CHAPTER 1

Numbers, proof and 'all that jazz'.

There is a fundamental difference between mathematics and other sciences. In most sciences, one does experiments to determine laws. A "law" will remain a law, only so long as it is not contradicted by experimental evidence. Newtonian physics was accepted as valid until it was contradicted by experiment, resulting in the discovery of the theory of relativity.

Mathematics, on the other hand, is based on *absolute certainty.* A mathematician may feel that some mathematical law is true on the basis of, say, a thousand experiments. He/she will not accept it as true, however, until it is absolutely certain that **it can never fail.** Achieving this kind of certainty requires constructing a logical argument showing the law's validity–i.e. constructing a proof.

There is, however, a problem with the notion that everything should be proved. If we insist on proving *everything* then we initially know *nothing*, and, if we know nothing, how can we prove anything? We have no place to begin. Clearly, we must have some body of information that we know to be true on which to base our proofs.

So what can we assume known and what must be proved? Before the time of Euclid, the answer to this question was personal and subjective. You were allowed to assume anything that you could bully your listener into believing. If I could get you to agree that all integers are even (which is false) I could use it to prove all sorts of other wonderful (and equally false) things. This often led to many mistakes, so much so that it was very difficult to know what was true and what was not.

Euclid solved this problem for geometry by stating an explicit collection of "self evident" properties (called axioms) which were assumed without proof. Furthermore, *the axioms are the only properties that were to be assumed without proof.* All other properties must be proved using either the axioms or their consequences.

Since the time of Euclid, lists of axioms for many fields of mathematics, such as set theory, logic, and numbers have been compiled. In these notes, we present one of the standard lists of axioms for the real numbers, which are the numbers used in calculus. Thus, we are stating "up front," those properties that we are allowed to assume without proof. As will be seen, the list is rather long and will be covered over several sections. We begin with the **field axioms**, which describe those properties of numbers that do not relate to inequalities.

In principle, every number fact we use should be proved using only our axioms. In fact, in these notes, we usually adopt a much looser standard. As the reader will see, proving everything directly from the axioms would take so long that we would never progress beyond this section! It is, however, important that the reader prove a number of basic number facts using only the axioms in order to appreciate their significance.

Before stating the number axioms, we state some properties of equality. These are not number axioms since they apply to things other than numbers, e.g. triangles, circles, functions, chairs, etc. They are really axioms of logic. We take these properties as given and do not typically indicate their use in our proofs. This is not to say that they are unimportant. For example, it is (EQ2) that allows us to write the distributive law (D1) as

$$ab + ac = a(b + c)$$

instead of

$$a(b + c) = ab + ac$$

which justifies factoring a out of $ab + ac$.

Properties of Equality

EQ1: (Identity) For all a, $a = a$.
EQ2: (Reflexive) If $a = b$, then $b = a$.
EQ3: (Transitive) If $a = b$ and $b = c$, then $a = c$.

The Field Axioms for the Real Numbers
Axioms for Addition

A0: (Existence of Addition) Addition is a well defined process which takes pairs of real numbers a and b and produces from then one single real number $a + b$.

A1: (Associativity) If a, b, and c are real numbers, then

$$a + (b + c) = (a + b) + c.$$

A2: (Additive Identity) There is a real number 0 such that for all real numbers a

$$a + 0 = a.$$

A3: (Additive Inverse) For every real number a there is a real number $-a$ such that

$$a + (-a) = 0.$$

A4: (Commutativity) If a and b are any real numbers, then

$$a + b = b + a.$$

Axioms for Multiplication

M0: (Existence of Multiplication) Multiplication is a well defined process which takes pairs of real numbers a and b and produces from then one single real number ab.

M1: (Associativity) If a, b, and c are any real numbers, then

$$a(bc) = (ab)c.$$

M2: (Multiplicative Identity) There is a real number 1 such that for all real numbers a

$$a1 = a.$$

M3: (Multiplicative Inverse) For every real number $a \neq 0$, there is a real number a^{-1} such that

$$aa^{-1} = 1.$$

M4: (Commutativity) If a and b are any real numbers, then

$$ab = ba.$$

Other Laws

D: (Distributive) For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

Z: (Non-triviality) $0 \neq 1$

Notice that the axioms mention neither subtraction nor division. This is because they may be expressed using addition and multiplication:

DEFINITION 1. *Let a and b be numbers. We define*

$$a - b = a + (-b)$$
$$\frac{a}{b} = ab^{-1} \qquad (b \neq 0)$$

EXAMPLE 1. Solve the following equality for x in a step-by-step manner, listing all of the axioms that you use.

$$3x + 5 = 12$$

Solution: We begin by adding -5 to both sides of the equality. Axiom (A3) guarantees the existence of the negative of any number. Axiom (A0) guarantees that adding -5 to both sides preserves the equality. We proceed as follows.

$$\begin{aligned} 3x + 5 &= 12 \\ (3x+5) + (-5) &= 12 + (-5) \qquad \text{(A3), (A0)} \\ 3x + (5 + (-5)) &= 7 \qquad \text{(A1)} \\ 3x + 0 &= 7 \qquad \text{(A3)} \\ 3x &= 7 \qquad \text{(A2)} \end{aligned}$$

Next we would like to divide both sides by 3. This is the same as multiplying by 1/3. 1/3 exists due to (M3) and multiplication preserves the equality due to (M0). Hence:

$$\begin{aligned} \frac{1}{3}(3x) &= \frac{7}{3} \qquad \text{(M0), (M3)} \\ (\frac{1}{3}\,3)x &= \frac{7}{3} \qquad \text{(M1)} \\ 1x &= \frac{7}{3} \qquad \text{(M3), (M4)} \\ x &= \frac{7}{3} \qquad \text{(M2), (M4)} \end{aligned}$$

We needed to use (M4) (Commutativity) in the third step because the (M3) tells us only that $aa^{-1} = 1$, and not that $a^{-1}a = 1$. Similarly, we needed (M4) in the last step because (M2) tells us only that $a1 = a$.

Conversely, if $x = \frac{7}{3}$,

$$3x + 5 = \frac{3 \cdot 7}{3} + 5 = 12$$

showing that $x = \frac{7}{3}$ does indeed solve the equality.

Remark: The last step in our solution, checking that $x = 7/3$ is a solution, was necessary. Our first sequence of equalities proved only that if x is a solution, then $x = 7/3$. It did not prove that $7/3$ actually is a solution.

Many familiar number properties do not appear in our axiom list. This is because they can be proved from the axioms. The following list contains some of the more common ones. **These properties are not axioms: they are consequences of the axioms.** We will leave most of the proofs to you.

THEOREM 1. *Let a, b, and c be real numbers. Then*

C1: $(a + b)c = ac + bc$.
C2: $0a = 0$
C3: $-a = (-1)a$.
C4: $-(ab) = (-a)b = a(-b)$.
C5: $-(-a) = a$.
C6: *If $a \neq 0 \neq b$, then $(ab)^{-1} = a^{-1}b^{-1}$.*
C7: $(a^{-1})^{-1} = a$.
C8: $-(a + b) = (-a) + (-b)$.

EXAMPLE 2. Prove (C2).

Solution: We note that

$$\begin{aligned} a + 0 \cdot a &= 1 \cdot a + 0 \cdot a \quad \text{(M2), (M4)} \\ &= (1 + 0)a \quad \text{(C1)} \\ &= 1 \cdot a \quad \text{(A2)} \\ &= a \quad \text{(M2), (M4)} \end{aligned} \tag{1}$$

Hence

$$a + 0 \cdot a = a$$

We solve this equation for $a \cdot 0$:

$$
\begin{aligned}
(-a) + (a + 0 \cdot a) &= -a + a = 0 \quad \text{(A0), (A3), (A4)} \\
((-a) + a) + 0 \cdot a &= 0 \quad \text{(A3), (A1)} \\
0 + 0 \cdot a &= 0 \quad \text{(A2), (A3), (A4)} \\
0 \cdot a &= 0 \quad \text{(A2), (A4)}
\end{aligned} \tag{2}
$$

EXAMPLE 3. Find all solutions to the following in a step-by-step manner, listing all of the properties that you use.

$$\frac{2x+1}{x} = 3$$

Solution: Suppose that x satisfies our equality. From the definition of fractions, this is equivalent with

$$(2x+1)x^{-1} = 3$$

Our solution proceeds as follows:

$$
\begin{aligned}
((2x+1)x^{-1})x &= 3x \quad \text{(M0)} \\
(2x+1)(x^{-1}x) &= 3x \quad \text{(M1)} \\
(2x+1)1 &= 3x \quad \text{(M3), (M4)} \\
2x+1 &= 3x \quad \text{(M2)}
\end{aligned}
$$

Next, we would like to cancel $3x$ by adding $-3x$ to both sides of the equation. If we do something to one side of an equality, we must do **exactly the same thing** to the other side. Thus, we add $-3x$ to the *left* of each side:

$$
\begin{aligned}
-(3x) + (2x+1) &= -(3x) + 3x \quad \text{(A0), (A3)} \\
&= 0 \quad \text{(A4), (A3)}
\end{aligned}
$$

We finish the computation:

$$
\begin{aligned}
(-(3x) + 2x) + 1 &= 0 \quad && \text{(A1)} \\
((-3)x + 2x) + 1 &= 0 \quad && \text{(C4)} \\
((-3) + 2)x + 1 &= 0 \quad && \text{(C1)} \\
(-1)x + 1 &= 0 \quad && \text{Number Fact}^{1} \\
-x + 1 &= 0 \quad && \text{(C3)} \\
x + (-x + 1) &= x + 0 \quad && \text{(A0)} \\
(x + (-x)) + 1 &= x \quad && \text{(A1), (A2)} \\
0 + 1 &= x \quad && \text{(A3)} \\
1 &= x \quad && \text{(A2), (A4)}
\end{aligned}
$$

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Conversely, if $x = 1$,

$$\frac{2x+1}{x} = \frac{2 \cdot 1 - 1}{1} = 1$$

showing that $x = 1$ does indeed solve the equality.

Clearly, putting in every step can be quite tedious, even in a simple calculation as in Example 3. Fortunately, it is not essential that you develop great skill at doing more complicated examples. The point here is only to stress that *all of the computations done in elementary algebra can all be justified using only the axioms for the real numbers.*

The next example demonstrates the necessity for checking the solution.

EXAMPLE 4. Find all solutions to the following equality

$$x = \sqrt{2 - x}$$

You need not indicate all of your steps.

[1] Example 6 demonstrates how to prove such "number facts".

Solution: We reason as follows. Suppose x satisfies the given inequality. Then

$$\begin{aligned} x &= \sqrt{2-x} \\ x^2 &= 2-x \\ x^2+x-2 &= 0 \\ (x-1)(x+2) &= 0 \end{aligned}$$

Hence, either $x = 1$ or $x = -2$. If $x = 1$

$$\sqrt{2-x} = \sqrt{1} = x.$$

However, if $x = -2$

$$\sqrt{2-x} = \sqrt{4} = 2 \neq x.$$

Hence the only solution is $x = 1$.

Remark: The symbol $\sqrt{a}$, by definition, is the **positive** square root. Hence $\sqrt{4} = 2$, not ± 2.

The axioms only discuss addition of pairs of numbers. We define addition of triples by

$$a + b + c = (a + b) + c$$

which, from the associative law, is the same as $a + (b + c)$. We define the sum of four numbers by

$$a + b + c + d = (a + b + c) + d$$

You will prove in the exercises that

$$(a + b + c) + d = (a + b) + (c + d) = a + (b + c + d)$$

Hence, the sum is the same no matter how we group the terms.

Similar comments apply to adding n numbers.

DEFINITION 2. *If $a_1, a_2, \ldots, a_n$ are n numbers, then we define*

$$a_1 + a_2 + \cdots + a_n = (a_1 + a_2 + \cdots + a_{n-1}) + a_n.$$

This is a **recursive** definition, in that it defines the sum n numbers, assuming that we already know how to sum $n - 1$ numbers. Thus, for example, it defines

$$a + b + c + d + e = (a + b + c + d) + e$$

where

$$a + b + c + d = (a + b + c) + d.$$

We define the product of n numbers similarly:

DEFINITION 3. *If $a_1, a_2, \dots, a_n$ are n numbers, then we define*

$$a_1\, a_2\, \dots\, a_n = (a_1\, a_2\, \dots\, a_{n-1})\, a_n.$$

You will prove specific instances of the following theorem in the exercises. By "grouping" we refer to the manner in which parentheses are put into the sum.

THEOREM 2. *The sum and product of n numbers are independent of both the order and the grouping of the terms.*

Theorem 2 covers all applications of both the commutative and associative laws in solving equalities.

EXAMPLE 5. Solve the following equality listing all of your steps.

$$4x - 9 = x - 2$$

Solution: Assume that x satisfies the given equality. Then

$$\begin{aligned}
4x - 9 + (-x) + 9 &= x - 2 + (-x) + 9 \quad \text{(A0)}\\
4x + (-x) + 9 + (-9) &= x + (-x) + 9 + (-2) \quad \text{(Thm. 2)}\\
4x + (-1)x + 0 &= 0 + 7 \quad \text{(A3), (C3)}\\
(4 + (-1))x &= 7 \quad \text{(A2), (C1)}\\
3x &= 7
\end{aligned}$$

The solution now proceeds exactly as at the end of Example 1 on page 8.

Theorem 2 allows us to prove some number facts that the reader learned in grade school. We define

$$\begin{aligned}
2 &= 1 + 1\\
3 &= 2 + 1\\
4 &= 3 + 1\\
5 &= 4 + 1\\
&\dots
\end{aligned}$$

The set $\mathbb{N}$ of **natural** numbers is, by definition, the numbers obtainable by adding 1 to itself any number of times. Thus,

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

EXAMPLE 6. Prove that

$$\begin{aligned} &(a) \qquad 2 \cdot 2 = 4 \\ &(b) \qquad -3 + 2 = -1 \end{aligned}$$

Solution:

Part (a):

$$\begin{aligned} 2 \cdot 2 &= (1+1)(1+1) \quad \text{Def. of 2} \\ &= 1(1+1) + 1(1+1) \quad \text{(C1)} \\ &= (1+1) + (1+1) \quad \text{(M2), (M4)} \\ &= 2 + (1+1) \quad \text{Def. of 2} \\ &= (2+1) + 1 \quad \text{(A1)} \\ &= 3 + 1 \quad \text{Def. of 3} \\ &= 4 \quad \text{Def. of 4} \end{aligned}$$

Part (b):

$$\begin{aligned} -3 + 2 &= -(2+1) + 2 \quad \text{Def. of 3} \\ &= -(1+2) + 2 \quad \text{(A4)} \\ &= ((-1) + (-2)) + 2 \quad \text{(C8)} \\ &= -2 + ((-2) + 2) \quad \text{(A1)} \\ &= -2 + 0 \quad \text{(A4), (A3)} \\ &= -2 \quad \text{(A2)} \end{aligned}$$

EXAMPLE 7. Prove that for real numbers a and b

$$(a+b)^2 = a^2 + 2ab + b^2$$

Solution:

$$
\begin{aligned}
(a+b)^2 &= (a+b)(a+b) = a(a+b) + b(a+b) \quad \text{(C1)} \\
&= a^2 + ab + ba + b^2 \quad \text{(D)} \\
&= a^2 + ab + ab + b^2 \quad \text{(M4)} \\
&= a^2 + 1(ab) + 1(ab) + b^2 \quad \text{(M2), (M4)} \\
&= a^2 + (1+1)ab + b^2 \quad \text{(C1)} \\
&= a^2 + 2ab + b^2 \quad \text{Def. of 2}
\end{aligned}
$$

Exercises:

(1) Solve the following equalities for x in a step-by-step manner, listing each property you use as in Examples 1 and 2. DO NOT USE Theorem 2.

(a) $7x - 5 = 19$

(b) $7x = 2x + 3$

(c) $\dfrac{3x+2}{x} = -1$

(2) Re-do Example 5 without using Theorem 2. You may stop once you have reduced the equation to $3x = 7$.

(3) The work below proves the following equality. Copy the proof onto your paper, giving reasons for each step. When using (A1), state what expression is being substituted for a, b, and c. DO NOT USE Theorem 2.

$$(x+y)(z+w) = (xz+yz) + (xw+yw)$$

$$
\begin{aligned}
(x+y)(z+w) &= x(z+w) + y(z+w) \\
&= (xz+xw) + (yz+yw) \\
&= xz + (xw + (yz+yw)) \\
&= xz + ((xw+yz) + yw) \\
&= xz + ((yz+xw) + yw) \\
&= xz + (yz + (xw+yw)) \\
&= (xz+yz) + (xw+yw)
\end{aligned}
$$

(4) Let x, y, z be real numbers. In the notes we defined

$$x + y + z = (x + y) + z$$

There are 6 different orders in which we could sum 3 numbers, all of which, of course, yield the same answer. As an illustration of this, use the axioms and the above definition to prove the following equalities. DO NOT USE Theorem 2.

(a) $x + y + z = z + x + y$.

(b) $x + y + z = z + y + x$

(5) Let x, y, z, and w be real numbers. Use (A1) and the definitions to prove the following equalities. In each case state what expression is being substituted for a, b, and c in (A1). DO NOT USE Theorem 2.

$$\begin{aligned} x + (y + z + w) &= (x + y) + (z + w) \\ &= (x + y + z) + w \end{aligned}$$

Remark: This exercise proves that the sum of four numbers is independent of how they are grouped.

(6) Let x, y, z, w, and u be real numbers. Use (A1) and the result of Exercise 5 to prove the following equalities. In each case state what expression is being substituted for a, b, and c in (A1). DO NOT USE Theorem 2.

$$\begin{aligned} x + (y + z + w + u) &= (x + y) + (z + w + u) \\ &= (x + y + z) + (w + u) \\ &= (x + y + z + w) + u \end{aligned}$$

Remark: This exercise proves that the sum of five numbers is independent of how they are grouped.

(7) State and solve analogue of Exercise 5 for multiplication.

(8) State and solve analogue of Exercise 6 for multiplication.

(9) Why was (M4) needed in the first equation in the sequence of equations (1)? How about in the fourth equation?

(10) Why was (A4) needed in the first equation in the sequence of equations (2)? How about in the fourth equation?

(11) Reason as in Example 6 to prove the following number facts.

(a) $2 + 2 = 4$

(b) $2 \cdot 3 = 6$ *Hint:* From Example 6 on page 14, $2 \cdot 2 = 4$.

(c) $3 - 4 = -1$

(d) $3 - 5 = -2$

(12) (a) Prove property (C1).
(b) Let c and d be real numbers. Suppose $c + d = 0$. Use the axioms to prove that $c = -d$. *Hint:* Solve $c + d = 0$ for c in a step-by-step manner.
(c) Use the result proved in (b) along with the axioms and property (C2) to prove (C3). *Hint:* $a + (-1)a = (1)a + (-1)a$.
(d) Use the result proved in (b) along with the axioms to prove (C5). *Hint:* $(-a) + a = ?$.
(e) Use the result proved in (b) along with the axioms and property (C2) to prove the first equality in (C4). *Hint:* $ab + (-a)b = ?$

(13) Let c and d be real numbers.
(a) Suppose $cd = 1$. Use the axioms to prove that $d = c^{-1}$. *Hint:* First explain why $c \neq 0$. Then solve $cd = 1$ for d in a step-by-step manner.
(b) Prove (C6). *Hint:* Simplify $(ab)(a^{-1}b^{-1})$ and apply the result proved in (a).
(c) Use the result proved in (a) along with the axioms to prove (C7). *Hint:* $a^{-1}a = ?$.

(14) If we wish to solve $x^2 + 3x + 2 = 0$, we factor, finding that $(x + 1)(x + 2) = 0$; hence $x = -1$ or $x = -2$. This is based on the property that if a and b are numbers and $ab = 0$, then either $a = 0$ or $b = 0$ (or both). Prove this property using the axioms. *Hint:* If $a = 0$, there is nothing to prove. Hence assume $a \neq 0$ and solve for b.

(15) Let a, b, c and d be numbers with b and d non-zero. Use the axioms and properties (C1)-(C8) to prove the following:
(a) $ab^{-1} = (ad)(bd)^{-1}$
(b) $ab^{-1} + cd^{-1} = (ad + bc)(bd)^{-1}$
(c) $(ab^{-1})(cd^{-1}) = (ac)(bd)^{-1}$
(d) $(ab^{-1})(cd^{-1})^{-1} = (ad)(bc)^{-1}$

(16) Each of the equalities in the preceding problem expresses a familiar law of fractions. Write each in fractional form.