37–42 |||| Find the horizontal and vertical asymptotes of each curve. Check your work by graphing the curve and estimating the asymptotes.

37. $y = \dfrac{x}{x + 4}$ **38.** $y = \dfrac{x^2 + 4}{x^2 - 1}$

39. $y = \dfrac{x^3}{x^2 + 3x - 10}$ **40.** $y = \dfrac{x^3 + 1}{x^3 + x}$

41. $h(x) = \dfrac{x}{\sqrt[4]{x^4 + 1}}$ **42.** $F(x) = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

43. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$$

$$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$$

44. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

45–48 |||| Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 11.

45. $y = x^2(x - 2)(1 - x)$

46. $y = (2 + x)^3(1 - x)(3 - x)$

47. $y = (x + 4)^5(x - 3)^4$

48. $y = (1 - x)(x - 3)^2(x - 5)^2$

□   □   □   □   □   □   □   □   □   □   □   □

49. (a) Use the Squeeze Theorem to evaluate $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

   (b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

50. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.
   (a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

   by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.
   (b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

51. Let P and Q be polynomials. Find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

52. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

   (i) $n = 0$ (ii) $n > 0$, n odd

   (iii) $n > 0$, n even   (iv) $n < 0$, n odd

   (v) $n < 0$, n even

Then use these sketches to find the following limits.

   (a) $\lim\limits_{x \to 0^+} x^n$ (b) $\lim\limits_{x \to 0^-} x^n$

   (c) $\lim\limits_{x \to \infty} x^n$ (d) $\lim\limits_{x \to -\infty} x^n$

53. Find $\lim_{x \to \infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

54. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

   (b) What happens to the concentration as $t \to \infty$?

55. In Chapter 9 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the *terminal velocity* of the raindrop.
   (a) Find $\lim_{t \to \infty} v(t)$.
   (b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s². How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

56. (a) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.
   (b) Can you solve part (a) without using a graphing device?

57. Use a graph to find a number N such that

$$\left| \frac{6x^2 + 5x - 3}{2x^2 - 1} - 3 \right| < 0.2 \qquad \text{whenever} \qquad x > N$$

58. For the limit

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

59. For the limit

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

60. For the limit

$$\lim_{x \to \infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

▸ ⒜ How large do we have to take x so that $1/x^2 < 0.0001$?
    ⒝ Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

▸ ⒜ How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
    ⒝ Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

▸ Use Definition 8 to prove that $\lim_{x \to -\infty} \dfrac{1}{x} = 0$.

▸ Prove, using Definition 9, that $\lim_{x \to \infty} x^3 = \infty$.

65. Use Definition 9 to prove that

$$\lim_{x \to \infty} e^x = \infty$$

66. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

67. Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

if these limits exist.

2.7   Tangents, Velocities, and Other Rates of Change

In Section 2.1 we guessed the values of slopes of tangent lines and velocities on the basis of numerical evidence. Now that we have defined limits and have learned techniques for computing them, we return to the tangent and velocity problems with the ability to calculate slopes of tangents, velocities, and other rates of change.

||| Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)
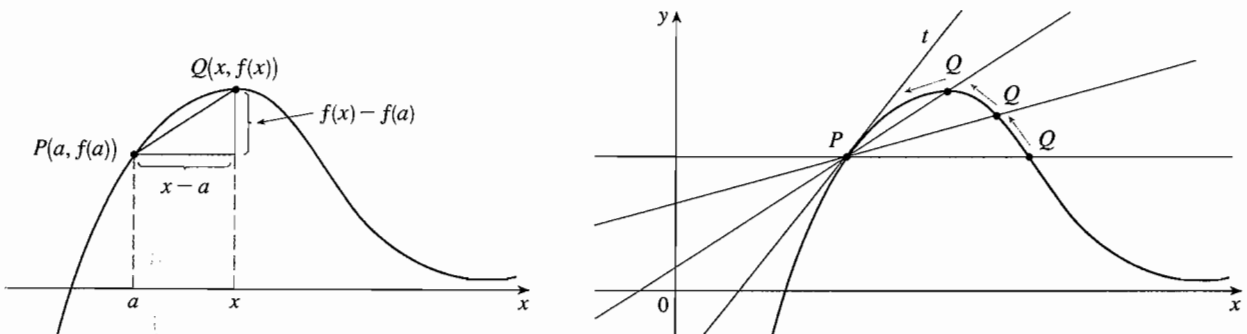

FIGURE 1