CHAPTER 11

Continuity

Consider the problem of measuring the side length of a square and then using the measured date to compute the area of the square. For example, if one side is measured to be 2.74 inches, then the area would be computed as $(2.74)^2 = 7.5076$ square inches. We know, of course, that no measurement is precise. There will undoubtedly be small errors in the measurement of the side length. This will result in errors in the computed area. We also know from experience that the errors in the area computation will be small as long as the errors in the side measurement are also small; values of s close to 2.74 will produce values of A close to $(2.74)^2$. In mathematical terms, this amounts to saying that

$$\lim_{s \to 2.74} s^2 = (2.74)^2.$$

This is a statement of the fact that the area function $A(s) = s^2$ is continuous at the point $s = 2.74$. In general, we define continuity as follows:

Definition 1. *A function f is continuous at a provided that*

$$\lim_{x \to a} f(x) = f(a).$$

Implicit in the above definition is the requirement that a belong to the domain of f. It is also implicit that $f(x)$ be defined for all x sufficiently close to a since otherwise, the limit would not exist. (We must be able to get $|f(x) - f(a)| < \epsilon$ for *all* x between $a - \delta$ and $a + \delta$.) Thus, the above definition requires that there is a $\delta > 0$ such that the interval $(a - \delta, a + \delta)$ belongs to the domain of f.

This, however, presents us with a difficulty. According to this definition, the function $y = \sqrt{x}$ is not continuous at $x = 0$:

$$\lim_{x \to 0} \sqrt{x}$$

does not exist since $\sqrt{x}$ is not defined for $x < 0$. The best we can say is that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Because of such examples, we are forced to amend our definition in the case that the domain of $f(x)$ is a closed (or half-closed) interval. Before doing so, however, we first give the "official" definition of left and right hand limits:

DEFINITION 2. Let $f(x)$ be a function. We say that

$$\lim_{x \to a^+} f(x) = L$$

provided that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

for all x satisfying

$$0 < |x - a| < \delta, \quad x > a.$$

The definition of $\lim_{x \to a^-} f(x) = L$ is identical, except that "$x > a$" in the last inequality above is replaced by "$x < a$."

Now our amended definition of continuity states:

DEFINITION 3. Suppose that the domain of $f(x)$ is the interval $[a, b]$. We say that $f(x)$ is continuous at a if

$$\lim_{x \to a^+} f(x) = f(a)$$

We say that $f(x)$ is continuous at b if

$$\lim_{x \to b^-} f(x) = f(b).$$

Intuitively, continuity may be described in the same terms as we did for the square function: values of x near a produce values of $f(x)$ near $f(a)$. (See Figure 1). It is very fortunate that most of the functions which arise in the real world are continuous. Otherwise, we would never be able to calculate anything!

You use continuity almost every time you evaluate a limit. For example, if you were to compute $\lim_{x \to 2} x^2$, you would probably just 'plug 2 in,' finding the answer $2^2 = 4$. What you are saying is that for the function $f(x) = x^2$, the limit as x approaches 2 of $f(x)$ is $f(2)$. Thus, another way of describing what continuity at a means is to say

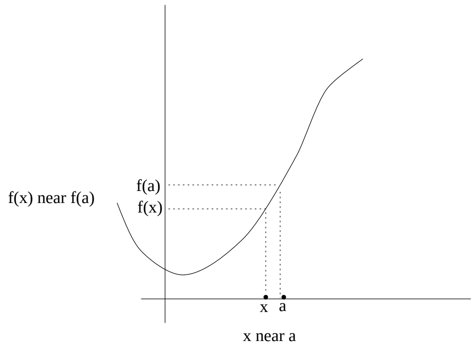


FIGURE 1

that the limit as x approaches a may be computed by 'plugging a into f'.

Figure 2 below show a few common types of discontinuities.

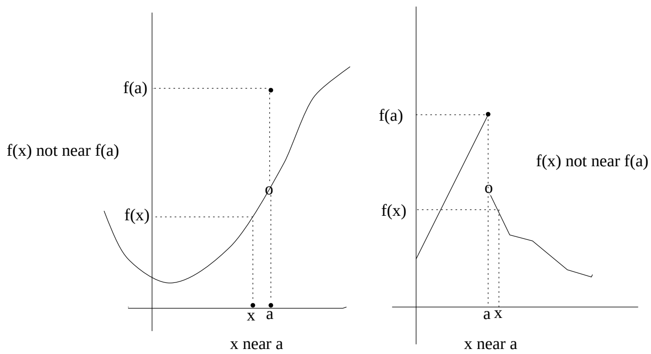


FIGURE 2

The first graph is an example of what is called a *removable discontinuity.* From the graph, $L = \lim_{x\to a} f(x)$ exists, but just does not happen to equal $f(a)$. If we were to *redefine* $f(a)$ by setting $f(a) = L$, then we would produce a new function which is continuous.

In the second graph, the continuity is not removable since $\lim_{x\to a} f(x)$ does not exist: we get different answers for the limit, depending upon whether we approach a from the left or from the right.

EXAMPLE 1. Let $f(x)$ be

$$f(x) = \frac{\sqrt{x} - 2}{x - 4}.$$

Then f is not continuous at $x = 4$ because $x = 4$ does not belong to the domain of f. Show that $x = 4$ is a removable discontinuity. How should $f(4)$ be defined so as to make f continuous?

Solution: To be continuous at $x = 4$ we require

$$f(4) = \lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}. \tag{1}$$

If we attempt to evaluate this limit by "plugging" $x = 4$ into the fraction, we get the indeterminant form $0/0$, which does not help.

There are several correct ways to evaluate this limit. Our favorite, perhaps, is to rationalize:

$$\begin{aligned} \frac{\sqrt{x} - 2}{x - 4} &= \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} \\ &= \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2}. \end{aligned}$$

The limit is $1/4$. Hence, we should define $f(4) = 1/4$.

The next example exhibits a much more 'serious' discontinuity.

Example 2. Let $f(x)$ be

$$\begin{aligned} f(x) &= \sin \frac{1}{x} \quad x \neq 0 \\ &= 0 \quad x = 0. \end{aligned}$$

Show that f is discontinuous at $x = 0$.

Solution: The function $f(x)$ is zero whenever $1/x = k\pi$ which is equivalent with $x = \frac{\pi}{k}$. Hence, f has an infinite number of zeros between π and 0. Between these zeros, f oscillates between 1 and -1. Thus, the graph of f looks something like that shown in Figure 3 below:

As $x \to 0$, $f(x)$ does not approach any single value, showing that f has a non-removable discontinuity at $x = 0$.

Proving that a given function is continuous at a given value is often quite easy.

Example 3. Prove that the function $f(x) = x^2$ is continuous at all $a \in \mathbb{R}$.

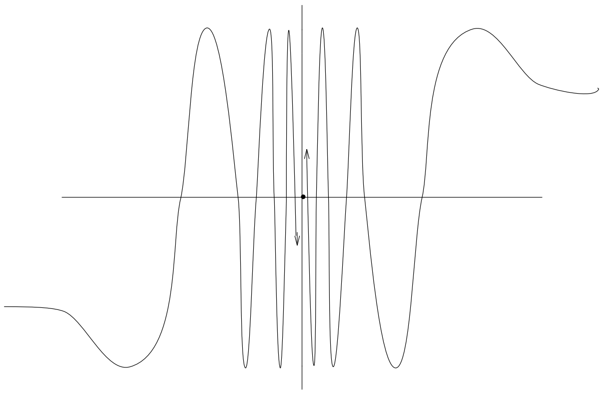


FIGURE 3

Solution: Let a be some real number. Then, from the product theorem for limits,

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = (\lim_{x\to a} x)(\lim_{x\to a} x) = aa = a^2 = f(a).$$

This simple problem illustrates the following theorem which is a direct consequence of the Product Theorem for limits of functions.

THEOREM 1 (Product). *Suppose that $f(x)$ and $g(x)$ are both continuous at a. Then $h(x) = f(x)g(x)$ is continuous at a.*

It follows from the product theorem that the function $f(x) = x^3$ is continuous for every a since $x^3 = x(x^2)$. The following proposition follows by similar reasoning:

PROPOSITION 1. *For $n \in \mathbb{N}$, function $f(x) = x^n$ is continuous at every $a \in \mathbb{R}$.*

Once we know the continuity of such functions, it is easy to prove the continuity of many other functions as well.

EXAMPLE 4. Prove that the function f below is continuous at every a in its domain:

$$f(x) = \frac{x^2+1}{x^2-3x+2}.$$

Solution: Let us first note that the denominator of f factors as $(x-1)(x-2)$. Hence the domain of f is all real x, $x \neq 1$ and $x \neq 2$.

Let a be an element of the domain of f. Since a is not equal to either 1 or 2, we see that as x approaches a, the denominator in f will not approach 0. This allows us to apply the quotient rule for limits:

$$\lim_{x\to a} \frac{x^2+1}{x^2-3x+2} = \frac{\lim_{x\to a} x^2 + \lim_{x\to a} 1}{\lim_{x\to a} x^2 - 3\lim_{x\to a} x + 2\lim_{x\to a}} = \frac{a^2+1}{a^2-3a+2}.$$

Since the final answer is what would have been obtained by plugging a into the formula for f, the continuity is proved.

The above example illustrates the following theorem which is a direct consequence of the quotient theorem for limits of functions.

Theorem 2 (Quotient). *Suppose that $f(x)$ and $g(x)$ are both continuous at a and that $g(a) \neq 0$. Then $h(x) = f(x)/g(x)$ is continuous at a.*

In a calculus class, one might compute a limit such as

$$\lim_{n\to\infty} \sqrt{\frac{n}{2n+1}}$$

as follows:

Let $x_n = \frac{n}{2n+1}$. Since

$$\lim_{n\to\infty} x_n = \frac{1}{2}$$

we see that

$$\begin{aligned} \lim_{n\to\infty} \sqrt{\frac{n}{2n+1}} &= \lim_{n\to\infty} \sqrt{x_n} \\ &= \lim_{x\to 1/2} \sqrt{x} = \sqrt{\frac{1}{2}}. \end{aligned}$$

This method is based on the continuity of $y = \sqrt{x}$ at $x = 1/2$. Specifically, it uses the following theorem:

Theorem 3 (Sequence). *Let $f(x)$ be continuous at a and let x_n be a sequence such that $\lim_{n\to\infty} x_n = a$. Then*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Proof Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = f(a)$, there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon. \tag{2}$$

for $|x - a| < \delta$, $x \neq a$. This inequality holds even if $x = a$ since in this case the left hand quantity is zero.

But, since $\lim_{n\to\infty} x_n = a$, there is an N such that

$$|x_n - a| < \delta$$

for all $n > N$. Replacing x with x_n in (2) shows that

$$|f(x_n) - f(a)| < \epsilon$$

for $n > N$, which proves our theorem.

Continuity is important for solving equations.

EXAMPLE 5. Show that the following equation has a solution $x \in [0, 1]$.

$$2x^3 + x^2 - 1 = 0$$

Solution: We compute that $f(0) = -1$ and $f(1) = 2$ which suggests that f has a zero somewhere in the interval $[0, 1]$. As a check we graph f over $[0, 1]$. The graph certainly seems to confirm the existence of a zero.

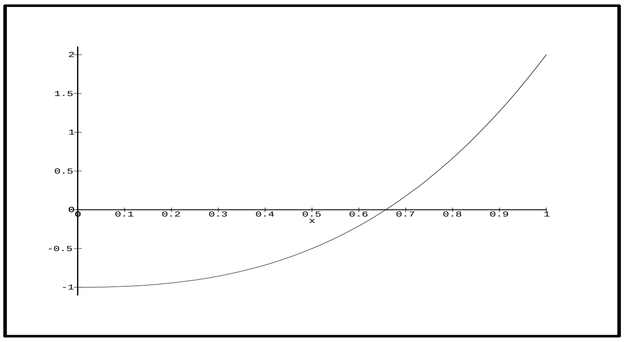


FIGURE 4

We stress, however, that the graph only *seems* to cross the axis. Indeed, on many graphing calculators, if you trace the graph, you will not find a value of x for which y is exactly zero. This is because the calculator only plots a finite number of points. That the graph actually *does* cross the x-axis is a consequence of Proposition 2 below. The last sentence in the statement of this proposition says that there is a *smallest* a such that $f(a) = 0$. Theorem 4 below, which is an immediate consequence of Proposition 2, is one of the fundamental results in analysis.

PROPOSITION 2. *Let f be continuous at every x in a closed interval $[b, c]$. Suppose that $f(b) < 0$ and $f(c) > 0$. Then there is an $a \in [b, c]$ such that $f(a) = 0$. We may choose a so that $f(x) < 0$ for all $x < a$, $x \in [b, c]$.*

Proof Let

$$S = \{x \in [b, c] \mid f(x) \geq 0\}$$

and let $a = \inf S$. (See Figure 5 below.) We will show that $f(a) = 0$. This will finish our proof since if $x \in [b, c]$ satisfies $x < a$, then $x \notin S$, showing that $f(x) < 0$.

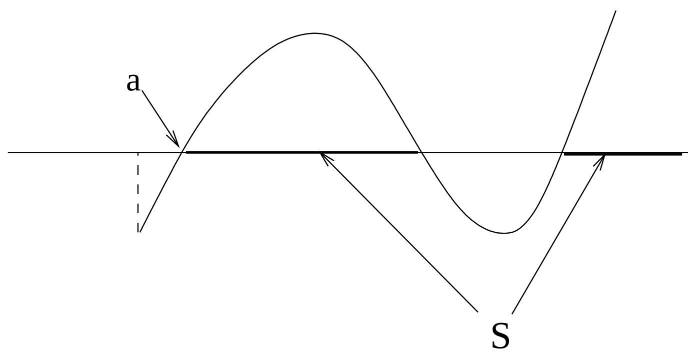


FIGURE 5

From Exercise 9 on page 95 in Chapter 6, there is a sequence $x_n \in S$ with $\lim_{n\to\infty} x_n = a$. From Theorem 3

$$f(a) = \lim_{n\to\infty} f(x_n).$$

Since $x_n \in S$, $f(x_n) \geq 0$. Hence $f(a) \geq 0$. (Exercise 29 on page 73 in Chapter 4.)

Suppose that $f(a) \neq 0$. Then $f(a) > 0$. We claim that it follows that there is a $\delta > 0$ such that $f(x) > 0$ for all $x \in [b, c]$ satisfying $|x - a| < \delta$. If our claim is true, then we have reached a contradiction, since it follows that $f(x)$ is positive on the interval $a - \delta < x < a + \delta$, which denies our observation that $f(x) < 0$ for $x < a$.

To prove the claim, let $\epsilon > 0$ be chosen so that

$$0 < \epsilon < f(a).$$

Since

$$\lim_{x \to a} f(x) = f(a)$$

there is a $\delta > 0$ such that for $0 < |x - a| < \delta$,

$$|f(x) - f(a)| < \epsilon$$
$$-\epsilon < f(x) - f(a) < \epsilon$$
$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

It is clear that the above inequalities are also valid for $x = a$. This proves or claim since $f(a) - \epsilon > 0$. Hence, our proposition follows. □

The following result follows by applying Proposition 2 to the function $f(x) = g(x) - D$. The details are left to the reader. (Exercise 11) As before, the last sentence in the statement of this theorem says that there is a *smallest* a such that $g(a) = D$.

THEOREM 4 (Intermediate Value (IVT)). *Let g be continuous at every x in a closed interval $[b, c]$. Suppose that $g(b) < D$ and $g(c) > D$. Then there is an a in the interval $[b, c]$ such that $g(a) = D$. This a may be chosen so that $g(x) < D$ for $x \in [b, c]$, $x < a$.*

Remark: The conclusion of the Theorem 4 holds if we assume instead that $g(b) > D$ and $g(c) < D$. This result follows by applying Proposition 2 to the function $f(x) = D - g(x)$. Again, we leave the details to the reader. (Exercise 12)

Remark: By definition, $\sqrt{2}$ is that positive number a such that

$$a^2 = 2.$$

How do we know that such a number exists? None of the axioms from Chapters 1 and 2 state that such a number exists. In fact, since $\sqrt{2}$ is irrational, the axioms from Chapters 1 and 2 *cannot*, by themselves, be used to prove the existence of $\sqrt{2}$: if we could prove its existence using these axioms then our proof would prove the existence of $\sqrt{2}$ in the rational numbers since these axioms all hold for both the real and the rational number systems. Thus, in the context of these notes, *we cannot prove the existence of $\sqrt{2}$ without using either the Least Upper Bound Axiom or one of its consequences*, such as the IVT.

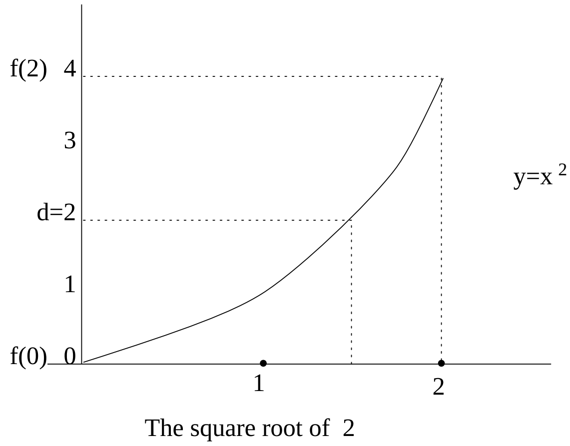


FIGURE 6

In fact, the existence of $\sqrt{2}$ is a simple consequence of the IVT. Consider the function $f(x) = x^2$ on the interval $[0, 2]$ shown in Figure 6. Proposition 1 Shows that $f(x)$ is continuous for all x. Also, $f(0) = 0$ and $f(2) = 4$. Since $0 < 2 < 4$, it follows from the IVT that there is a number $a \in [0, 2]$ such that $2 = f(a) = a^2$, proving the existence of $\sqrt{2}$. In fact, in precisely the same manner we can prove that every positive number d has a positive square root.

There is a deep and important difference between the way continuous functions behave on closed intervals and other types of intervals. Consider for example the function $f(x) = x^2$ on the interval $(0, 2)$. The maximum value of $f(x)$ appears to 4; except 4 is not a value of $f(x)$ at all since $2 \notin (0, 2)$. Rather 4 is a sup. This function has no maximum over $(0, 2)$. Similarly f has only an inf over $(0, 2)$ since $0 \notin (0, 2)$. Even worse, consider the function $f(x) = 1/x$ on the same interval. This function isn't even bounded on this interval.

On the other hand, our intuition tells us that this kind of "misbehavior" cannot happen for a continuous function over a closed interval. Such a function should have both a maximum and a minimum.

THEOREM 5. *Let f be continuous at every x in a closed interval $[a, b]$. Then there is a value $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.*

Proof The proof breaks down into two steps:

(1) Prove that there is a number M such that $f(x) \leq M$ for all $x \in [a, b]$. (We say that $f(x)$ *is bounded from above.*)

(2) Prove the existence of c.

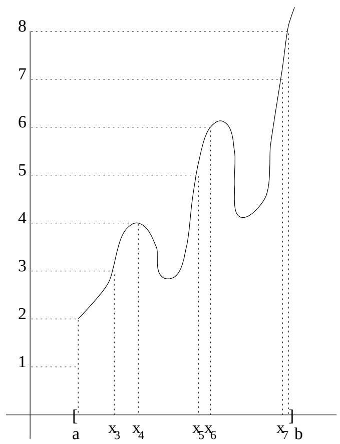


FIGURE 7

To prove (1), assume that it is false. Then for each $M \in \mathbb{R}$, there is an $x \in [a, b]$ such that $f(x) > M$. In particular, for each $n \in \mathbb{N}$, $n > f(a)$, there is an $x \in [a, b]$ such that

$$f(x) > n > f(a).$$

It then follows from the MVT that there is a smallest value $x_n \in [a, b]$ such that

$$f(x_n) = n.$$

(See Figure 7.)

Figure 7 suggests that the x_n are increasing. This is indeed true: Since $f(x_{n+1}) = n + 1 > n > f(a)$, there is a value of x between a and x_{n+1} such that $f(x) = n$. Since x_n is the first such x, we see that $x_n \leq x \leq x_{n+1}$, as claimed.

From the Bounded Increasing Theorem, $x = \lim_{n\to\infty} x_n$ exists. But then

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = \infty$$

which is nonsense, proving that f is bounded.

Now let

$$y_{max} = \sup\{f(x) \mid x \in [a, b]\}.$$

This exists since, as we just showed, f is bounded from above. We want to prove that there is a $c \in [a, b]$ such that

$$f(c) = y_{max}.$$

Suppose that this is false. Then the function

$$g(x) = y_{max} - f(x)$$

is positive on $[a,b]$. Hence, from Theorem 2

$$h(x) = \frac{1}{y_{max} - f(x)}$$

is continuous on $[a,b]$. Thus, from the argument done to prove part (1), $h(x)$ is bounded from above–i.e. there is a number M' such that

$$h(x) \le M' \tag{3}$$

for all $x \in [a,b]$.

On the other hand, since y_{max} is the sup of the y-values of $f(x)$ over $[a,b]$, there is a sequence $x_n \in [a,b]$ such that

$$y_{max} = \lim_{n\to\infty} f(x_n).$$

(Exercise 9 on page 95 in Chapter 6.)

There is then an N such that for all $n \ge N$,

$$|y_{max} - f(x_n)| < \frac{1}{M'}.$$

This implies that $h(x_n) > M'$, contradicting inequality (3). This finishes the proof of our theorem. □

Remark: There is of course a Min Theorem. Being lazy, and not liking to type, we shall leave both the statement and the proof to the reader.

Exercises:

(1) Let $f(x) = (\sin x)/x$ of $x \ne 0$ and let $f(0) = 1$. Why is f continuous at $x = 0$?

(2) Let g be the function defined by

$$g(x) = \frac{x^{100} - 2^{100}}{x - 2} \quad x \ne 2.$$

How should $g(2)$ be defined so as to make g continuous for all real numbers x.

(3) Let $f(x)$ be differentiable at $x = a$. How should the following function be defined at $x = a$ to make it continuous.

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

(4) Let $f(x) = x\sin(1/x)$. How should $f(0)$ be defined so as to make f continuous.

(5) Let f be the function defined by

$$f(x) = x \quad x < 1$$
$$f(x) = a - x \quad x \geq 1$$

where a is some real number. How should a be chosen so as to make f continuous for all real x?

(6) Find an example (reader's choice) of a function which in not continuous at

$$1, \quad \frac{1}{2}, \quad \frac{1}{3} \quad \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

but is continuous for all other values of x, including $x = 0$. *Hint*: Let $f(x) = 0$ if $x \neq 1/n$. How should $f(1/n)$ be defined?

(7) Suppose that we define a function f by saying that $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Thus, $f(\pi) = -1$ and $f(2/3) = 1$. Graph f. For which values of x is f continuous? Explain.

(8) Define a function f as follows. Suppose first that x is rational, $x = p/q$ where p and q are integers with $q > 0$ and p and q have no common factors. In this case, we define $f(x) = 1/q$. If x is irrational, we define $f(x) = 0$. Thus

$$f(\frac{3}{4}) = f(\frac{1}{4}) = \frac{1}{4}$$
$$f(\frac{4}{18}) = f(\frac{2}{9}) = fn19f(\frac{32}{17}) \quad = \frac{1}{17}$$
$$f(\sqrt{2}) = f(\pi) = 0.$$

(a) Compute $f(3)$, $f(3.1)$, $f(3.14)$ and $f(3.141)$. Do you think f is continuous at $x = \pi$? Explain.

(b) Compute $f(3.1)$, $f(3.01)$, $f(3.001)$ and $f(3.0001)$. Do you think f is continuous at $x = 3$? Explain.

(c) For which values of x do you think $f(x)$ continuous? Explain.

(9) Consider the function $f(x) = 1/x$. Then $f(1) = 1 > 0$ and $f(-1) = -1 < 0$. Theorem 1 would seem to say that there is an $a \in [-1, 1]$ such that $f(a) = 0$. This, of course, is false. Why is this not a contradiction to the IVT?

(10) Prove that there is a value of x such that $x^3 - x = 10$. Find the value of x to within $\pm.005$. Prove your answer.
(11) Write a careful proof of the IVT (Theorem 4) using Proposition 2.
(12) Write a complete statement of the theorem implied by the remark immediately following the statement of the IVT (Theorem 4). Then use Proposition 2 to prove this theorem.
(13) Prove that there is an $x \in [0, 1]$ such that $\cos x = x$ and find x to within $\pm.001$. *Hint*: Let $f(x) = \cos x - x$. Consider $f(0)$ and $f(1)$.
(14) Suppose that f is continuous at every x in $[0, 1]$ and that for all x in this interval, $0 \leq f(x) \leq 1$. Prove that there is an $x \in [0, 1]$ such that $f(x) = x$. *Hint*: This is similar to Exercise 13.
(15) Suppose that f is continuous at every x in $[0, 1]$ and that for all x in this interval, $0 \leq f(x) \leq 1$. Prove that there is an $x \in [0, 1]$ such that $f(x) = 1 - x$. *Hint*: This is similar to Exercise 13.
(16) Find both points of intersection of the curves curves $y = e^x$ and $y = 3x + 1$. Give an answer accurate to within $\pm.01$. (Note: If you were asked to find the area between these curves, you would need to find these points before integration. There is no algebraic way to solve for these points.)
(17) Prove than any cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has at least one real zero. For this you should consider $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
(18) Draw a graph which represents a one-to-one function $f(x)$ which is defined for all real numbers x and which is increasing for some values of x and decreasing for other values of x. What 'bad' property does your graph exhibit. Prove that any such example must necessarily have this same 'bad' property.

    (Recall that one-to-one means that for each y-value there is at most one x such that $f(x) = y$.)
(19) Let f be a continuous at every x in a closed interval $[a, b]$. Prove that the range of f is also a closed interval. *Hint*: Prove that the range is $[c, d]$ where $c = \min f(x)$ and $d = \max f(x)$
(20) Let f be a one-to-one function and let g be the inverse function. (Hence, $g(f(x)) = x$ for all x in the domain of f.)

Prove that $f(g(y)) = y$ for all y in the range of f. *Hint:* Since y is in the range of f, $y = f(x)$ for some x.

(21) Let f be a one-to-one function which is increasing. Prove that f^{-1} is also increasing. *Hint* Suppose that there are numbers $a < b$ such that $g(a) \geq g(b)$. What do you know about the effect of applying f to inequalities?

(22) Let f be a continuous, increasing function defined for all real numbers and let $g(x) = f^{-1}(x)$. Below is a rather poorly written proof of the continuity of $g(x)$. Rewrite this proof in a more acceptable form. Specifically

(a) You will need to begin with a statement defining ϵ followed by a statement defining δ.

(b) You will need to prove that the value of δ defined in (5) is positive. *Hint:* Apply $f(x)$ to the inequality $g(a)+\epsilon > g(a) > g(a) - \epsilon$.

(c) The given proof is a "backwards" proof. You will need to reverse it.

(d) You will need to include a statement between (4) and (5) defining x such as "Let $0 < |x - a| < \delta$."

(e) You will need to explain how (4) follows from your definition of δ.

(f) You will need to explain how (2) follows from (3).

(g) You will need to explain how (1) follows from (2). *Hint:* See Exercise 22 above.

(h) You will need to put in a "bottom line" statement indicating that you have done what was necessary.

Proof

$|g(x) - g(a)| < \epsilon$

$$g(a) - \epsilon < g(x) < g(a) + \epsilon \qquad (1)$$

$$f(g(a) - \epsilon) < f(g(x)) < f(g(a) + \epsilon) \qquad (2)$$

$$f(g(a) - \epsilon) < x < f(g(a) + \epsilon) \qquad (3)$$

$$f(g(a) - \epsilon) - a < x - a < f(g(a) + \epsilon) - a \qquad (4)$$

Let δ be the smaller of

$$a - f(g(a) - \epsilon) \text{ and } f(g(a) + \epsilon) - a. \qquad (5)$$

(23) State carefully a theorem relating to minimums which is analogous to Theorem 5. Give a careful proof of your theorem using a similar line of reasoning as was used in proving Theorem 5.

(24) Let

$$f(x) = 2 \quad x > 3$$
$$f(x) = 1 \quad x \leq 3$$

Graph f. What is the largest x such that $f(x) \leq 1$? What is the smallest x such that $f(x) \geq 2$?

(25) Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose also that $f(a) = f(b)$. Prove that there is a point x_o , $a < x_o < b$, such that $f'(x_o) = 0$. For your proof you may assume the theorem that states that $f(x)$ has either a max or a min at $x_o \in (a, b)$, then $f'(x_o) = 0$.

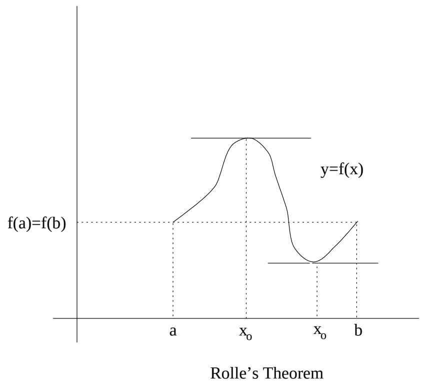
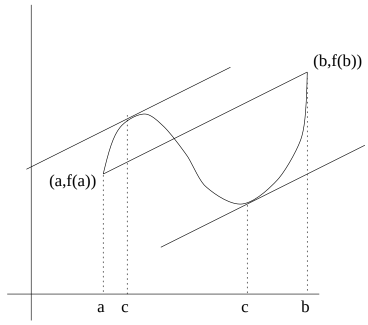


FIGURE 8. Exercise 25

(26) Rolle's theorem is important for one, and only one, reason: It is used in proving the Mean Value Theorem. The Mean Value Theorem is pictured below. Pictorially, it says that given a secant line for some differentiable curve, there is a point at which the slope of the tangent line is equal to that of the secant line.

In writing, the MVT says

THEOREM 6 (Mean Value Theorem MVT). *Let f be continuous on the closed interval $[a, b]$ and differentiable on the*



The Mean Value Theorem

FIGURE 9. Exercise 26

open interval (a, b). *Then there is a value* c, $a < c < b$, *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

In this problem, we request that you answer the questions below and, hence, prove the MVT.

(a) Let l be the line which passes through the points $(a, f(a))$ and $(b, f(b))$ in the figure above. This is the secant line. Compute a formula for l. Express your formula in the form $y = mx + B$.

(b) Let $h(x) = f(x) - (mx + B)$ where $mx + B$ is from (a) above. Indicate on a graph similar to the one above what quantity $h(x)$ measures.

(c) Let h be as in (b) above. Show that $h(a) = h(b)$ and $h'(x) = f'(x) - m$. What, explicitly, does Rolle's Theorem tell you about h? The MVT should drop out!

(27) If f is a continuous function defined over a closed interval $[a, b]$, we define the 'average value' of f to be

$$A = \frac{1}{b - a} \int_a^b f(x) dx.$$

The reason that this is thought of as an average is that the integral is thought of as summing the values of $f(x)$ for $x \in [a, b]$ and $(b - a)$, in some sense, represents the number of x in $[a, b]$.

Now, suppose that f is increasing (and continuous) over $[a, b]$. We expect that $f(a)$ is 'below average' and $f(b)$ is 'above average'. There should, then, exist some value c between a and b where $f(c)$ is exactly average. This is the content of the 'Mean Value Theorem for Integrals'.

THEOREM 7 (Integral Mean Value). *Let f be a continuous function over the interval $[a, b]$. Then there is a c between a and b such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

In this exercise, you are asked to prove this important theorem in the case where $f(a) > 0$ and f is increasing over the interval $[a, b]$. Four your proof, you should use geometric reasoning involving area to prove that

$$f(a) \leq \frac{1}{b-a}\int_a^b f(x)dx$$

($f(a)$ is 'below average') and

$$f(b) \geq \frac{1}{b-a}\int_a^b f(x)dx.$$

($f(b)$ is above average.) How does the Integral Mean Value Theorem follow? How have you used the continuity of f? *Hint*: Put a rectangle of height $f(a)$ under the curve and a rectangle of height $f(b)$ over the curve.

(28) In the above exercise, you needed to assume that f was increasing. You can avoid this if you deal with the minimum and maximum values of f instead of $f(a)$ and $f(b)$. Explicitly, use geometric reasoning to prove that the minimum value of f is 'below average' and the maximum is 'above average'. How does the theorem follow?

(29) One of the most important uses of the Integral Mean Value Theorem is to prove the Fundamental Theorem of Calculus. Let f be a continuous function defined for all real numbers. Let

$$F(x) = \int_0^x f(t)dt.$$

The Fundamental Theorem says that for all a,

$$F'(a) = f(a).$$

In this exercise, we request that you use the Mean Value Theorem for Integrals to prove the Fundamental Theorem. The proof is based upon

$$F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}.$$

The most important step is to prove that

$$\frac{F(x) - F(a)}{x - a} = \frac{1}{x - a} \int_a^x f(t)dt.$$

Once you have done this, you apply the Mean Value Theorem for Integrals and let $x \to a$. Write out in detail how this all works.