In *A Preview of Calculus* (page 2) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." Thus, a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. How can this idea be made precise?

Locate tangents interactively and explore them numerically.
Resources / Module 1
/ Tangents
/ What Is a Tangent?

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows two lines l and t passing through a point P on a curve C. The line l intersects C only once, but it certainly does not look like what we think of as a tangent. The line t, on the other hand, looks like a tangent but it intersects C twice.

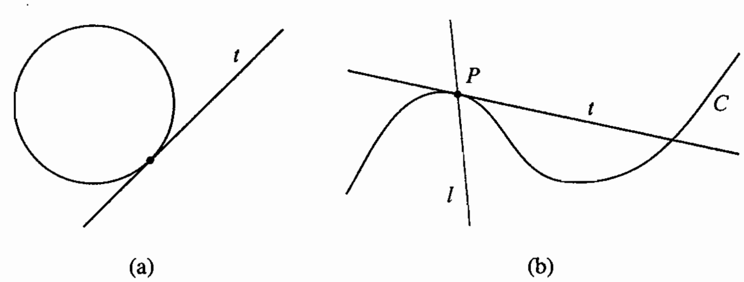


FIGURE 1

To be specific, let's look at the problem of trying to find a tangent line t to the parabola $y = x^2$ in the following example.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

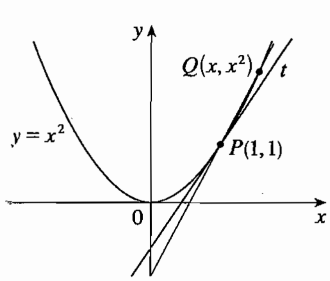


FIGURE 2

SOLUTION We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line t should be $m = 2$.

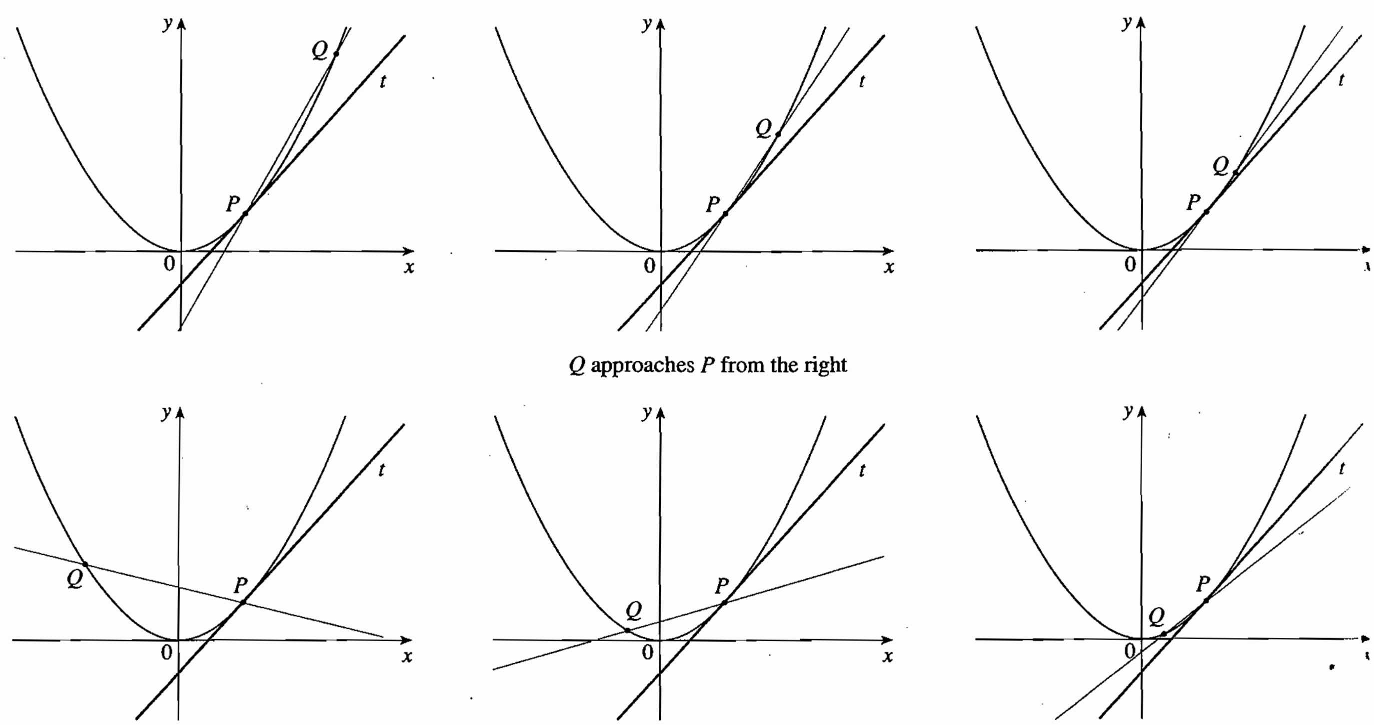
We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line (see Appendix B) to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Figure 3 illustrates the limiting process that occurs in this example. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.



Q approaches P from the right

Q approaches P from the left

FIGURE 3

In Module 2.1 you can see how the process in Figure 3 works for five additional functions.

Many functions that occur in science are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

EXAMPLE 2 The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data at the left describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

SOLUTION In Figure 4 we plot the given data and use them to sketch a curve that approximates the graph of the function.

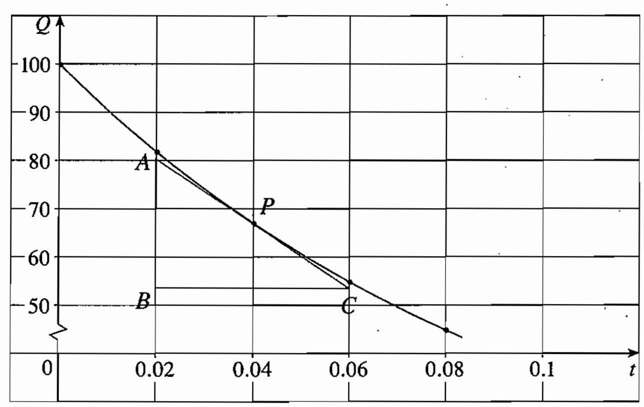


FIGURE 4

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675.

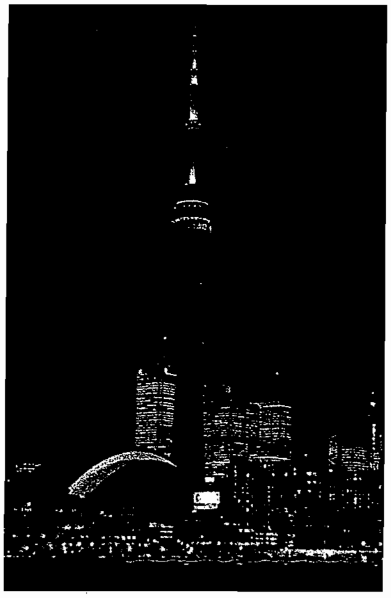
Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 4. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about −670 microamperes.

The Velocity Problem

If you watch the speedometer of a car as you travel in city traffic, you see that the needle doesn't stay still for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined? Let's investigate the example of a falling ball.



The CN Tower in Toronto is currently the tallest freestanding building in the world.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leq t \leq 6$	53.9
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus, the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the problem of finding velocities. If we draw the graph of the distance function of the ball (as in Figure 5) and we consider the points $P(a, 4.9a^2)$ and $Q(a + h, 4.9(a + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(a + h)^2 - 4.9a^2}{(a + h) - a}$$

which is the same as the average velocity over the time interval $[a, a + h]$. Therefore, the velocity at time $t = a$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

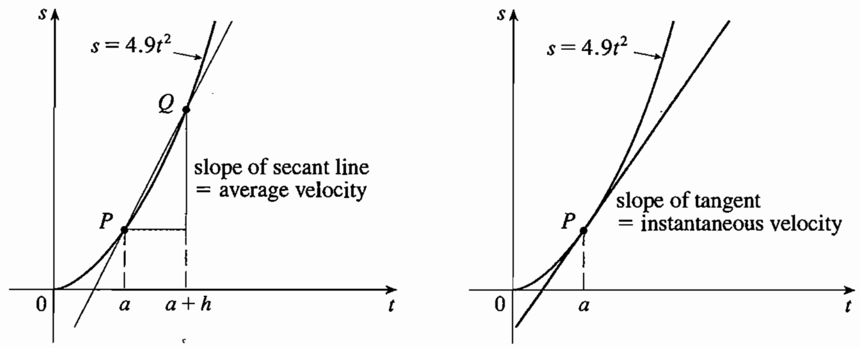


FIGURE 5

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next five sections, we will return to the problems of finding tangents and velocities in Section 2.7.

2.1 Exercises

1. A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5$, 10, 20, 25, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$

What are your conclusions?

3. The point $P(1, \frac{1}{2})$ lies on the curve $y = x/(1 + x)$.

(a) If Q is the point $(x, x/(1 + x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 0.5 (ii) 0.9
(iii) 0.99 (iv) 0.999
(v) 1.5 (vi) 1.1
(vii) 1.01 (viii) 1.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(1, \frac{1}{2})$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(1, \frac{1}{2})$.

4. The point $P(2, \ln 2)$ lies on the curve $y = \ln x$.

(a) If Q is the point $(x, \ln x)$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 1.5 (ii) 1.9
(iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1
(vii) 2.01 (viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, \ln 2)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, \ln 2)$.
(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 40 ft/s, its height in feet after t seconds is given by $y = 40t - 16t^2$.
(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second
(b) Find the instantaneous velocity when $t = 2$.

6. If an arrow is shot upward on the moon with a velocity of 58 m/s, its height in meters after t seconds is given by $h = 58t - 0.83t^2$.
(a) Find the average velocity over the given time intervals:
(i) $[1, 2]$ (ii) $[1, 1.5]$ (iii) $[1, 1.1]$
(iv) $[1, 1.01]$ (v) $[1, 1.001]$
(b) Find the instantaneous velocity after one second.

7. The displacement (in feet) of a certain particle moving in a straight line is given by $s = t^3/6$, where t is measured in seconds.
(a) Find the average velocity over the following time periods:
(i) $[1, 3]$ (ii) $[1, 2]$
(iii) $[1, 1.5]$ (iv) $[1, 1.1]$
(b) Find the instantaneous velocity when $t = 1$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities found in part (a).
(d) Draw the tangent line whose slope is the instantaneous velocity from part (b).

8. The position of a car is given by the values in the table.

t (seconds)	0	1	2	3	4	5
s (feet)	0	10	32	70	119	178

(a) Find the average velocity for the time period beginning when $t = 2$ and lasting
(i) 3 seconds (ii) 2 seconds (iii) 1 second
(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 2$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.
(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9. Do the slopes appear to be approaching a limit?
(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.
(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function f defined by $f(x) = x^2 - x + 2$ for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2, but not equal to 2.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

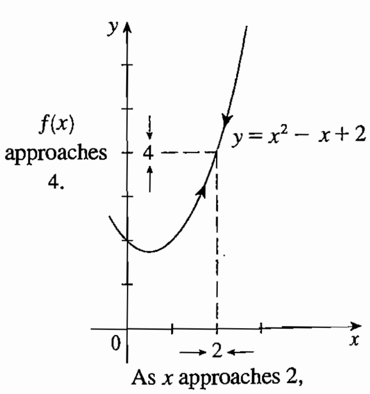


FIGURE 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

1 Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. A more precise definition will be given in Section 2.4.

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.

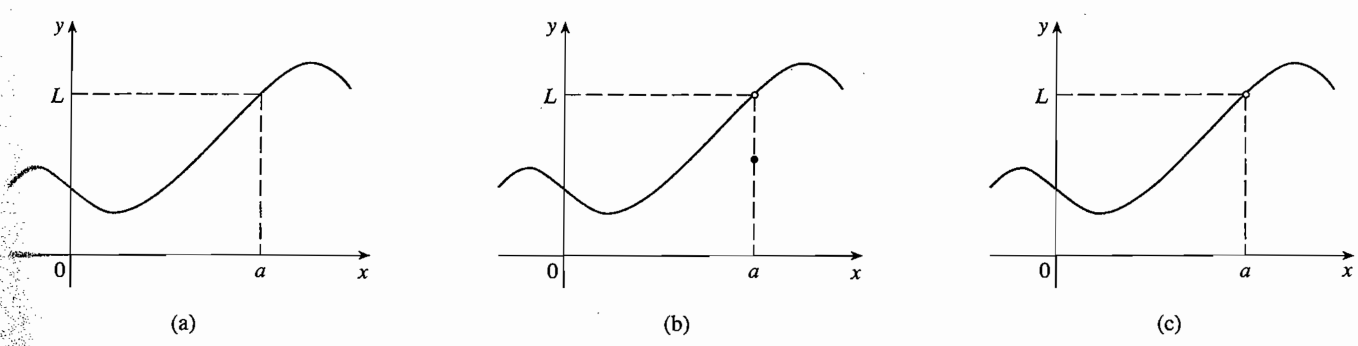


FIGURE 2 $\lim_{x \to a} f(x) = L$ in all three cases

EXAMPLE 1 Guess the value of $\displaystyle\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$.

SOLUTION Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

of x that are close to a but not equal to a. The tables at the left give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the tables, we make the guess that

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

Example 1 is illustrated by the graph of f in Figure 3. Now let's change f slightly by giving it the value 2 when $x = 1$ and calling the resulting function g:

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

This new function g still has the same limit as x approaches 1 (see Figure 4).

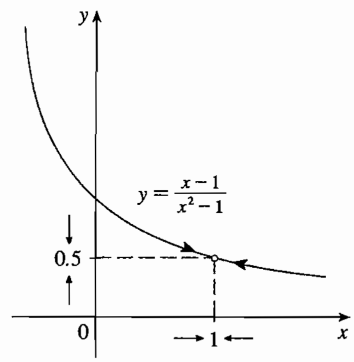


FIGURE 3

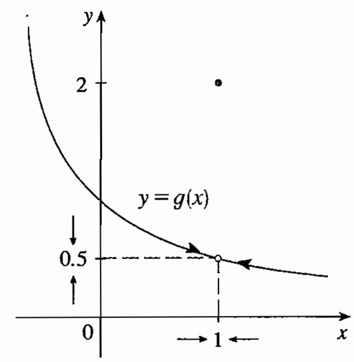


FIGURE 4

EXAMPLE 2 Estimate the value of $\lim_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667

As t approaches 0, the values of the function seem to approach $0.1666666\ldots$ and so we guess that

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

In Example 2 what would have happened if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

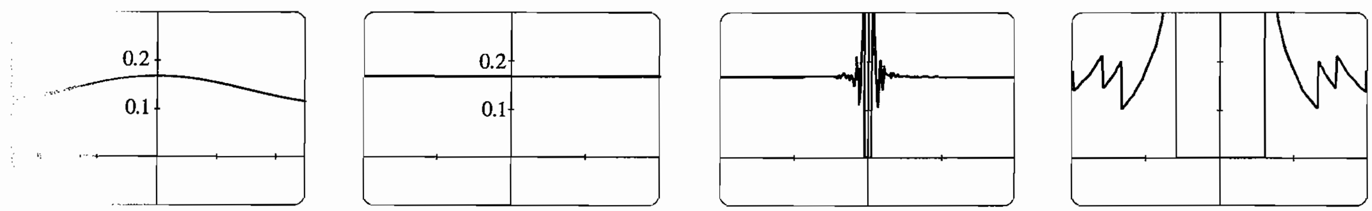
If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000... to as many digits as the calculator is capable of carrying.)

For a further explanation of why calculators sometimes give false values, see the web site www.stewartcalculus.com. Click on *Additional Topics* and then on *Lies My Calculator and Computer Told Me*. In particular, see the section called The Perils of Subtraction.

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

of Example 2 on a graphing calculator or computer. Parts (a) and (b) of Figure 5 show quite accurate graphs of f, and when we use the trace mode (if available) we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too far, as in parts (c) and (d), then we get inaccurate graphs, again because of problems with subtraction.



(a) $[-5, 5]$ by $[-0.1, 0.3]$ (b) $[-0.1, 0.1]$ by $[-0.1, 0.3]$ (c) $[-10^{-6}, 10^{-6}]$ by $[-0.1, 0.3]$ (d) $[-10^{-7}, 10^{-7}]$ by $[-0.1, 0.3]$

FIGURE 5

EXAMPLE 3 Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct the following table of values correct to eight decimal places. From the table and the graph in Figure 6 we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 3 using a geometric argument.

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

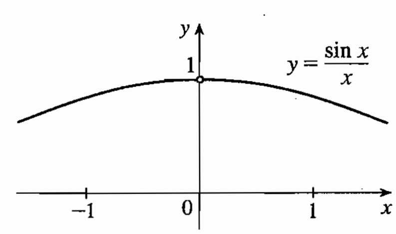


FIGURE 6

|||| COMPUTER ALGEBRA SYSTEMS
Computer algebra systems (CAS) have commands that compute limits. In order to avoid the types of pitfalls demonstrated in Examples 2, 4, and 5, they don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. If you have access to a CAS, use the limit command to compute the limits in the examples of this section and to check your answers in the exercises of this chapter.

EXAMPLE 4 Investigate $\lim_{x\to 0} \sin\dfrac{\pi}{x}$.

SOLUTION Again the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin\pi = 0 \qquad f(\tfrac{1}{2}) = \sin 2\pi = 0$$
$$f(\tfrac{1}{3}) = \sin 3\pi = 0 \qquad f(\tfrac{1}{4}) = \sin 4\pi = 0$$
$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x\to 0} \sin\frac{\pi}{x} = 0$$

but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. [In fact, $\sin(\pi/x) = 1$ when

$$\frac{\pi}{x} = \frac{\pi}{2} + 2n\pi$$

and, solving for x, we get $x = 2/(4n + 1)$.] The graph of f is given in Figure 7.

Listen to the sound of this function trying to approach a limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Does Not Exist

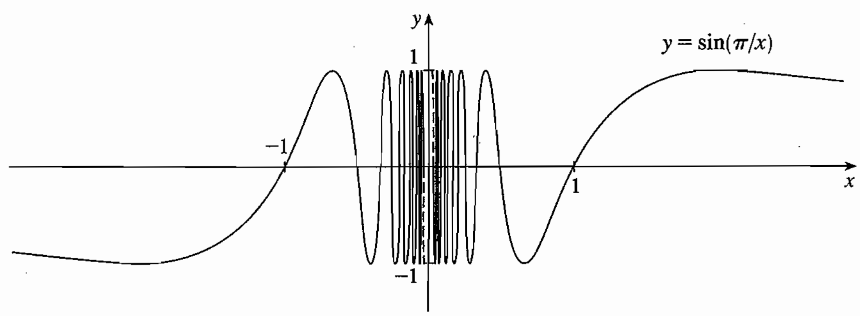


FIGURE 7

Module 2.2 helps you explore limits at points where graphs exhibit unusual behavior.

The dashed lines indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0 (see Exercise 37). Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x\to 0} \sin\frac{\pi}{x} \text{ does not exist}$$

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

EXAMPLE 5 Find $\lim_{x\to 0} \left(x^3 + \dfrac{\cos 5x}{10{,}000}\right)$.

SOLUTION As before, we construct a table of values. From the table in the margin it appears that

$$\lim_{x\to 0} \left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0$$

x	$x^3 + \dfrac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

But if we persevere with smaller values of x, the second table suggests that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.000100 = \frac{1}{10{,}000}$$

Later we will see that $\lim_{x \to 0} \cos 5x = 1$; then it follows that the limit is 0.0001.

Examples 4 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give the wrong values. In the next two sections, however, we will develop foolproof methods for calculating limits.

EXAMPLE 6 The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

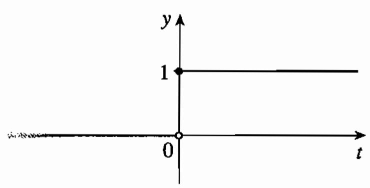


FIGURE 8

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 8.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore, $\lim_{t \to 0} H(t)$ does not exist.

||| One-Sided Limits

We noticed in Example 6 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0. Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

2 Definition We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus, the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.

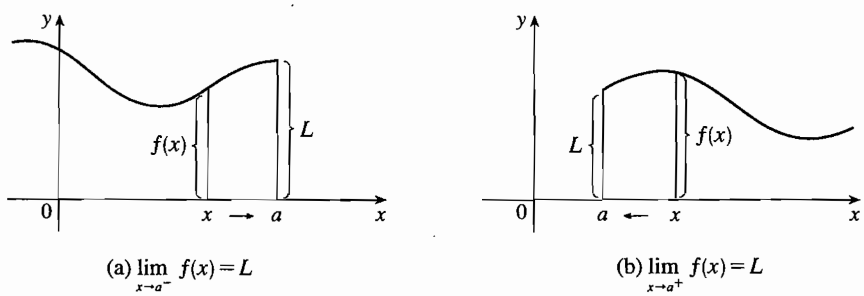


FIGURE 9

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

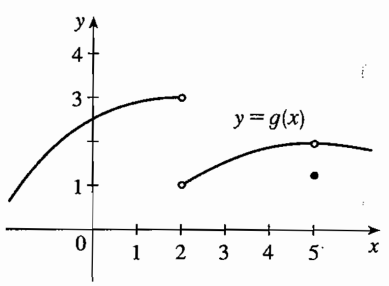


FIGURE 10

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

SOLUTION From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)} \lim_{x \to 2^-} g(x) = 3 \quad \text{and} \quad \text{(b)} \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \to 2} g(x)$ does not exist.

The graph also shows that

$$\text{(d)} \lim_{x \to 5^-} g(x) = 2 \quad \text{and} \quad \text{(e)} \lim_{x \to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

|||| Infinite Limits

EXAMPLE 8 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table on the next page.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 11 that the values of $f(x)$ can be made arbitrarily large

x	$\frac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000

by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x\to 0} (1/x^2)$ does not exist.

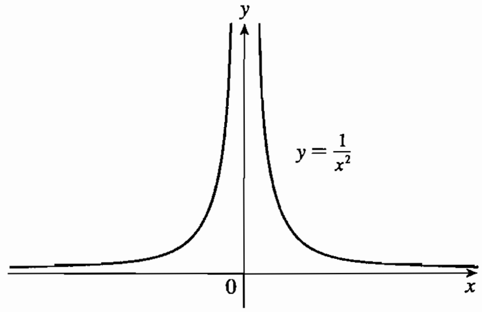


FIGURE 11

To indicate the kind of behavior exhibited in Example 8, we use the notation

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

⊘ This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x\to a} f(x) = \infty$$

to indicate that the values of $f(x)$ become larger and larger (or "increase without bound") as x becomes closer and closer to a.

Explore infinite limits interactively.
Resources / Module 2
/ Limits that Are Infinite
/ Examples A and B

> **4 Definition** Let f be a function defined on both sides of a, except possibly at a itself. Then
>
> $$\lim_{x\to a} f(x) = \infty$$
>
> means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x\to a} f(x) = \infty$ is

$$f(x) \to \infty \qquad \text{as} \qquad x \to a$$

Again the symbol ∞ is not a number, but the expression $\lim_{x\to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 12.

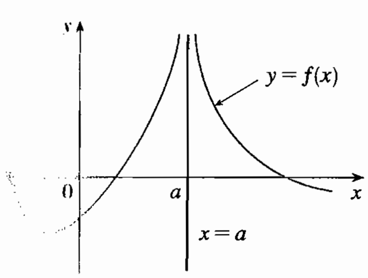


FIGURE 12

$\lim_{x\to a} f(x) = \infty$

A similar sort of limit, for functions that become large negative as x gets close to a, is defined in Definition 5 and is illustrated in Figure 13.

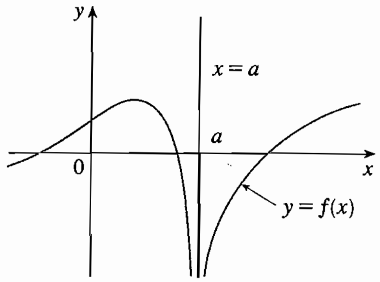


FIGURE 13
$\lim_{x \to a} f(x) = -\infty$

5 Definition Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

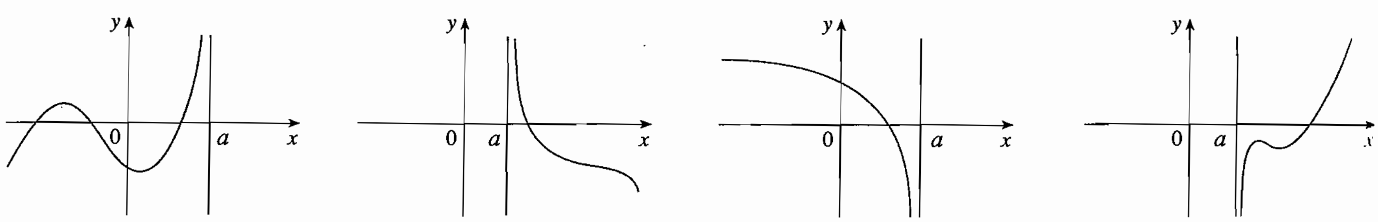
$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of x that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$. Illustrations of these four cases are given in Figure 14.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 14

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 14 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 9 Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$.

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. Thus, intuitively we see that

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

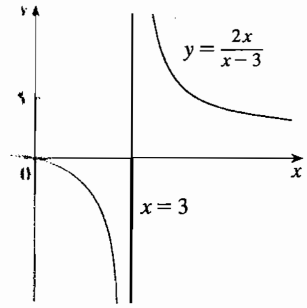


FIGURE 15

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote.

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

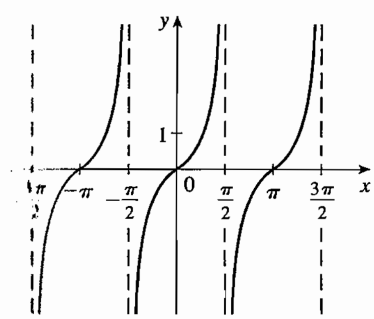


FIGURE 16
$y = \tan x$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this.

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$. From Figure 17 we see that

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_a x$ provided that $a > 1$. (See Figures 11 and 12 in Section 1.6.)

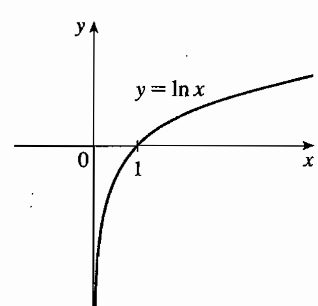


FIGURE 17
The y-axis is a vertical asymptote of the natural logarithmic function.

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

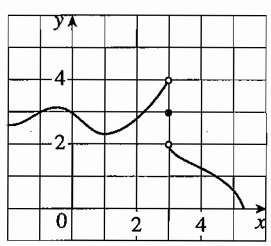
2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

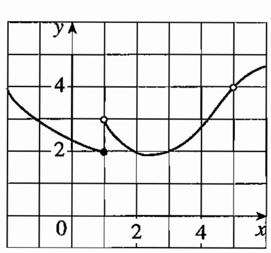
In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

3. Explain the meaning of each of the following.
(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

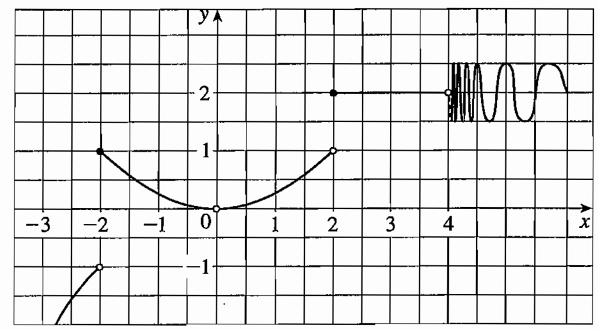
4. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 3^-} f(x)$
(c) $\lim_{x \to 3^+} f(x)$ (d) $\lim_{x \to 3} f(x)$
(e) $f(3)$



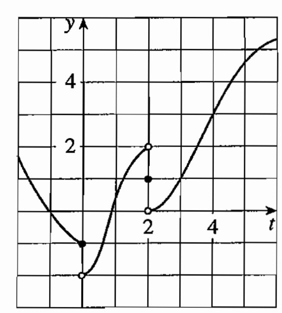
5. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$ (c) $\lim_{x \to 1} f(x)$
(d) $\lim_{x \to 5} f(x)$ (e) $f(5)$



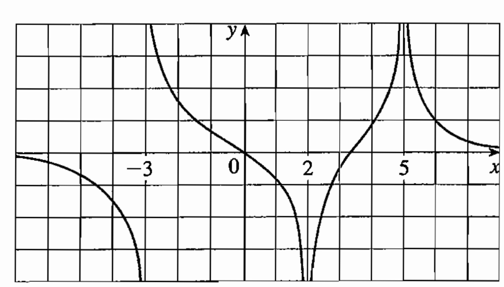
6. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{x \to -2^-} g(x)$ (b) $\lim_{x \to -2^+} g(x)$ (c) $\lim_{x \to -2} g(x)$
(d) $g(-2)$ (e) $\lim_{x \to 2^-} g(x)$ (f) $\lim_{x \to 2^+} g(x)$
(g) $\lim_{x \to 2} g(x)$ (h) $g(2)$ (i) $\lim_{x \to 4^+} g(x)$
(j) $\lim_{x \to 4^-} g(x)$ (k) $g(0)$ (l) $\lim_{x \to 0} g(x)$



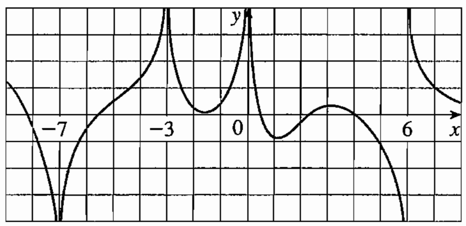
7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.
(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$
(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$
(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$



8. For the function R whose graph is shown, state the following.
(a) $\lim_{x \to 2} R(x)$ (b) $\lim_{x \to 5} R(x)$
(c) $\lim_{x \to -3^-} R(x)$ (d) $\lim_{x \to -3^+} R(x)$
(e) The equations of the vertical asymptotes.



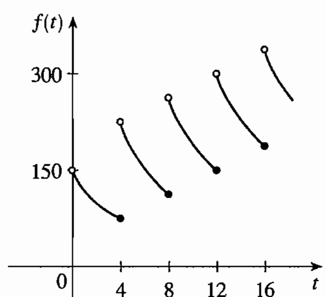
9. For the function f whose graph is shown, state the following.
(a) $\lim_{x\to -7} f(x)$ (b) $\lim_{x\to -3} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 6^-} f(x)$ (e) $\lim_{x\to 6^+} f(x)$
(f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. (Later we will be able to compute the dosage and time interval to ensure that the concentration of the drug does not reach a harmful level.) Find

$$\lim_{t\to 12^-} f(t) \quad \text{and} \quad \lim_{t\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11. Use the graph of the function $f(x) = 1/(1 + e^{1/x})$ to state the value of each limit, if it exists. If it does not exist, explain why.
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

12. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x-1)^2 & \text{if } x \geqslant 1 \end{cases}$$

13–14 |||| Sketch the graph of an example of a function f that satisfies all of the given conditions.

13. $\lim_{x\to 3^+} f(x) = 4$, $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$,
$f(3) = 3$, $f(-2) = 1$

14. $\lim_{x\to 0^-} f(x) = 1$, $\lim_{x\to 0^+} f(x) = -1$, $\lim_{x\to 2^-} f(x) = 0$
$\lim_{x\to 2^+} f(x) = 1$, $f(2) = 1$, $f(0)$ is undefined

15–18 |||| Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

15. $\lim_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2}$, $x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
$1.9, 1.95, 1.99, 1.995, 1.999$

16. $\lim_{x\to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$, $x = 0, -0.5, -0.9, -0.95, -0.99,$
$-0.999, -2, -1.5, -1.1, -1.01, -1.001$

17. $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$, $x = \pm 1, \pm 0.5, \pm 0.1, \pm 0.05, \pm 0.01$

18. $\lim_{x\to 0^+} x \ln(x + x^2)$, $x = 1, 0.5, 0.1, 0.05, 0.01, 0.005, 0.001$

19–22 |||| Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

19. $\lim_{x\to 0} \dfrac{\sqrt{x+4} - 2}{x}$

20. $\lim_{x\to 0} \dfrac{\tan 3x}{\tan 5x}$

21. $\lim_{x\to 1} \dfrac{x^6 - 1}{x^{10} - 1}$

22. $\lim_{x\to 0} \dfrac{9^x - 5^x}{x}$

23–30 |||| Determine the infinite limit.

23. $\lim_{x\to 5^+} \dfrac{6}{x - 5}$

24. $\lim_{x\to 5^-} \dfrac{6}{x - 5}$

25. $\lim_{x\to 1} \dfrac{2 - x}{(x - 1)^2}$

26. $\lim_{x\to 0} \dfrac{x - 1}{x^2(x + 2)}$

27. $\lim_{x\to -2^+} \dfrac{x - 1}{x^2(x + 2)}$

28. $\lim_{x\to \pi^-} \csc x$

29. $\lim_{x\to (-\pi/2)^-} \sec x$

30. $\lim_{x\to 5^+} \ln(x - 5)$

31. Determine $\lim_{x\to 1^-} \dfrac{1}{x^3 - 1}$ and $\lim_{x\to 1^+} \dfrac{1}{x^3 - 1}$
(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,
(b) by reasoning as in Example 9, and
(c) from a graph of f.

32. (a) Find the vertical asymptotes of the function

$$y = \frac{x}{x^2 - x - 2}$$

(b) Confirm your answer to part (a) by graphing the function.

33. (a) Estimate the value of the limit $\lim_{x\to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?
(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

34. The slope of the tangent line to the graph of the exponential function $y = 2^x$ at the point $(0, 1)$ is $\lim_{x\to 0} (2^x - 1)/x$. Estimate the slope to three decimal places.

35. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1$, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05, and guess the value of

$$\lim_{x \to 0} \left(x^2 - \frac{2^x}{1000} \right)$$

(b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001. Guess again.

36. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1, 0.5, 0.1, 0.05$, 0.01, and 0.005.

(b) Guess the value of $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach 0 values for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.4 a method for evaluating the limit will be explained.)

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

37. Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

38. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light. What happens as $v \to c^-$?

39. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \qquad -\pi \leq x \leq \pi$$

Then find the exact equations of these asymptotes.

40. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

(b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

2.3 Calculating Limits Using the Limit Laws

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law
1. The limit of a sum is the sum of the limits.

Difference Law
2. The limit of a difference is the difference of the limits.

Constant Multiple Law
3. The limit of a constant times a function is the constant times the limit of the function.

Product Law
4. The limit of a product is the product of the limits.

Quotient Law
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 2.4 we give a precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

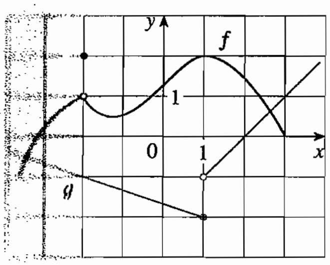


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2} [f(x) + 5g(x)]$ (b) $\lim_{x\to 1} [f(x)g(x)]$ (c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x\to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to -2} g(x) = -1$$

Therefore, we have

$$\begin{aligned}\lim_{x\to -2} [f(x) + 5g(x)] &= \lim_{x\to -2} f(x) + \lim_{x\to -2} [5g(x)] && \text{(by Law 1)}\\ &= \lim_{x\to -2} f(x) + 5 \lim_{x\to -2} g(x) && \text{(by Law 3)}\\ &= 1 + 5(-1) = -4\end{aligned}$$

(b) We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4. The given limit does not exist, since the left limit is not equal to the right limit.

(c) The graphs show that

$$\lim_{x\to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x\to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law
6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$	**8.** $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 2.4.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 37 in Section 2.4.)

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law, which is proved as a consequence of Law 10 in Section 2.5.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Explore limits like these interactively.
Resources / Module 2
/ The Essential Examples
/ Examples D and E

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)

$$\begin{aligned} \lim_{x \to 5} (2x^2 - 3x + 4) &= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 && \text{(by Laws 2 and 1)} \\ &= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{(by 3)} \\ &= 2(5^2) - 3(5) + 4 && \text{(by 9, 8, and 7)} \\ &= 39 \end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x}=\frac{\lim\limits_{x\to-2}(x^3+2x^2-1)}{\lim\limits_{x\to-2}(5-3x)} \qquad \text{(by Law 5)}$$

$$=\frac{\lim\limits_{x\to-2}x^3+2\lim\limits_{x\to-2}x^2-\lim\limits_{x\to-2}1}{\lim\limits_{x\to-2}5-3\lim\limits_{x\to-2}x} \qquad \text{(by 1, 2, and 3)}$$

$$=\frac{(-2)^3+2(-2)^2-1}{5-3(-2)} \qquad \text{(by 9, 8, and 7)}$$

$$=-\frac{1}{11}$$

NOTE ◦ If we let $f(x)=2x^2-3x+4$, then $f(5)=39$. In other words, we would have gotten the correct answer in Example 2(a) by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 53 and 54). We state this fact as follows.

> **Direct Substitution Property** If f is a polynomial or a rational function and a is in the domain of f, then
>
> $$\lim_{x\to a}f(x)=f(a)$$

Functions with the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated by direct substitution, as the following examples show.

EXAMPLE 3 Find $\lim\limits_{x\to1}\dfrac{x^2-1}{x-1}$.

SOLUTION Let $f(x)=(x^2-1)/(x-1)$. We can't find the limit by substituting $x=1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

The numerator and denominator have a common factor of $x-1$. When we take the limit as x approaches 1, we have $x\neq1$ and so $x-1\neq0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to1}\frac{x^2-1}{x-1}=\lim_{x\to1}\frac{(x-1)(x+1)}{x-1}$$

$$=\lim_{x\to1}(x+1)$$

$$=1+1=2$$

The limit in this example arose in Section 2.1 when we were trying to find the tangent to the parabola $y=x^2$ at the point $(1, 1)$.

[illegible]WTON AND LIMITS

[illegible] Newton was born on Christmas Day in [illegible] the year of Galileo's death. When he [illegible] Cambridge University in 1661 Newton [illegible] know much mathematics, but he learned [illegible] by reading Euclid and Descartes and [illegible]tending the lectures of Isaac Barrow. [illegible] was closed because of the plague in [illegible] and 1666, and Newton returned home to [illegible] on what he had learned. Those two years [illegible] amazingly productive for at that time he [illegible] four of his major discoveries: (1) his [illegible]ntation of functions as sums of infinite [illegible] including the binomial theorem; (2) his [illegible] on differential and integral calculus; (3) his [illegible] of motion and law of universal gravitation; [illegible] his prism experiments on the nature of [illegible] and color. Because of a fear of controversy [illegible]ticism, he was reluctant to publish his dis[illegible] and it wasn't until 1687, at the urging of [illegible] astronomer Halley, that Newton published [illegible] *Mathematica*. In this work, the greatest [illegible] treatise ever written, Newton set forth [illegible] of calculus and used it to investigate [illegible], fluid dynamics, and wave motion, [illegible] explain the motion of planets and comets.

[illegible] beginnings of calculus are found in the [illegible] of areas and volumes by ancient [illegible] scholars such as Eudoxus and Archimedes. [illegible] aspects of the idea of a limit are [illegible] in their "method of exhaustion," Eudoxus [illegible] Archimedes never explicitly formulated the [illegible] of a limit. Likewise, mathematicians [illegible] as Cavalieri, Fermat, and Barrow, the imme[illegible] precursors of Newton in the development [illegible], did not actually use limits. It was [illegible] Newton who was the first to talk explicitly [illegible] limits. He explained that the main idea [illegible] limits is that quantities "approach nearer [illegible] any given difference." Newton stated [illegible] limit was the basic concept in calculus, [illegible] left to later mathematicians like [illegible] to clarify his ideas about limits.

NOTE ▫ In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, with the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, if $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

EXAMPLE 4 Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 2$$

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

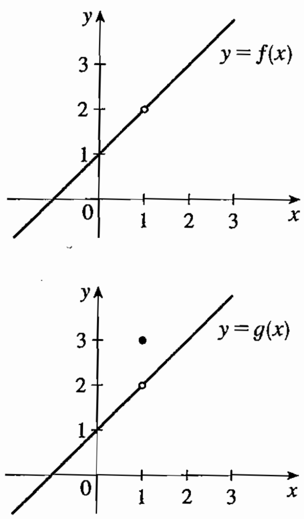


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

EXAMPLE 5 Evaluate $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$

Explore a limit like this one interactively.
Resources / Module 2
/ The Essential Examples
/ Example C

EXAMPLE 6 Find $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned} \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2 + 9} + 3\right)} = \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2 + 9} + 3\right)} \\ &= \lim_{t \to 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 2.2.

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x\to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 3.

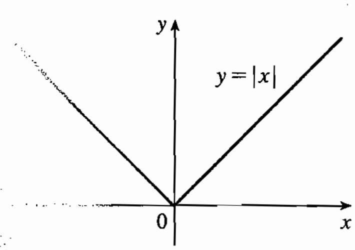


FIGURE 3

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x\to 0} |x| = 0$$

EXAMPLE 8 Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

SOLUTION

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

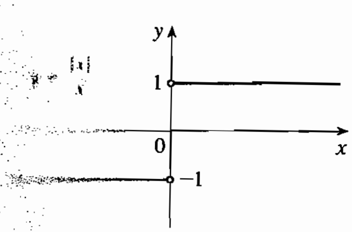


FIGURE 4

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x\to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x\to 4} f(x)$ exists.

It is shown in Example 3 in Section 2.4 that $\lim_{x\to 0^+} \sqrt{x} = 0$.

SOLUTION Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus, the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 5. ■

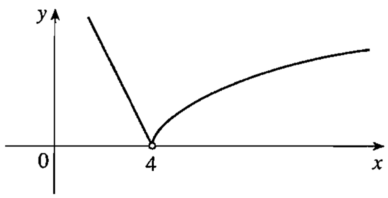


FIGURE 5

EXAMPLE 10 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

|||| Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1. ■

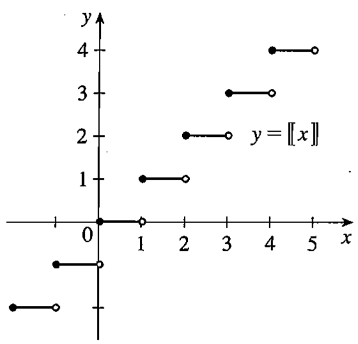


FIGURE 6
Greatest integer function

The next two theorems give two additional properties of limits. Their proofs can be found in Appendix F.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

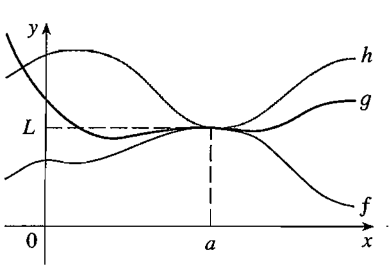


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we *cannot* use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

Watch an animation of a similar limit.
Resources / Module 2
/ Basics of Limits
/ Sound of a Limit that Exists

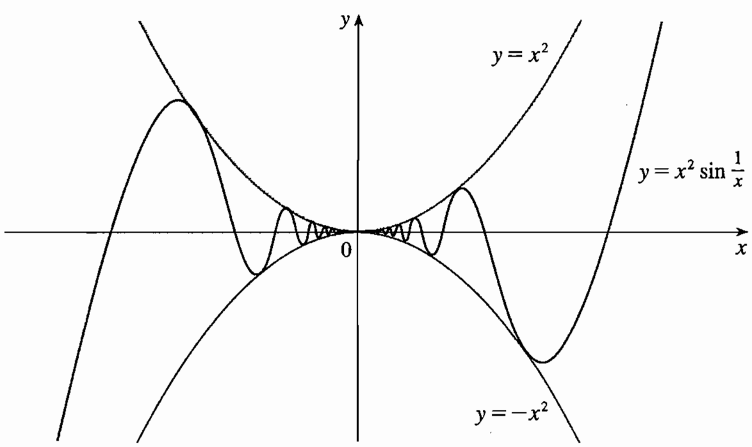


FIGURE 8

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

2.3 Exercises

1. Given that

$$\lim_{x \to a} f(x) = -3 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 8$$

find the limits that exist. If the limit does not exist, explain why.

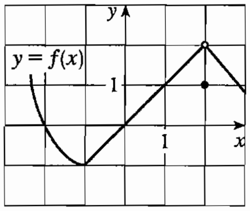
(a) $\lim_{x \to a} [f(x) + h(x)]$ (b) $\lim_{x \to a} [f(x)]^2$

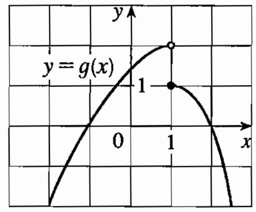
(c) $\lim_{x \to a} \sqrt[3]{h(x)}$ (d) $\lim_{x \to a} \dfrac{1}{f(x)}$

(e) $\lim_{x \to a} \dfrac{f(x)}{h(x)}$ (f) $\lim_{x \to a} \dfrac{g(x)}{f(x)}$

(g) $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ (h) $\lim_{x \to a} \dfrac{2f(x)}{h(x) - f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x\to 2}[f(x) + g(x)]$ (b) $\lim_{x\to 1}[f(x) + g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ (d) $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} x^3 f(x)$ (f) $\lim_{x\to 1}\sqrt{3 + f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to -2}(3x^4 + 2x^2 - x + 1)$

4. $\lim_{x\to 2}\dfrac{2x^2 + 1}{x^2 + 6x - 4}$

5. $\lim_{x\to 3}(x^2 - 4)(x^3 + 5x - 1)$

6. $\lim_{t\to -1}(t^2 + 1)^3(t + 3)^5$

7. $\lim_{x\to 1}\left(\dfrac{1 + 3x}{1 + 4x^2 + 3x^4}\right)^3$

8. $\lim_{u\to -2}\sqrt{u^4 + 3u + 6}$

9. $\lim_{x\to 4^-}\sqrt{16 - x^2}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2}(x + 3)$$

is correct.

11–30 Evaluate the limit, if it exists.

11. $\lim_{x\to 2}\dfrac{x^2 + x - 6}{x - 2}$

12. $\lim_{x\to -4}\dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$

13. $\lim_{x\to 2}\dfrac{x^2 - x + 6}{x - 2}$

14. $\lim_{x\to 4}\dfrac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim_{t\to -3}\dfrac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim_{x\to -1}\dfrac{x^2 - 4x}{x^2 - 3x - 4}$

17. $\lim_{h\to 0}\dfrac{(4 + h)^2 - 16}{h}$

18. $\lim_{x\to 1}\dfrac{x^3 - 1}{x^2 - 1}$

19. $\lim_{h\to 0}\dfrac{(1 + h)^4 - 1}{h}$

20. $\lim_{h\to 0}\dfrac{(2 + h)^3 - 8}{h}$

21. $\lim_{t\to 9}\dfrac{9 - t}{3 - \sqrt{t}}$

22. $\lim_{h\to 0}\dfrac{\sqrt{1 + h} - 1}{h}$

23. $\lim_{x\to 7}\dfrac{\sqrt{x + 2} - 3}{x - 7}$

24. $\lim_{x\to 2}\dfrac{x^4 - 16}{x - 2}$

25. $\lim_{x\to -4}\dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

26. $\lim_{t\to 0}\left(\dfrac{1}{t} - \dfrac{1}{t^2 + t}\right)$

27. $\lim_{x\to 9}\dfrac{x^2 - 81}{\sqrt{x} - 3}$

28. $\lim_{h\to 0}\dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

29. $\lim_{t\to 0}\left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t}\right)$

30. $\lim_{x\to 1}\dfrac{\sqrt{x} - x^2}{1 - \sqrt{x}}$

31. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

32. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

33. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2\cos 20\pi x = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2\cos 20\pi x$, and $h(x) = x^2$ on the same screen.

34. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3 + x^2}\sin\frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

35. If $1 \le f(x) \le x^2 + 2x + 2$ for all x, find $\lim_{x\to -1} f(x)$.

36. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x\to 1} f(x)$.

37. Prove that $\lim_{x\to 0} x^4\cos\dfrac{2}{x} = 0$.

38. Prove that $\lim_{x\to 0^+}\sqrt{x}\,e^{\sin(\pi/x)} = 0$.

39–44 Find the limit, if it exists. If the limit does not exist, explain why.

39. $\lim_{x\to -4}|x + 4|$

40. $\lim_{x\to -4^-}\dfrac{|x + 4|}{x + 4}$

41. $\lim_{x \to 2} \frac{|x-2|}{x-2}$

42. $\lim_{x \to 1.5} \frac{2x^2 - 3x}{|2x-3|}$

43. $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

44. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

45. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \to 0^-} \operatorname{sgn} x$

(iii) $\lim_{x \to 0} \operatorname{sgn} x$ (iv) $\lim_{x \to 0} |\operatorname{sgn} x|$

46. Let

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \leqslant 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

(a) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.

(b) Does $\lim_{x \to 2} f(x)$ exist?

(c) Sketch the graph of f.

47. Let $F(x) = \frac{x^2 - 1}{|x - 1|}$.

(a) Find

(i) $\lim_{x \to 1^+} F(x)$ (ii) $\lim_{x \to 1^-} F(x)$

(b) Does $\lim_{x \to 1} F(x)$ exist?

(c) Sketch the graph of F.

48. Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leqslant 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.

(i) $\lim_{x \to 0^+} h(x)$ (ii) $\lim_{x \to 0} h(x)$ (iii) $\lim_{x \to 1} h(x)$

(iv) $\lim_{x \to 2^-} h(x)$ (v) $\lim_{x \to 2^+} h(x)$ (vi) $\lim_{x \to 2} h(x)$

(b) Sketch the graph of h.

49. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x \to -2^+} [\![x]\!]$ (ii) $\lim_{x \to -2} [\![x]\!]$ (iii) $\lim_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} [\![x]\!]$ (ii) $\lim_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim_{x \to a} [\![x]\!]$ exist?

50. Let $f(x) = x - [\![x]\!]$.

(a) Sketch the graph of f.

(b) If n is an integer, evaluate

(i) $\lim_{x \to n^-} f(x)$ (ii) $\lim_{x \to n^+} f(x)$

(c) For what values of a does $\lim_{x \to a} f(x)$ exist?

51. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

52. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

53. If p is a polynomial, show that $\lim_{x \to a} p(x) = p(a)$.

54. If r is a rational function, use Exercise 53 to show that $\lim_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

55. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

56. Show by means of an example that $\lim_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

57. Show by means of an example that $\lim_{x \to a} [f(x)g(x)]$ may exist even though neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

58. Evaluate $\lim_{x \to 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

59. Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

60. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?