CHAPTER 2

Inequalities

In this section we add the axioms describe the behavior of inequalities (the **order axioms**) to the list of axioms begun in Chapter 1. A thorough mastery of this section is essential as analysis is based on inequalities.

Before describing the additional axioms, however, let us first ask, "What, exactly, is an inequality?" Addition is a binary operation; it takes two numbers a and b and produces a third, $a + b$. Less than is a **binary relation**: it takes two numbers a and b and produces either the value 'true' or 'false'. Mathematically, we would say that $<$ is a function whose domain is the set of all pairs of real numbers and whose range is the set $\{\text{true, false}\}$. Thus $2 < 3$ produces 'true' and $3 < 2$ produces 'false'. If we write $a < b$ without explanation, we are asserting that $a < b$ is true.

Order Axioms

I1: (Trichotomy) For real numbers a and b, one and only one, of the following statements must hold:
(1) $a < b$
(2) $b < a$
(3) $a = b$.

I2: (Transitivity) If $a < b$ and $b < c$, then $a < c$.

I3: (Additivity) If $a < b$ and c is any real number, then $a + c < b + c$.

I4: (Multiplicativity) If $a < b$ and $c > 0$, then $ac < bc$.

Important! *Throughout this text, in our proofs, we will typically only give reasons for material from the current chapter. Hence, in doing proofs with inequalities, we will typically not explicitly indicate* *the use of field axioms such as associativity, commutativity, etc. Similarly, in Chapter 3, we will not typically indicate the use of the order axioms in our proofs.*

We define $a > b$ to mean $b < a$. The statement $a \leq b$ is a compound statement. It is true if either $a < b$ or if $a = b$. Thus $2 \leq 3$ and $3 \leq 3$ are both true statements. The symbol '$\geq$' is defined similarly.

There are many rules for studying inequalities which are derivable from the axioms. The reader will be asked to prove many of them in the exercises. **These are not axioms.**

THEOREM 1. *Let a, b, c, and d be real numbers. Then*

E1: (Inequalities add) If $a < b$ and $c < d$, then $a + c < b + d$.
E2: (Positive inequalities multiply) If $0 < a < b$ and $0 < c < d$, then $0 < ac < bd$.
E3: (Multiplication by negatives reverses inequalities) If $a < b$ and $c < 0$, then $ac > bc$.
E4: (Inversion reverses inequalities) If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b} > 0$.
E5: (The product to two negatives is positive) If $a < 0$ and $b < 0$ then $ab > 0$.
E6: If $ab > 0$ then either both a and b are positive or they are both negative.
E7: For all a, $a^2 \geq 0$.
E8: If $a \in \mathbb{N}$, $a > 0$. (Recall that $\mathbb{N}$ is the set of natural numbers.)

Remark: In the following example we use interval notation familiar from calculus. Thus, if a and b are real numbers with $a < b$, then (a, b) is the set of x such that $a < x < b$. Use of a bracket instead of a parenthesis indicates that the corresponding end point is included. Hence, for example, $[a, b)$ is the set of x such that $a \leq x < b$. Use of ∞ as a right end point, or $-\infty$ as a left endpoint, indicates that the interval has no endpoint on that side. Note, however, that ∞ is NOT A NUMBER! Thus, for example, there is no interval "$(-1, \infty]$."

In general, a set is just a collection of objects. The objects in the set are the *elements* of the set. We write "$x \in A$" as shorthand for "x is an element of A." Hence, $x \in (2, 5]$ is equivalent with $2 < x \leq 5$.

EXAMPLE 1. Find the solution set to the following inequality. Do your work in a step-by-step manner so as to demonstrate the order axioms and properties used.

$$2x + 1 < 3x + 2 \tag{1}$$

Solution: Suppose that x satisfies the given inequality. Then

$$\begin{aligned} 2x + 1 &< 3x + 2 \\ 2x &< 3x + 1 \quad \text{(I3)} \\ -x &< 1 \quad \text{(I3)} \\ x &> -1 \quad \text{(E3)} \end{aligned}$$

Hence, if x satisfies inequality (1), then $x \in (-1, \infty)$.

Conversely, suppose that $x \in (-1, \infty)$. Then

$$\begin{aligned} x &> -1 \\ -x &< 1 \quad \text{(E3)} \\ 2x &< 3x + 1 \quad \text{(I3)} \\ 2x + 1 &< 3x + 2 \quad \text{(I3)} \end{aligned}$$

Hence, x satisfies the given inequality, showing that the solution set is $(-1, \infty)$.

Note that our proof in the second part of the solution was just the proof from the first part "run in reverse."

Remark: Notice that our solution required two proofs: we first proved that if x solves (1), then $x \in (-1, \infty)$. We next proved that if $x \in (-1, \infty)$, then it solves (1). Both parts are necessary because the solution set is a SET. Two sets A and B are equal if they consist of exactly the same elements–i.e. every element of A is an element of B AND every element of B is an element of A. Thus, in principal, *proving the validity of a solution set to an inequality will always involve two proofs.* We first show that if some number solves the inequality, then it belongs to a certain set. Next we show that every element of this set solves the inequality. This second proof is often just the first proof reversed. In fact, we often skip the second proof altogether since it is usually clear that our steps do reverse. *However, in this section we will insist that you do the reverse proof since we* *want to stress that both parts are necessary.* Sometimes, in fact, the steps do not reverse, in which case your solution may require some modification, as in the following example.

EXAMPLE 2. Solve the following inequality and prove your answer.

$$x \geq \sqrt{2-x} \tag{2}$$

Solution: We begin by noting that inequality (2) is meaningless if $x > 2$, since then $\sqrt{2-x}$ in undefined. Thus, we assume that $x \leq 2$. We now reason as follows[1]:

$$\begin{aligned} x &\geq \sqrt{2-x} \\ x^2 &\geq 2-x \quad \text{(We squared both sides}^1\text{)} \\ x^2+x-2 &\geq 0 \quad \text{(I3)} \\ (x-1)(x+2) &\geq 0 \end{aligned}$$

From (E6), this inequality holds if $(x-1)$ and $(x+2)$ both have the same sign (or are both zero) which holds if either $x \geq 1$ (both terms ≥ 0) or $x \leq -2$ (both terms ≤ 0). Since we have already assumed $x \leq 2$, our solution appears to be $(-\infty, -2] \cup [1, 2]$. [2] This, however, is wrong. For example, if $x = -2$, inequality (2) says $-2 \geq \sqrt{4} = 2$.

To find our mistake, we attempt to reverse our sequence of inequalities:

Suppose that $x \in (-\infty, -2] \cup [1, 2]$. Then $x - 1$ and $x + 2$ both have the same sign (or are both zero). Hence

$$\begin{aligned} (x-1)(x+2) &\geq 0 \\ x^2+x-2 &\geq 0 \\ x^2 &\geq 2-x \end{aligned} \tag{3}$$

We would like to take the square root of both sides of this inequality. We must, however, be careful. For negative x, it is not true that $\sqrt{x^2} = x$. For example

$$\sqrt{(-2)^2} = \sqrt{4} = 2 \neq -2$$

[1]We discuss the validity of operations such as squaring and square rooting inequalities after the discussion of this example.

[2]In set theory, $A \cup B$ is the set of elements which belong to A or B or both.

Rather $\sqrt{x^2} = |x|$. Thus, "square-rooting" both sides of inequality (3) produces

$$|x| > \sqrt{2-x}$$

which is not equivalent with equation (2).

In fact, since square roots can never be negative, only non-negative x can satisfy inequality (2). Thus, the interval $(-\infty, -2]$ cannot be part of our solution set. For x in $[1, 2]$, the final inequality in formula (3) can be square-rooted, showing that our solution set is just $[1, 2]$.

Our first step in solving Example 2 was to square both sides of the inequality (2). Is this allowed? More generally, if we do the same thing to both sides of an inequality, is the inequality preserved? The answer to this last question is, "NO!" If we multiply both sides by -1, the inequality reverses: $2 < 3$ but $-2 > -3$. If we take the inverse of both sides the inequality can also reverse: $2 < 3$ but $\frac{1}{2} > \frac{1}{3}$. On the other hand, adding the same number to both sides preserves the inequality: $2+1 < 3+1$. So when does doing the same thing to both sides preserve the inequality and when does it reverse it?

The answer comes from calculus. Recall that a function $y = f(x)$ is said to be increasing if y gets larger as x gets larger–i.e. $x_1 < x_2$ implies $f(x_1) < f(x_2)$. This means that applying an increasing function to both sides of an inequality preserves it. The function $y = x^2$ is increasing for $x \geq 0$. (Figure 1) Hence squaring both sides of an inequality will be valid as long as both sides are non-negative. Since square roots are non-negative, inequality (2) is only meaningful if both sides are non-negative. Hence, squaring both sides was indeed valid.

Similarly, applying a decreasing function to both sides of an inequality will reverse it. For example, the function $y = x^2$ is decreasing for $x < 0$. Hence, squaring inequalities involving *negative* numbers will reverse the inequality. For example $-3 > -4$ but $9 < 16$.

EXAMPLE 3. Find an interval I on which $a < b$ implies

$$ae^{-a} > be^{-b}$$

Solution: Let $f(x) = xe^{-x}$. The example asks for an interval I on which application of $f(x)$ to both sides of $a < b$ reverses the inequality. This will be true for any interval on which f is decreasing.

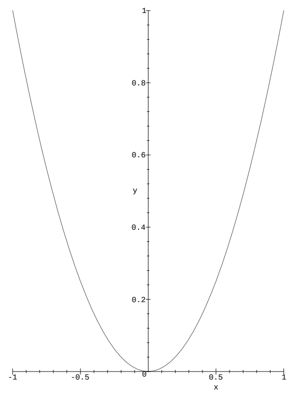

FIGURE 1

From calculus, $f(x)$ will be decreasing on any interval where $f'(x) < 0$. We compute

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

which is negative for $x > 1$. Hence we may take $I = (1, \infty)$. To illustrate our answer, note that 2 and 3 belong to I and $2 < 3$, but $2e^{-2} = .271$ which is larger than $3e^{-3} = .149$.

Remark: In a formal proof, whenever you apply a function to both sides of an inequality, you must justify your work in terms of the increasing or decreasing nature of the function in question.

When solving inequalities, one must be careful when multiplying both sides by a quantity which might potentially be negative.

EXAMPLE 4. Solve the following inequality and prove your answer.

$$\frac{x}{x+1} \geq 1 \tag{4}$$

Solution: If we are not careful, we will not find any such x. Specifically, we might reason as follows:

$$\begin{aligned} \frac{x}{x+1} &\geq 1 \\ x &\geq x+1 \quad \text{(I4)} \\ 0 &\geq 1 \quad \text{(I3)} \end{aligned}$$

Since $0 < 1$, there are no such x.

But this is wrong. For $x = -2$,

$$\frac{-2}{-2+1} = 2 \geq 1$$

Our mistake lay in the first step of our solution where we multiplied both sides of the given inequality by $x + 1$ without reversing the inequality. This is valid only if $x + 1$ is positive.

If $x + 1 < 0$, (i.e. $x < -1$)

$$\begin{aligned} \frac{x}{x+1} &\geq 1 \\ x &\leq x+1 \quad \text{(E3)} \\ 0 &\leq 1 \quad \text{(I3)} \end{aligned}$$

Since $1 = 1^2$, (E7) implies that $0 \leq 1$; hence our inequality yields no additional restriction on x. Thus, we guess that the inequality is valid for all $x < -1$. We can prove this by repeating the above inequalities in reverse order:

$$\begin{aligned} 0 &\leq 1 \quad \text{(E7) and } 1 = 1^2 \\ x &\leq x+1 \quad \text{(I3)} \\ \frac{x}{x+1} &\geq 1 \quad \text{(E3)} \end{aligned}$$

Thus, the solution to our inequality is $x < -1$. (Note that the inequality is meaningless if $x = -1$ since division by 0 is not allowed.)

When multiplying (or dividing) an inequality by a quantity that can be either positive or negative, it is often necessary to the cases where the quantity may be positive separate from the cases where it may be negative, as in the next example.

EXAMPLE 5. Solve the following inequality and prove your answer.

$$\frac{x^2}{(x+1)} < x+1 \tag{5}$$

Solution: We begin by multiplying by $x+1$. Since this quantity may be positive or negative, we split the argument into the corresponding cases:

Case 1: $x > -1$.

Assume that x satisfies the given inequality and $x > -1$. Then

$$\begin{aligned}
\frac{x^2}{(x+1)} &< x+1 \\
x^2 &< (x+1)^2 \quad \text{(I4)} \\
x^2 &< x^2 + 2x + 1 \\
0 &< 2x + 1 \quad \text{(I3)} \\
-1 &< 2x \quad \text{(I3)} \\
-\frac{1}{2} &< x \quad \text{(I4)}
\end{aligned}$$

Conversely, if $x > -\frac{1}{2}$, then $x > -1$. Hence, we may reverse the above sequence of inequalities to see that inequality (5) holds. We conclude that in Case 1, our inequality holds if and only if $x \in (-\frac{1}{2}, \infty)$.

Case 2: $x < -1$.

Assume that x satisfies the given inequality and $x < -1$. Then

$$\begin{aligned}
\frac{x^2}{(x+1)} &< x+1 \\
x^2 &> (x+1)^2 \quad \text{(E3)} \\
x^2 &> x^2 + 2x + 1 \\
0 &> 2x + 1 \quad \text{(I3)} \\
-1 &> 2x \quad \text{(I3)} \\
-\frac{1}{2} &> x \quad \text{(I4)}
\end{aligned}$$

Since $-\frac{1}{2} > -1$, this is true for all x satisfying the assumptions of Case 1. Conversely, if $x < -1$, then $x < -\frac{1}{2}$. Hence, we may reverse the above sequence of inequalities to see that inequality (5) holds. We conclude that our inequality for all x satisfying the hypotheses of Case 2.

Conclusion: Putting Case 1 and 2 together, we see that the solution set of the inequality is $(-\infty, -1) \cup (-\frac{1}{2}, \infty)$.

Many students learn to do proofs by starting with what they wish to prove and reasoning until they obtain a true statement. *Without further qualification, this is not a valid proof technique.* For example, if we square both sides of

$$-2 = 2$$

we obtain

$$4 = 4$$

which is true. This certainly does not prove that $-2 = 2$.

To be valid, a proof must, in principle, begin with known facts and end with what you want to prove. The bottom line must be what you want to prove. A proof that begins with what you want to prove and ends with a true statement is called **backwards.**

Often, as the following example illustrates, a backwards proof can be reversed to produce a valid proof. In fact, it is a standard proof technique to begin (on a piece of scratch paper) with what you want to prove, reason until you reach a known statement, and then produce the formal proof by reversing the argument.

In this example, as elsewhere in these notes, we write our "scratch work" in *italics* **to distinguish it from the work we might hand in if this were a homework assignment.**

EXAMPLE 6. Let a and b be positive numbers. Prove that

$$a + b \leq \sqrt{2}(a^2 + b^2)^{1/2}.$$

Scratch Work: *Squaring both sides and simplifying produces:*

$$\begin{aligned}(a+b)^2 &\le 2(a^2+b^2)\\ a^2+2ab+b^2 &\le 2a^2+2b^2\\ 0 &\le a^2-2ab+b^2 \quad \textit{(Subtract } a^2+2ab+b^2 \textit{ from both sides.)}\\ 0 &\le (a-b)^2\end{aligned}$$

which is true from (E7). Our formal proof will be obtained by reversing the above steps.

Proof:

$$\begin{aligned}0 &\le (a-b)^2 \quad \text{(E7)}\\ 0 &\le a^2-2ab+b^2\\ a^2+2ab+b^2 &\le 2a^2+2b^2 \qquad \text{(I3): Add } a^2+2ab+b^2 \text{ to both sides}\\ (a+b)^2 &\le 2(a^2+b^2)\\ a+b &\le \sqrt{2}(a^2+b^2)^{1/2} \quad y=\sqrt{x} \text{ is a increasing function}\end{aligned}$$

as desired.

Note that in our scratch work we squared both sides of the inequality whereas in the actual proof we took the square root of both sides.

Remark: A backwards argument becomes a valid proof if we are careful to mention (and check) the reversibility of each step in the backwards argument. *In this section we will insist that all "backwards proofs" be reversed. Later we will allow you to simply note that the steps do reverse.*

We will often need to study inequalities involving absolute values. By definition,

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

Proving theorems about absolute values often requires the consideration of several cases which depend on the sign of the quantities involved.

EXAMPLE 7. Prove that for all x and y

$$|xy| = |x|\,|y|$$

Solution: There are 3 cases: (1) $x > 0$, (2) $x = 0$ and (3)$x < 0$. We will begin case (1) and leave the remaining cases as exercises.

Case (1) splits into three subcases: (2.1) $y > 0$, (2.2) $y = 0$, and (2.3) $y < 0$.

We will do only (2.3). In this case, $|x| = x$ and $|y| = -y$ so

$$|x|\,|y| = -xy$$

Also, multiplication of the inequality $x > 0$ by y reverses it, showing that $xy < 0$. Hence

$$|xy| = -xy$$

Hence, in this case, $|xy| = |x|\,|y|$

The kind of absolute value inequalities we need are typically of the form

$$|x| < a \tag{6}$$

which is equivalent with

$$-a < x < a$$

The inequality

$$a < |x|$$

is equivalent with

$$x > a \quad \text{or} \quad -x > a$$

EXAMPLE 8. Find all x such that

$$|4 - 3x| < .1 \tag{7}$$

Solution: We reason

$$-.1 < 4 - 3x < .1$$
$$-4.1 < -3x < -3.9$$
$$\frac{4.1}{3} > x > \frac{3.9}{3} = 1.3$$

This shows that if x satisfies (7), then $x \in (1.3, \frac{4.1}{3})$. We leave it as an exercise to show that conversely, every x in this interval does satisfy (7).

A very important absolute value inequality is the triangle inequality which states that for all real numbers x and y

$$|x+y| \leq |x| + |y| \tag{8}$$

We leave the case-by-case proof as an exercise.

Exercises

In doing inequality problems, you should quote the relevant inequality axioms or properties in use, (I1-I4, E1-E8) but you need not quote any of the axioms or properties from Chapter 1.

(1) In each part, solve the inequality and prove your answer. Justify any application of functions (such as taking logs, exponentials, square roots, etc.) in terms of increasing and decreasing.
  (a) $2x+7 < 3x+4$ *ans.* $(3, \infty)$
  (b) $|2y-8| < .0002$ *ans.* $(3.9999, 4.0001)$
  (c) $\ln(3-4t) < 7$ *ans.* $((3-e^7)/4, 3/4)$. (Note: lnx is defined only if $x > 0$.)
  (d) $\frac{x}{3x+1} > \frac{1}{3}$ *ans.* $(-\infty, -\frac{1}{3})$
  (e) $((.5)^x - 3)^{1/3} < 5$ *ans.* $(\frac{\ln(128)}{\ln(.5)}, \infty)$
  (f) $(2x-1)^{-1/3} > 2$ *ans.* $(\frac{1}{2}, \frac{9}{16})$
  (g) $0 \leq \arccos(3x+1) < \frac{\pi}{3}$ *ans.* $(-\frac{1}{6}, 0]$ *Hint:* Graph $y = \cos x$ over $[0, \frac{\pi}{3}]$.
  (h) $0 \leq \arcsin(3x+1) < \frac{\pi}{4}$ *ans.* $[-\frac{1}{3}, \frac{\sqrt{2}-2}{6})$ *Hint:*Graph $y = \sin x$ over $[0, \frac{\pi}{4}]$.
  (i) $\frac{1}{\arctan x} < .001$ *ans.* $(-\infty, 0) \cup (\tan(1000), \infty)$
  (j) $(.4)^{2x-1} < 7^x$ *ans.* $(\frac{\ln .4}{2\ln .4 - \ln 7}, \infty)$
  (k) $\frac{x^2-3}{x} > x-2$ *ans.* $(-\infty, 0) \cup (\frac{3}{2}, \infty)$. (See Example 5 on page 25.)
  (l) $\frac{x^2}{x+1} < x+2$. (See Example 5 on page 25.)

In the following 7 exercises, you are asked to prove properties (E1)-(E8). In these exercises, *you may only use one of the properties (E1)-(E8) if it was proved in one of the preceding exercises.* **Otherwise, you should use only the axioms.**

(2) Prove property (E1). *Hint:* Try adding c onto both sides of $a < b$ and b onto both sides of $c < d$.

(3) Prove property (E2). Show by example that (E2) can fail if $c < 0$. *Hint:* Reason as in the preceding exercise, using multiplication instead of addition.
(4) Can we subtract inequalities? I.e. if $a < b$ and $c < d$ does it follow that $a - c < b - d$? If so, prove it. If not, find a counter example.
(5) Can we divide positive inequalities? I.e. if $0 < a < b$ and $0 < c < d$ does it follow that $a/c < b/d$? If so, prove it. If not, find a counter example.
(6) Suppose that $a < b$.
    (a) Prove that $-b < -a$. *Hint:* Begin by adding $-b$ onto both sides of $a < b$.
    (b) Prove (E3). *Hint:* From (a), $-c > 0$.
    (c) Prove (E5).
    (d) Prove (E7). *Hint:* Consider three cases: $a > 0$, $a = 0$, and $a < 0$. Which axiom allows you to break this up into these three cases?
    (e) Use (E7) to prove that $1 > 0$. Then prove that 2, 3, and 4 are all positive.
(7) (a) Prove that if $b > 0$, then $b^{-1} > 0$. *Hint:* Either $b^{-1} > 0$ or $b^{-1} = 0$ or $b^{-1} < 0$. If you can show that the latter two conditions are impossible, then the first must hold.
    (b) Prove (E4). *Hint:* Begin by multiplying both sides of $a < b$ by b^{-1}.

In the remaining exercises you may use both the axioms and properties (E1)-(E8).

(8) Prove that the average of two numbers lies between them I.e. if $a < b$ then $a < \frac{a+b}{2} < b$. *Hint:* Do each inequality separately.
(9) Let a and b be positive numbers. Prove that

$$\text{(i)} \qquad \sqrt{ab} \leq \frac{a+b}{2}$$

$$\text{(ii)} \qquad \sqrt{a^2+b^2} \leq a+b$$

$$\text{(iii)} \qquad \frac{a}{b} + \frac{b}{a} \geq 2$$

$$\text{(iv)} \qquad \text{If } 0 < a < b \text{ then } a^2 < ab < b^2$$

(10) Do the remaining subcases from case (1) of Example 6 in the text.
(11) Do Case (2) from Example 6.
(12) Do Case (3) from Example 6.
(13) Prove that for all x and y, $y \neq 0$,

$$|\frac{x}{y}| = \frac{|x|}{|y|}$$

Split your proof into three cases as in Example 6. Your instructor may assing you to do only certain of these cases.

(14) The purpose of this exercise is to understand why the triangle inequality holds.
  (a) Find a non-zero value of y such that $|2 + y| = 2 + |y|$ and a value such that $|2 + y| < 2 + |y|$.
  (b) Describe (i) the set of all y such that $|2 + y| = 2 + |y|$ and (ii) the set of all y such that $|2 + y| < 2 + |y|$.
  (c) Suppose $x > 0$. For which y is $|x + y| = |x| + |y|$? $|x + y| < |x| + |y|$? Note that from Axiom I1, this accounts for all y.
  (d) Answer the questions from (c) under the assumption that $x = 0$. Next answer them under the assumption that $x < 0$.

Remark This all amounts, more or less, to a proof of inequality (7). The reason that it is only "more or less" is that you weren't ask you to prove the answers to (c) and (d).

(15) Graph the set of points (x, y) defined by

$$\begin{array}{ll} \text{(a)} & |x| = |y| \\ \text{(b)} & |x| + |y| = 1 \\ \text{(c)} & |xy| = 2 \\ \text{(d)} & |x| - |y| = 2 \end{array}$$

(16) In parts (a)-(g), find all intervals I for which the stated inequality holds for all elements x and y of I, $x < y$. Illustrate your answer with a specific value of x and y. Prove your answer using calculus.
  (a) $x^3 < y^3$
  (b) $x^{-1/3} < y^{-1/3}$
  (c) $x^4 < y^4$

(d) $\ln x < \ln y$
(e) $ye^{-y^2} < xe^{-x^2}$
(f) $y^3 + 6y^2 + 9y < x^3 + 6x^2 + 9x$
(g) $\frac{x}{x^2+1} < \frac{y}{y^2+1}$