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Appendix D

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21. Use the data in the table to model the population of the world in the 20th century by a cubic function. Then use your model to estimate the population in the year 1925.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

22. The table shows the mean (average) distances d of the planets from the Sun (taking the unit of measurement to be the

distance from Earth to the Sun) and their periods T (time of revolution in years).

Planet	d	T
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881
Jupiter	5.203	11.861
Saturn	9.541	29.457
Uranus	19.190	84.008
Neptune	30.086	164.784
Pluto	39.507	248.350

- (a) Fit a power model to the data.
 (b) Kepler's Third Law of Planetary Motion states that "The square of the period of revolution of a planet is proportional to the cube of its mean distance from the Sun." Does your model corroborate Kepler's Third Law?

1.3 New Functions from Old Functions

In this section we start with the basic functions we discussed in Section 1.2 and obtain new functions by shifting, stretching, and reflecting their graphs. We also show how to combine pairs of functions by the standard arithmetic operations and by composition.

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs. Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y -coordinate is increased by the same number c). Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x). Therefore, the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Now let's consider the **stretching** and **reflecting** transformations. If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y -coordinate is multiplied by the same number c). The graph of

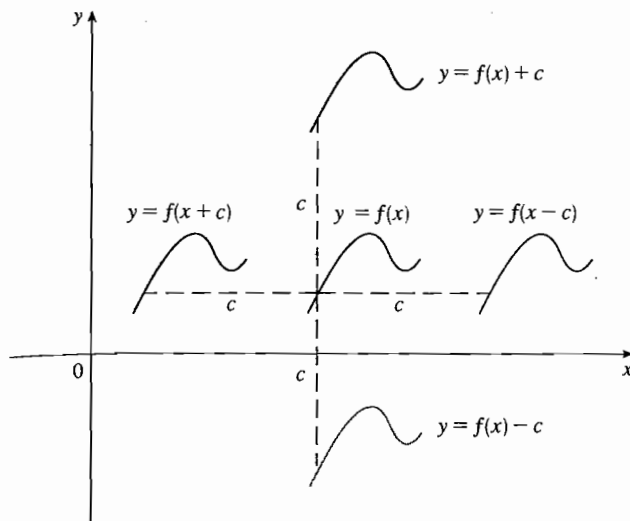


FIGURE 1
Translating the graph of f

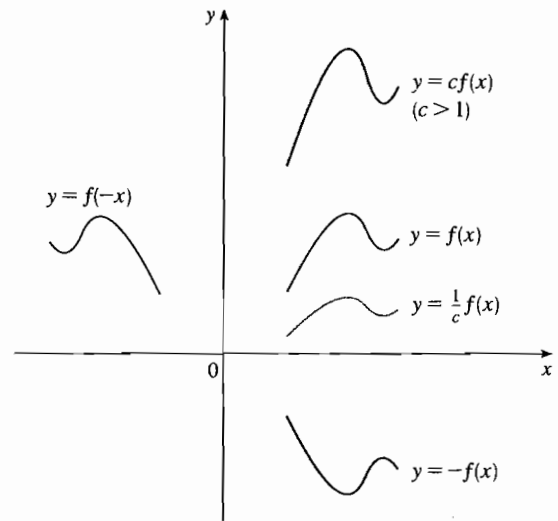


FIGURE 2
Stretching and reflecting the graph of f

$y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis because the point (x, y) is replaced by the point $(x, -y)$. (See Figure 2 and the following chart, where the results of other stretching, compressing, and reflecting transformations are also given.)

In Module 1.3 you can see the effect of combining the transformations of this section.

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Figure 3 illustrates these stretching transformations when applied to the cosine function with $c = 2$. For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y -coordinate of each point on the graph of $y = \cos x$ by 2. This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

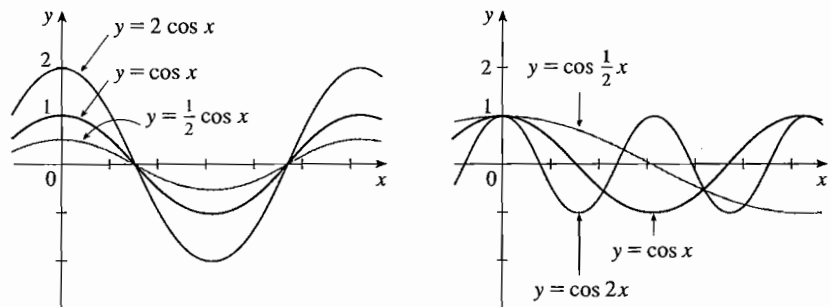


FIGURE 3

EXAMPLE 1 Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION The graph of the square root function $y = \sqrt{x}$, obtained from Figure 13 in Section 1.2, is shown in Figure 4(a). In the other parts of the figure we sketch $y = \sqrt{x} - 2$ by shifting 2 units downward, $y = \sqrt{x - 2}$ by shifting 2 units to the right, $y = -\sqrt{x}$ by reflecting about the x -axis, $y = 2\sqrt{x}$ by stretching vertically by a factor of 2, and $y = \sqrt{-x}$ by reflecting about the y -axis.

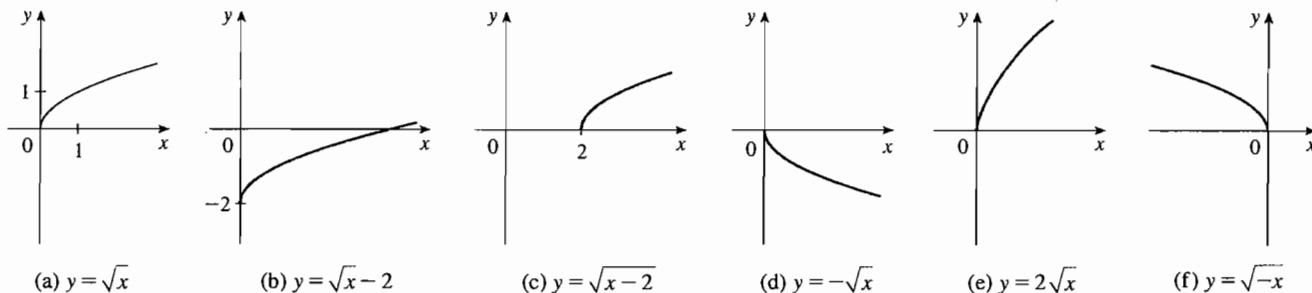


FIGURE 4

EXAMPLE 2 Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 5).

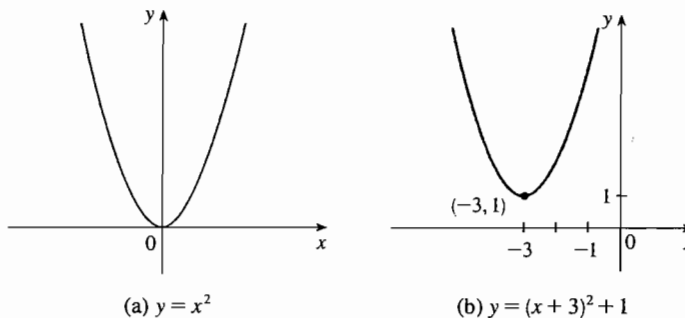


FIGURE 5

EXAMPLE 3 Sketch the graphs of the following functions.

(a) $y = \sin 2x$

(b) $y = 1 - \sin x$

SOLUTION

(a) We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2 (see Figures 6 and 7). Thus, whereas the period of $y = \sin x$ is 2π , the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

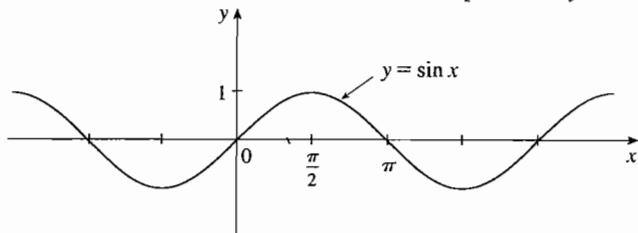


FIGURE 6

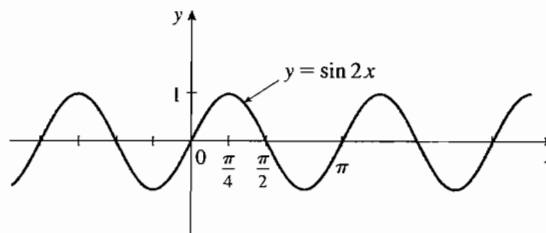


FIGURE 7

(b) To obtain the graph of $y = 1 - \sin x$, we again start with $y = \sin x$. We reflect about the x -axis to get the graph of $y = -\sin x$ and then we shift 1 unit upward to get $y = 1 - \sin x$. (See Figure 8.)

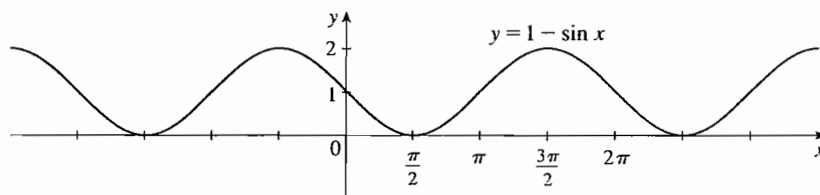


FIGURE 8

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

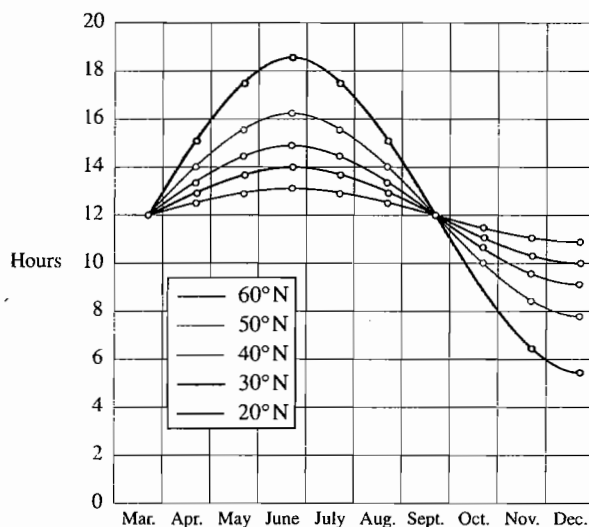


FIGURE 9

Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Lucia C. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935) page 40.

SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π , so the horizontal stretching factor is $c = 2\pi/365$.

We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore, we model the length of daylight in Philadelphia on the t th day of the year by the function

$$L(t) = 12 + 2.8 \sin \left[\frac{2\pi}{365}(t - 80) \right]$$

Another transformation of some interest is taking the absolute value of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$. This tells us how to get the graph of $y = |f(x)|$ from the graph

of $y = f(x)$: The part of the graph that lies above the x -axis remains the same; the part that lies below the x -axis is reflected about the x -axis.

EXAMPLE 5 Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ in Figure 10(a) by shifting the parabola $y = x^2$ downward 1 unit. We see that the graph lies below the x -axis when $-1 < x < 1$, so we reflect that part of the graph about the x -axis to obtain the graph of $y = |x^2 - 1|$ in Figure 10(b).

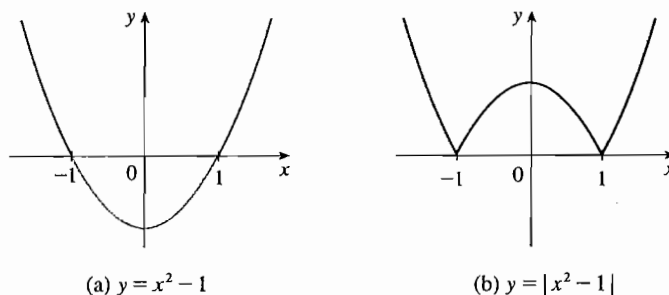


FIGURE 10

||| Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

If we define the sum $f + g$ by the equation

$$\boxed{\text{1}} \quad (f + g)(x) = f(x) + g(x)$$

then the right side of Equation 1 makes sense if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g . If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$.

Notice that the $+$ sign on the left side of Equation 1 stands for the operation of addition of *functions*, but the $+$ sign on the right side of the equation stands for addition of the *numbers* $f(x)$ and $g(x)$.

Similarly, we can define the difference $f - g$ and the product fg , and their domains are also $A \cap B$. But in defining the quotient f/g we must remember not to divide by 0.

Algebra of Functions Let f and g be functions with domains A and B . Then the functions $f + g$, $f - g$, fg , and f/g are defined as follows:

$$(f + g)(x) = f(x) + g(x) \quad \text{domain} = A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

EXAMPLE 6 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4-x^2}$, find the functions $f+g$, $f-g$, fg , and f/g .

SOLUTION The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. The domain of $g(x) = \sqrt{4-x^2}$ consists of all numbers x such that $4-x^2 \geq 0$, that is, $x^2 \leq 4$. Taking square roots of both sides, we get $|x| \leq 2$, or $-2 \leq x \leq 2$, so the domain of g is the interval $[-2, 2]$. The intersection of the domains of f and g is

$$[0, \infty) \cap [-2, 2] = [0, 2]$$

Thus, according to the definitions, we have

$$(f+g)(x) = \sqrt{x} + \sqrt{4-x^2} \quad 0 \leq x \leq 2$$

$$(f-g)(x) = \sqrt{x} - \sqrt{4-x^2} \quad 0 \leq x \leq 2$$

$$(fg)(x) = \sqrt{x}\sqrt{4-x^2} = \sqrt{4x-x^3} \quad 0 \leq x \leq 2$$

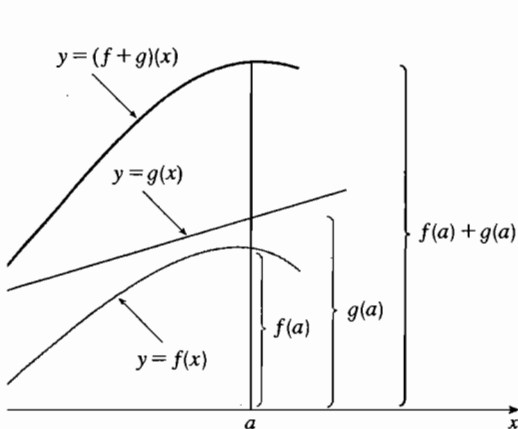
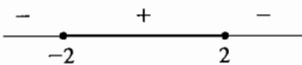
$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{4-x^2}} = \sqrt{\frac{x}{4-x^2}} \quad 0 \leq x < 2$$

Notice that the domain of f/g is the interval $[0, 2)$; we have to exclude $x = 2$ because $g(2) = 0$.

The graph of the function $f+g$ is obtained from the graphs of f and g by **graphical addition**. This means that we add corresponding y -coordinates as in Figure 11. Figure 12 shows the result of using this procedure to graph the function $f+g$ from Example 6.

Another way to solve $4-x^2 \geq 0$:

$$(2-x)(2+x) \geq 0$$



RE 11

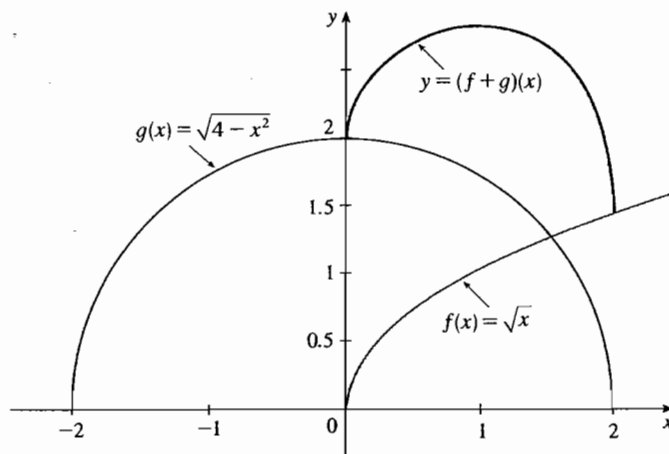


FIGURE 12

||| Composition of Functions

There is another way of combining two functions to get a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

In general, given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$. The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

Definition Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. The best way to picture $f \circ g$ is by either a machine diagram (Figure 13) or an arrow diagram (Figure 14).

FIGURE 13

The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

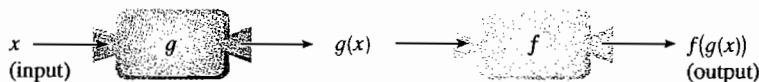
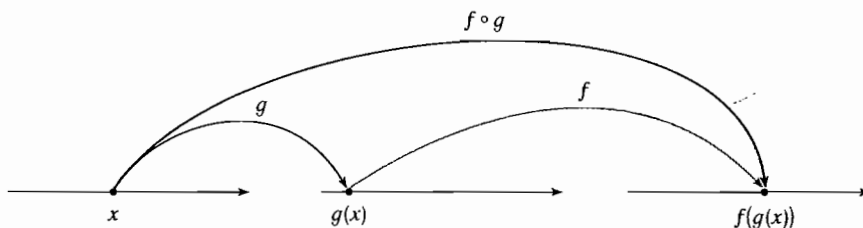


FIGURE 14

Arrow diagram for $f \circ g$



EXAMPLE 7 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

NOTE You can see from Example 7 that, in general, $f \circ g \neq g \circ f$. Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second. In Example 7, $f \circ g$ is the function that *first* subtracts 3 and *then* squares; $g \circ f$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 8 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

SOLUTION

$$(a) \quad (f \circ g)(x) = f(g(x)) = f(\sqrt{2 - x}) = \sqrt{\sqrt{2 - x}} = \sqrt[4]{2 - x}$$

The domain of $f \circ g$ is $\{x \mid 2 - x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

If $0 \leq a \leq b$, then $a^2 \leq b^2$.

$$(b) \quad (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus, we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

$$(c) \quad (f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

$$(d) \quad (g \circ g)(x) = g(g(x)) = g(\sqrt{2 - x}) = \sqrt{2 - \sqrt{2 - x}}$$

This expression is defined when $2 - x \geq 0$, that is, $x \leq 2$, and $2 - \sqrt{2 - x} \geq 0$. This latter inequality is equivalent to $\sqrt{2 - x} \leq 2$, or $2 - x \leq 4$, that is, $x \geq -2$. Thus, $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE 9 Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\ &= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1} \end{aligned}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to decompose a complicated function into simpler ones, as in the following example.

EXAMPLE 10 Given $F(x) = \cos^2(x + 9)$, find functions f , g , and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \quad g(x) = \cos x \quad f(x) = x^2$$

Then

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x) \end{aligned}$$

1.3 Exercises

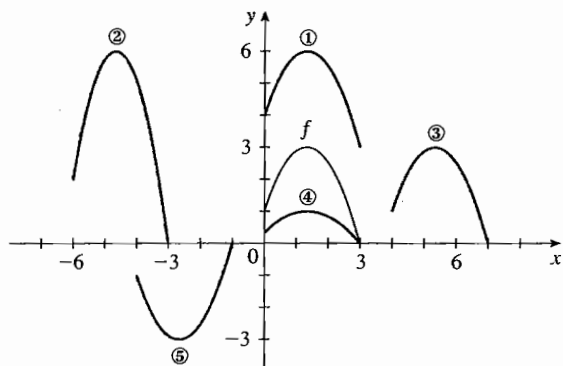
Suppose the graph of f is given. Write equations for the graphs that are obtained from the graph of f as follows.

- Shift 3 units upward.
- Shift 3 units downward.
- Shift 3 units to the right.

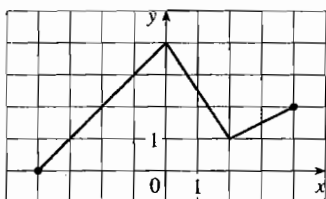
- Shift 3 units to the left.
- Reflect about the x -axis.
- Reflect about the y -axis.
- Stretch vertically by a factor of 3.
- Shrink vertically by a factor of 3.

2. Explain how the following graphs are obtained from the graph of $y = f(x)$.
- (a) $y = 5f(x)$ (b) $y = f(x - 5)$
 (c) $y = -f(x)$ (d) $y = -5f(x)$
 (e) $y = f(5x)$ (f) $y = 5f(x) - 3$

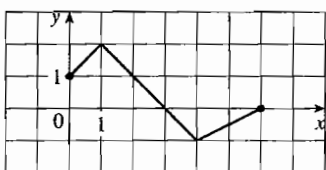
3. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.
- (a) $y = f(x - 4)$ (b) $y = f(x) + 3$
 (c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
 (e) $y = 2f(x + 6)$



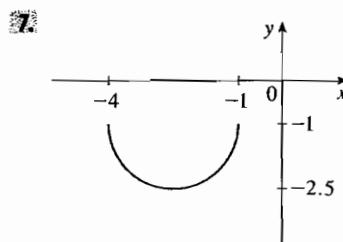
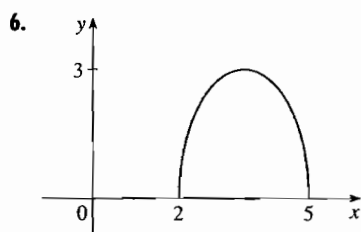
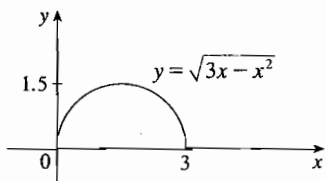
4. The graph of f is given. Draw the graphs of the following functions.
- (a) $y = f(x + 4)$ (b) $y = f(x) + 4$
 (c) $y = 2f(x)$ (d) $y = -\frac{1}{2}f(x) + 3$



5. The graph of f is given. Use it to graph the following functions.
- (a) $y = f(2x)$ (b) $y = f(\frac{1}{2}x)$
 (c) $y = f(-x)$ (d) $y = -f(-x)$



- 6-7 ||| The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



8. (a) How is the graph of $y = 2 \sin x$ related to the graph of $y = \sin x$? Use your answer and Figure 6 to sketch the graph of $y = 2 \sin x$.
 (b) How is the graph of $y = 1 + \sqrt{x}$ related to the graph of $y = \sqrt{x}$? Use your answer and Figure 4(a) to sketch the graph of $y = 1 + \sqrt{x}$.

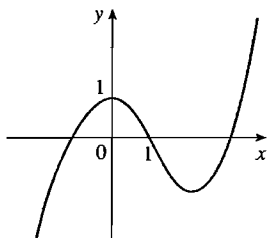
9-24 ||| Graph the function, not by plotting points, but by starting with the graph of one of the standard functions given in Section 1.2, and then applying the appropriate transformations.

9. $y = -x^3$ 10. $y = 1 - x^2$
 11. $y = (x + 1)^2$ 12. $y = x^2 - 4x + 3$
 13. $y = 1 + 2 \cos x$ 14. $y = 4 \sin 3x$
 15. $y = \sin(x/2)$ 16. $y = \frac{1}{x - 4}$
 17. $y = \sqrt{x + 3}$ 18. $y = (x + 2)^4 + 3$
 19. $y = \frac{1}{2}(x^2 + 8x)$ 20. $y = 1 + \sqrt[3]{x - 1}$
 21. $y = \frac{2}{x + 1}$ 22. $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$
 23. $y = |\sin x|$ 24. $y = |x^2 - 2x|$

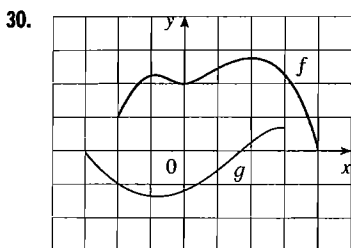
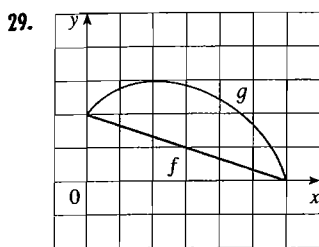
25. The city of New Orleans is located at latitude 30°N . Use Figure 9 to find a function that models the number of hours of daylight at New Orleans as a function of the time of year. Use the fact that on March 31 the Sun rises at 5:51 A.M. and sets at 6:18 P.M. in New Orleans to check the accuracy of your model.

26. A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between periods of maximum brightness is 5.4 days, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

27. (a) How is the graph of $y = f(|x|)$ related to the graph of f ?
 (b) Sketch the graph of $y = \sin |x|$.
 (c) Sketch the graph of $y = \sqrt{|x|}$.
28. Use the given graph of f to sketch the graph of $y = 1/f(x)$. Which features of f are the most important in sketching $y = 1/f(x)$? Explain how they are used.



29-30 III Use graphical addition to sketch the graph of $f + g$.



31-32 III Find $f + g$, $f - g$, fg , and f/g and state their domains.

31. $f(x) = x^3 + 2x^2$, $g(x) = 3x^2 - 1$

32. $f(x) = \sqrt{1+x}$, $g(x) = \sqrt{1-x}$

33-34 III Use the graphs of f and g and the method of graphical addition to sketch the graph of $f + g$.

33. $f(x) = x$, $g(x) = 1/x$ 34. $f(x) = x^3$, $g(x) = -x^2$

35-40 III Find the functions $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

35. $f(x) = 2x^2 - x$, $g(x) = 3x + 2$

36. $f(x) = 1 - x^3$, $g(x) = 1/x$

37. $f(x) = \sin x$, $g(x) = 1 - \sqrt{x}$

38. $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$

39. $f(x) = x + \frac{1}{x}$, $g(x) = \frac{x+1}{x+2}$

40. $f(x) = \sqrt{2x+3}$, $g(x) = x^2 + 1$

41-44 III Find $f \circ g \circ h$.

41. $f(x) = x + 1$, $g(x) = 2x$, $h(x) = x - 1$

42. $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = 1 - x$

43. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 2$, $h(x) = x + 3$

44. $f(x) = \frac{2}{x+1}$, $g(x) = \cos x$, $h(x) = \sqrt{x+3}$

45-50 III Express the function in the form $f \circ g$.

45. $F(x) = (x^2 + 1)^{10}$

46. $F(x) = \sin(\sqrt{x})$

47. $G(x) = \frac{x^2}{x^2 + 4}$

48. $G(x) = \frac{1}{x+3}$

49. $u(t) = \sqrt{\cos t}$

50. $u(t) = \frac{\tan t}{1 + \tan t}$

51-53 III Express the function in the form $f \circ g \circ h$.

51. $H(x) = 1 - 3^{x^2}$

52. $H(x) = \sqrt[3]{\sqrt{x-1}}$

53. $H(x) = \sec^4(\sqrt{x})$

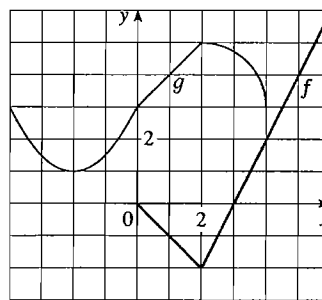
54. Use the table to evaluate each expression.

- (a) $f(g(1))$ (b) $g(f(1))$ (c) $f(f(1))$
 (d) $g(g(1))$ (e) $(g \circ f)(3)$ (f) $(f \circ g)(6)$

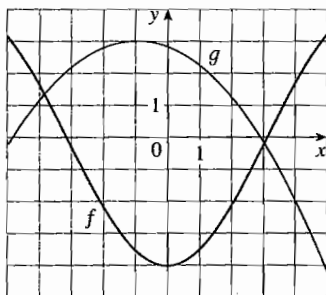
x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

55. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.

- (a) $f(g(2))$ (b) $g(f(0))$ (c) $(f \circ g)(0)$
 (d) $(g \circ f)(6)$ (e) $(g \circ g)(-2)$ (f) $(f \circ f)(4)$



56. Use the given graphs of f and g to estimate the value of $f(g(x))$ for $x = -5, -4, -3, \dots, 5$. Use these estimates to sketch a rough graph of $f \circ g$.



57. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.
- Express the radius r of this circle as a function of the time t (in seconds).
 - If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.
58. An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.
- Express the horizontal distance d (in miles) that the plane has flown as a function of t .
 - Express the distance s between the plane and the radar station as a function of d .
 - Use composition to express s as a function of t .

59. The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

- Sketch the graph of the Heaviside function.

- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.
- Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

60. The Heaviside function defined in Exercise 59 can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit.
- Sketch the graph of the ramp function $y = tH(t)$.
 - Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.
 - Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.
61. (a) If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
- (b) If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.
62. If $f(x) = x + 4$ and $h(x) = 4x - 1$, find a function g such that $g \circ f = h$.
63. Suppose g is an even function and let $h = f \circ g$. Is h always an even function?
64. Suppose g is an odd function and let $h = f \circ g$. Is h always an odd function? What if f is odd? What if f is even?

||| 1.4 Graphing Calculators and Computers

In this section we assume that you have access to a graphing calculator or a computer with graphing software. We will see that the use of such a device enables us to graph more complicated functions and to solve more complex problems than would otherwise be possible. We also point out some of the pitfalls that can occur with these machines.

Graphing calculators and computers can give very accurate graphs of functions. But we will see in Chapter 4 that only through the use of calculus can we be sure that we have uncovered all the interesting aspects of a graph.

A graphing calculator or computer displays a rectangular portion of the graph of a function in a **display window** or **viewing screen**, which we refer to as a **viewing rectangle**. The default screen often gives an incomplete or misleading picture, so it is important to choose the viewing rectangle with care. If we choose the x -values to range from a minimum value of $Xmin = a$ to a maximum value of $Xmax = b$ and the y -values to range from

D Trigonometry

Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360° , which is the same as 2π rad. Therefore



$$\pi \text{ rad} = 180^\circ$$

and



$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \quad 1^\circ = \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad}$$

EXAMPLE 1

- (a) Find the radian measure of 60° . (b) Express $5\pi/4$ rad in degrees.

SOLUTION

(a) From Equation 1 or 2 we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60^\circ = 60 \left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4} \left(\frac{180}{\pi}\right) = 225^\circ$$

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Figure 1 shows a sector of a circle with central angle θ and radius r subtending an arc with length a . Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π , we have

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a , we obtain



$$\theta = \frac{a}{r}$$

$$a = r\theta$$

Remember that Equations 3 are valid only when θ is measured in radians.

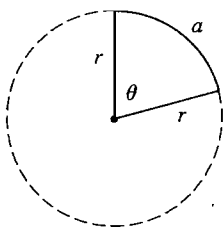


FIGURE 1

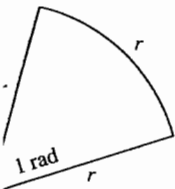


FIGURE 2

In particular, putting $a = r$ in Equation 3, we see that an angle of 1 rad is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle (see Figure 2).

EXAMPLE 2

- (a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
 (b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of $3\pi/8$ rad?

SOLUTION

- (a) Using Equation 3 with $a = 6$ and $r = 5$, we see that the angle is

$$\theta = \frac{6}{5} = 1.2 \text{ rad}$$

- (b) With $r = 3$ cm and $\theta = 3\pi/8$ rad, the arc length is

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8} \text{ cm}$$

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x -axis as in Figure 3. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation as in Figure 4.

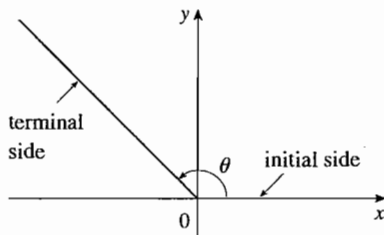
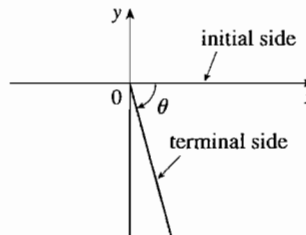
FIGURE 3 $\theta \geq 0$ FIGURE 4 $\theta < 0$

Figure 5 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$, and $11\pi/4$ have the same initial and terminal sides because

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \quad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

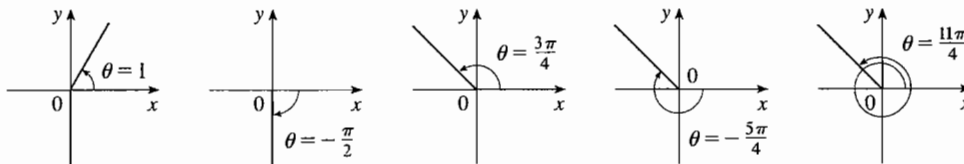


FIGURE 5
Angles in standard position

||| The Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 6).

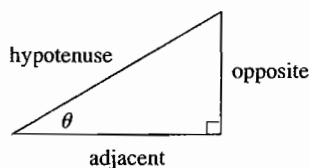


FIGURE 6



$\sin \theta = \frac{\text{opp}}{\text{hyp}}$	$\csc \theta = \frac{\text{hyp}}{\text{opp}}$
$\cos \theta = \frac{\text{adj}}{\text{hyp}}$	$\sec \theta = \frac{\text{hyp}}{\text{adj}}$
$\tan \theta = \frac{\text{opp}}{\text{adj}}$	$\cot \theta = \frac{\text{adj}}{\text{opp}}$

This definition doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 7. Then we define

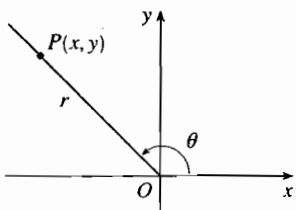


FIGURE 7



$\sin \theta = \frac{y}{r}$	$\csc \theta = \frac{r}{y}$
$\cos \theta = \frac{x}{r}$	$\sec \theta = \frac{r}{x}$
$\tan \theta = \frac{y}{x}$	$\cot \theta = \frac{x}{y}$

Since division by 0 is not defined, $\tan \theta$ and $\sec \theta$ are undefined when $x = 0$ and $\csc \theta$ and $\cot \theta$ are undefined when $y = 0$. Notice that the definitions in (4) and (5) are consistent when θ is an acute angle.

If θ is a number, the convention is that $\sin \theta$ means the sine of the angle whose radian measure is θ . For example, the expression $\sin 3$ implies that we are dealing with an angle of 3 rad. When finding a calculator approximation to this number we must remember to set our calculator in radian mode, and then we obtain

$$\sin 3 \approx 0.14112$$

If we want to know the sine of the angle 3° we would write $\sin 3^\circ$ and, with our calculator in degree mode, we find that

$$\sin 3^\circ \approx 0.05234$$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 8. For instance,

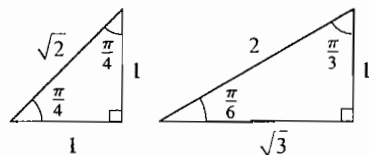


FIGURE 8

$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\sin \frac{\pi}{6} = \frac{1}{2}$	$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\cos \frac{\pi}{3} = \frac{1}{2}$
$\tan \frac{\pi}{4} = 1$	$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$	$\tan \frac{\pi}{3} = \sqrt{3}$

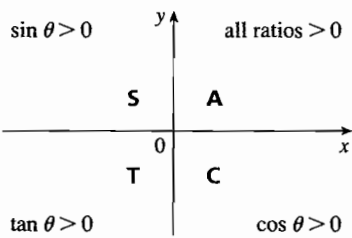


FIGURE 9

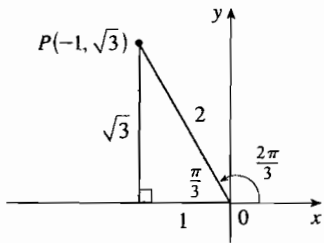


FIGURE 10

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule “All Students Take Calculus” shown in Figure 9.

EXAMPLE 3 Find the exact trigonometric ratios for $\theta = 2\pi/3$.

SOLUTION From Figure 10 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P(-1, \sqrt{3})$. Therefore, taking

$$x = -1 \quad y = \sqrt{3} \quad r = 2$$

in the definitions of the trigonometric ratios, we have

$$\begin{aligned} \sin \frac{2\pi}{3} &= \frac{\sqrt{3}}{2} & \cos \frac{2\pi}{3} &= -\frac{1}{2} & \tan \frac{2\pi}{3} &= -\sqrt{3} \\ \csc \frac{2\pi}{3} &= \frac{2}{\sqrt{3}} & \sec \frac{2\pi}{3} &= -2 & \cot \frac{2\pi}{3} &= -\frac{1}{\sqrt{3}} \end{aligned}$$

The following table gives some values of $\sin \theta$ and $\cos \theta$ found by the method of Example 3.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

EXAMPLE 4 If $\cos \theta = \frac{2}{5}$ and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ .

SOLUTION Since $\cos \theta = \frac{2}{5}$, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 11. If the opposite side has length x , then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$. We can now use the diagram to write the other five trigonometric functions:

$$\begin{aligned} \sin \theta &= \frac{\sqrt{21}}{5} & \tan \theta &= \frac{\sqrt{21}}{2} \\ \csc \theta &= \frac{5}{\sqrt{21}} & \sec \theta &= \frac{5}{2} & \cot \theta &= \frac{2}{\sqrt{21}} \end{aligned}$$

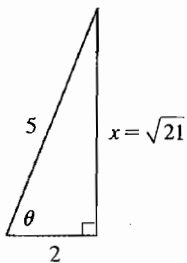


FIGURE 11

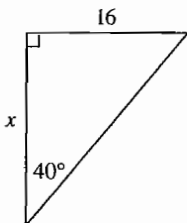


FIGURE 12

EXAMPLE 5 Use a calculator to approximate the value of x in Figure 12.

SOLUTION From the diagram we see that

$$\tan 40^\circ = \frac{16}{x}$$

Therefore

$$x = \frac{16}{\tan 40^\circ} \approx 19.07$$

||| Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following, which are immediate consequences of the definitions of the trigonometric functions.

$$\begin{aligned} \text{6} \quad \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} \\ \tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

For the next identity we refer back to Figure 7. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$\sin^2 \theta + \cos^2 \theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\text{7} \quad \sin^2 \theta + \cos^2 \theta = 1$$

If we now divide both sides of Equation 7 by $\cos^2 \theta$ and use Equations 6, we get

$$\text{8} \quad \tan^2 \theta + 1 = \sec^2 \theta$$

Similarly, if we divide both sides of Equation 7 by $\sin^2 \theta$, we get

$$\text{9} \quad 1 + \cot^2 \theta = \csc^2 \theta$$

The identities

$$\text{10a} \quad \sin(-\theta) = -\sin \theta$$

$$\text{10b} \quad \cos(-\theta) = \cos \theta$$

||| Odd functions and even functions are discussed in Section 1.1.

show that \sin is an odd function and \cos is an even function. They are easily proved by drawing a diagram showing θ and $-\theta$ in standard position (see Exercise 39).

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have

$$\text{11} \quad \sin(\theta + 2\pi) = \sin \theta \quad \cos(\theta + 2\pi) = \cos \theta$$

These identities show that the sine and cosine functions are periodic with period 2π .

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

12a

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

12b

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

The proofs of these addition formulas are outlined in Exercises 85, 86, and 87.

By substituting $-y$ for y in Equations 12a and 12b and using Equations 10a and 10b, we obtain the following **subtraction formulas**:

13a

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

13b

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in Equations 12 or Equations 13, we obtain the corresponding formulas for $\tan(x \pm y)$:

14a

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

14b

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas (12), we get the **double-angle formulas**:

15a

$$\sin 2x = 2 \sin x \cos x$$

15b

$$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

16a

$$\cos 2x = 2 \cos^2 x - 1$$

16b

$$\cos 2x = 1 - 2 \sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

17a

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

17b

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from Equations 12 and 13:

12a

$$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$$

12b

$$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$

12c

$$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus. If you forget any of them, remember that they can all be deduced from Equations 12a and 12b.

EXAMPLE 6 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.

SOLUTION Using the double-angle formula (15a), we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \quad \text{or} \quad \sin x(1 - 2 \cos x) = 0$$

Therefore, there are two possibilities:

$$\sin x = 0 \quad \text{or} \quad 1 - 2 \cos x = 0$$

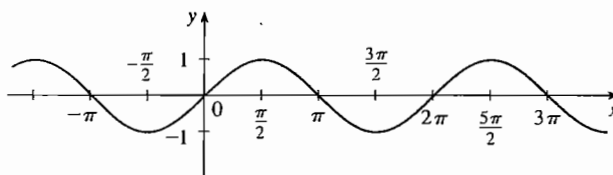
$$x = 0, \pi, 2\pi \quad \cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

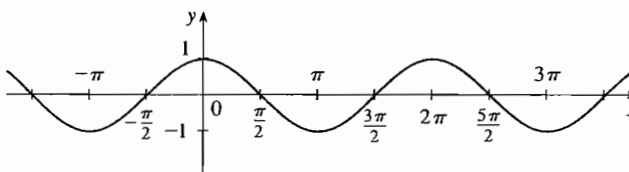
The given equation has five solutions: $0, \pi/3, \pi, 5\pi/3,$ and 2π .

Graphs of Trigonometric Functions

The graph of the function $f(x) = \sin x$, shown in Figure 13(a), is obtained by plotting points for $0 \leq x \leq 2\pi$ and then using the periodic nature of the function (from Equation 11) to complete the graph. Notice that the zeros of the sine function occur at the



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

FIGURE 13

integer multiples of π , that is,

$$\sin x = 0 \quad \text{whenever } x = n\pi, \quad n \text{ an integer}$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

(which can be verified using Equation 12a), the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 13(b)]. Note that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x , we have

$$-1 \leq \sin x \leq 1 \quad -1 \leq \cos x \leq 1$$

The graphs of the remaining four trigonometric functions are shown in Figure 14 and their domains are indicated there. Notice that tangent and cotangent have range $(-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π , whereas cosecant and secant have period 2π .

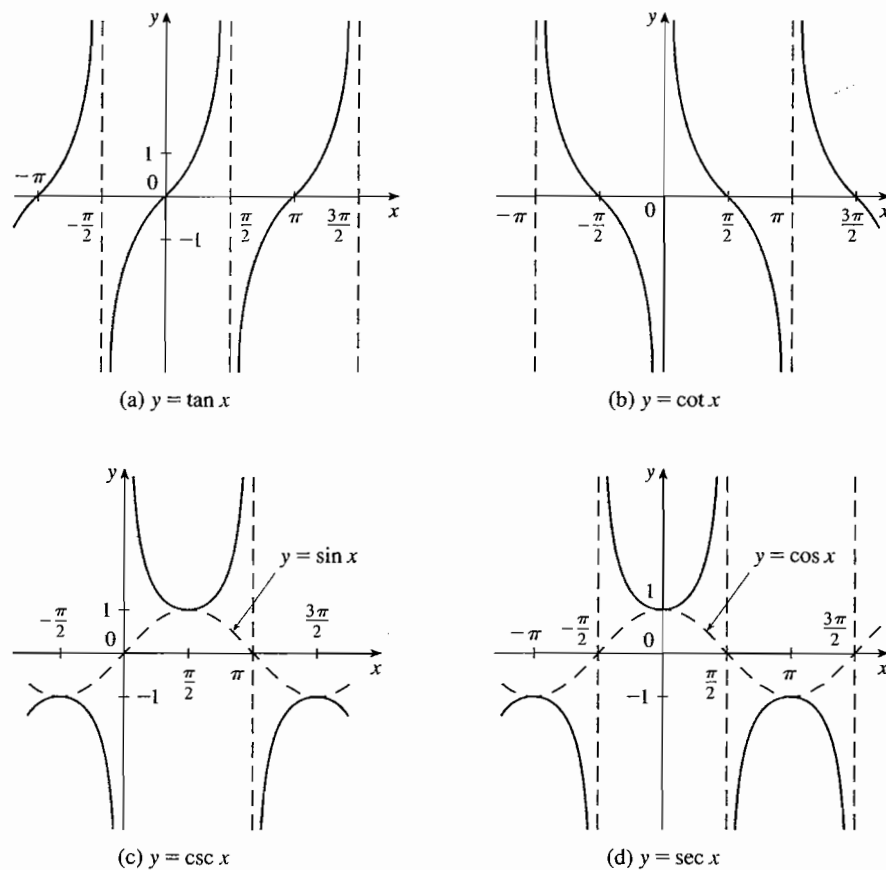


FIGURE 14

D Exercises

1-6 Convert from degrees to radians.

1. 210° 2. 300° 3. 9°
 4. -315° 5. 900° 6. 36°

7-12 Convert from radians to degrees.

7. 4π 8. $-\frac{7\pi}{2}$ 9. $\frac{5\pi}{12}$
 10. $\frac{8\pi}{3}$ 11. $-\frac{3\pi}{8}$ 12. 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.
 14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of 72° .
 15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?
 16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17-22 Draw, in standard position, the angle whose measure is given.

17. 315° 18. -150° 19. $-\frac{3\pi}{4}$ rad
 20. $\frac{7\pi}{3}$ rad 21. 2 rad 22. -3 rad

23-28 Find the exact trigonometric ratios for the angle whose radian measure is given.

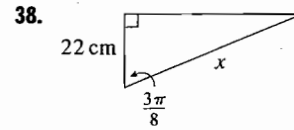
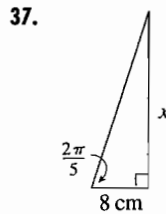
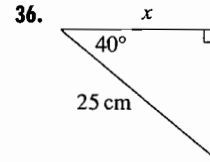
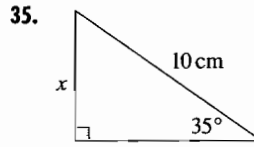
23. $\frac{3\pi}{4}$ 24. $\frac{4\pi}{3}$ 25. $\frac{9\pi}{2}$
 26. -5π 27. $\frac{5\pi}{6}$ 28. $\frac{11\pi}{4}$

29-34 Find the remaining trigonometric ratios.

29. $\sin \theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$
 30. $\tan \alpha = 2$, $0 < \alpha < \frac{\pi}{2}$
 31. $\sec \phi = -1.5$, $\frac{\pi}{2} < \phi < \pi$
 32. $\cos x = -\frac{1}{3}$, $\pi < x < \frac{3\pi}{2}$
 33. $\cot \beta = 3$, $\pi < \beta < 2\pi$

34. $\csc \theta = -\frac{4}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$

35-38 Find, correct to five decimal places, the length of the side labeled x .



39-41 Prove each equation.

39. (a) Equation 10a (b) Equation 10b
 40. (a) Equation 14a (b) Equation 14b
 41. (a) Equation 18a (b) Equation 18b
 (c) Equation 18c

42-58 Prove the identity.

42. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
 43. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$ 44. $\sin(\pi - x) = \sin x$
 45. $\sin \theta \cot \theta = \cos \theta$ 46. $(\sin x + \cos x)^2 = 1 + \sin 2x$
 47. $\sec y - \cos y = \tan y \sin y$
 48. $\tan^2 \alpha - \sin^2 \alpha = \tan^2 \alpha \sin^2 \alpha$
 49. $\cot^2 \theta + \sec^2 \theta = \tan^2 \theta + \csc^2 \theta$
 50. $2 \csc 2t = \sec t \csc t$ 51. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
 52. $\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta$
 53. $\sin x \sin 2x + \cos x \cos 2x = \cos x$
 54. $\sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y)$
 55. $\frac{\sin \phi}{1 - \cos \phi} = \csc \phi + \cot \phi$

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56. $\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}$

57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

59-64 ||| If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x+y)$ 60. $\cos(x+y)$

61. $\cos(x-y)$ 62. $\sin(x-y)$

63. $\sin 2y$ 64. $\cos 2y$

65-72 ||| Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$ 66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$ 68. $|\tan x| = 1$

69. $\sin 2x = \cos x$ 70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$ 72. $2 + \cos 2x = 3 \cos x$

73-76 ||| Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leq \frac{1}{2}$ 74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$ 76. $\sin x > \cos x$

77-82 ||| Graph the function by starting with the graphs in Figures 13 and 14 and applying the transformations of Section 1.3 where appropriate.

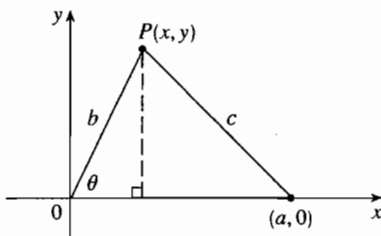
77. $y = \cos\left(x - \frac{\pi}{3}\right)$ 78. $y = \tan 2x$

79. $y = \frac{1}{3} \tan\left(x - \frac{\pi}{2}\right)$ 80. $y = 1 + \sec x$

81. $y = |\sin x|$ 82. $y = 2 + \sin\left(x + \frac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths a , b , and c , and θ is the angle between the sides with lengths a and b , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

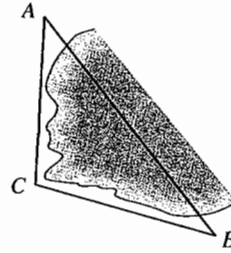


[Hint: Introduce a coordinate system so that θ is in standard position as in the figure. Express x and y in terms of θ and then use the distance formula to compute c .]

84. In order to find the distance $|AB|$ across a small inlet, a point C is located as in the figure and the following measurements were recorded:

$$\angle C = 103^\circ \quad |AC| = 820 \text{ m} \quad |BC| = 910 \text{ m}$$

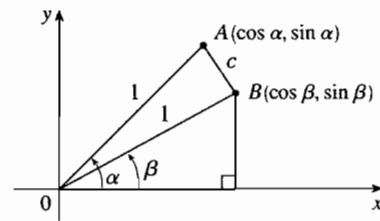
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

[Hint: Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \quad \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$$

to prove the subtraction formula for the sine function.

88. Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \frac{1}{2} ab \sin \theta$$

89. Find the area of triangle ABC , correct to five decimal places, if

$$|AB| = 10 \text{ cm} \quad |BC| = 3 \text{ cm} \quad \angle ABC = 107^\circ$$