CHAPTER 3

Rates of Growth

Inequalities are very useful in comparing rates of growth of functions.

DEFINITION 1. *If f and g are two functions, we say that f* **dominates** *g if there is a value N such that*

$$f(x) > g(x)$$

for all $x > N$.

(See Figure 1.)

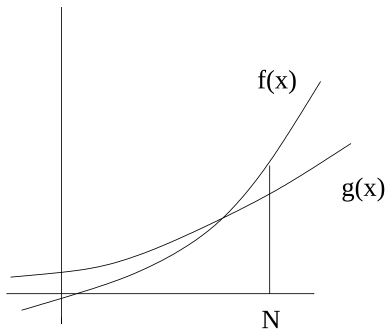

FIGURE 1. f grows faser than g

The function $f(x) = Cx^a$ where C and a are both positive, exhibits **power growth** in that it grows like a power of x. The following example illustrates that the larger the power, the faster the growth, regardless of the value of C.

EXAMPLE 1. Prove that $(.01)x^4$ dominates x^2.

Scratch work: *According to Definition 1, we need to find a value N such that*

$$(.01)x^4 > x^2$$

for all $x > N$. Division by $(.01)x^2$ followed by taking the square root produces

$$x^2 > 100$$
$$x > 10$$

Thus, we can take $N = 10$.

For our formal solution, we state the value of N and reverse the above steps to prove that the stated value works.

Solution: Let $N = 10$. Assume $10 > x$. We may square this inequality because $y = x^2$ is an increasing function for $x > 0$, obtaining

$$x^2 > 100$$

We may multiply by $(.01)x^2$ since this term is positive, showing that

$$(.01)x^4 > x^2$$

for $x > 10$, fulfilling the requirements of Definition 1.

Example 2. For each pair of functions below (i) determine which is the dominant function, (ii) prove your answer by finding a number N fulfilling the requirements of Definition 1. Prove that your value of n really works.

(1) $3x^2 + 5x - 3, \quad x^3$
(2) $x^5, \quad x^6 + 3x^2 - 2x + 1$

Solution

Scratch work(1):

For polynomials, the highest power of x determines the rate of growth. Hence, in (1), we expect x^3 to dominate. To prove our answer we must find an N such that

$$3x^2 + 5x - 3 < x^3$$

for all $x > N$.

We see no way of solving this inequality. Instead we seek a simple quantity "?" and a value N for which

$$3x^2 + 5x - 3 <? < x^3$$

for all $x > N$. In other words, we wish to replace $3x^2 + 5x - 3$ by a larger, simpler quantity, which is still dominated by x^3. We begin by dropping negative terms:

$$3x^2 + 5x - 3 < 3x^2 + 5x.$$

Since

$$\frac{1}{2}x^3 + \frac{1}{2}x^3 = x^3$$

this latter quantity will be less than x^3 provided

$$(1) \qquad 3x^2 < \frac{1}{2}x^3 \quad \textit{and} \quad 5x < \frac{1}{2}x^3.$$

Solving these inequalities yields

$$(2) \qquad 6 < x \quad \textit{and} \quad \sqrt{10} < x.$$

Since $\sqrt{10} < 6$, both of (1) hold for $x > 6$.

Proof (1): Let $N = 6$ and assume $x > N$. Then (2) both hold. Hence (1) both hold. It follows as in the scratch work that

$$3x^2 + 5x - 3 < x^3$$

for all $x > 6$, showing that x^3 dominates $3x^2 + 5x - 3$.

Scratch Work (2):

Now we expect $x^6 + 3x^2 - 2x + 1$ to dominate. To prove this we must find an N such that for $x > N$,

$$x^5 < x^6 + 3x^2 - 2x + 1.$$

Again we seek a quantity "?" satisfying

$$x^5 <? < x^6 + 3x^2 - 2x + 1,$$

meaning that we want to make $x^6 + 3x^2 - 2x + 1$ smaller. *Hence, we drop* positive *quantities producing*

$$x^5 < x^6 - 2x < x^6 + 3x^2 - 2x + 1.$$

We cannot drop the $-2x$ term since this would make x^6-2x larger. We can, however, replace $2x$ by a larger *quantity, such as $\frac{1}{2}x^6$, since* *subtracting a larger quantity produces a smaller result.*[1] *We note that*

$$2x < \frac{1}{2}x^6 \tag{3}$$

for $4 < x^5$*–i.e. for* $4^{1/5} < x$*. For such* x

$$x^6 - 2x > x^6 - \frac{1}{2}x^6 = \frac{1}{2}x^6.$$

Finally

$$\frac{1}{2}x^6 > x^5 \tag{4}$$

provided $x > 2$*. Since* $2 > 4^{1/5}$ *we choose* $N = 2$*.*

Proof (2): Let $N = 2$ and assume that $x > N$. Then (3) and (4) both hold. It follows as in the scratch work that

$$x^5 < x^6 + 3x^2 - 2x + 1$$

for all $x > 2$, showing that $x^6 + 3x^2 - 2x + 1$ dominates x^5.

In studying rates of growth, it is useful to note the following proposition which is proved in the exercises.

PROPOSITION 1. *Suppose that* $0 < a < b$*. Then*

$$x^b > x^a$$

for $x > 1$*.*

One of the goals in the study of rates of growth is to get information how fast a function grows by comparing it to another function whose growth we understand.

DEFINITION 2. *Let* f *and* g *be functions. We say that* f *grows like a multiple of* g *if there is are constants* $C > 0$*,* $D > 0$ *and* N *such that for all* $x > N$*,*

$$Cg(x) < f(x) < Dg(x).$$

(See Figure 2)

EXAMPLE 3. Show that $f(x) = x^3 - 3x^2 + 5x - 1$ grows like a multiple of x^3.

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[1] We choose a multiple of x^6 in order to obtain a simple expression. We used $\frac{1}{2}x^6$ in order to obtain a positive result; no negative multiple of x^6 would dominate x^5.

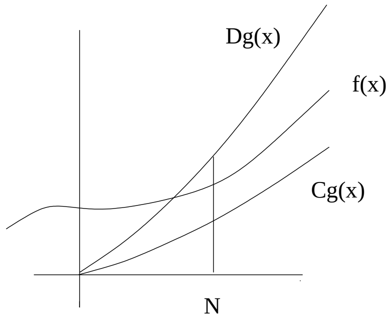

FIGURE 2. f grows like a multiple of g

Solution: According to Definition 2, we need to find positive constants C, D, and N such that

$$Cx^3 < x^3 - 3x^2 + 5x - 1 < Dx^3$$

for all $x > N$.

Deleting negative terms makes a sum larger. Thus, for $x > 1$,

$$x^3 - 3x^2 + 5x - 1 < x^3 + 5x < x^3 + 5x^3 = 6x^3 \tag{5}$$

Hence, for the right hand inequality, we may take $D = 6$ and $N = 1$.

Finding C is harder. Deleting positive terms makes a sum smaller. Hence, for $x > 0$,

$$x^3 - 3x^2 + 5x - 1 > x^3 - 3x^2 - 1 \tag{6}$$

Subtracting more also makes a quantity smaller. Hence, for $x > 1$,

$$x^3 - 3x^2 - 1 > x^3 - 3x^3 - x^3 = -3x^3$$

Unfortunately, we cannot use -3 as C since a positive value is required. To avoid this problem, we replace $3x^2$ and 1 by sufficiently multiples of x^3.

We choose N so that both of the following hold for $x > N$:

$$\begin{aligned} 3x^2 &< \frac{1}{3}x^3 \\ 1 &< \frac{1}{3}x^3 \end{aligned} \tag{7}$$

The first inequality is valid for $x > 9$ and the second for $x > 3^{1/3} \approx 1.44$. For $x > 9$, both are valid and

$$x^3 - 3x^2 - 1 > x^3 - \frac{1}{3}x^3 - \frac{1}{3}x^3 = \frac{1}{3}x^3 \tag{8}$$

Hence we may choose $N = 9$ and $C = 1/3$. This value of N also works for the D inequality with $D = 6$ because $9 > 1$.

It is a general property of ratios of positive numbers that the larger the denominator, the smaller the number. Thus, for example

$$\frac{3}{9} < \frac{3}{5}$$

because $5 < 9$. This principle is the basis for the next two examples.

Example 4. Find a value of m such that the following function grows like a multiple of x^m. Prove your answer.

$$f(x) = \frac{3x^5}{x^3 - 3x^2 + 5x - 1}$$

Solution: We need to find $C > 0$, $D > 0$, m, and N such that

$$Cx^m < \frac{3x^5}{x^3 - 3x^2 + 5x - 1} < Dx^m.$$

for all $x > N$.

Finding m is easy. The numerator is a multiple of x^5 while, from Example 2, the denominator grows like a multiple of x^3. Hence the whole fraction should grow like a multiple of $x^5/x^3 = x^2$, implying $m = 2$. More precisely, from the work done in Example 2, we know that for $x > 9$

$$\frac{1}{3}x^3 < x^3 - 3x^2 + 5x - 1 < 6x^3 \tag{9}$$

We may invert this inequality since, for $x > 9$, $x^3 > 0$. We find

$$\frac{3}{x^3} > \frac{1}{x^3 - 3x^2 + 5x - 1} > \frac{1}{6x^3}$$

We may multiply by $3x^5$ since, again, for $x > 9$, $x^5 > 0$. We find

$$9x^2 > \frac{3x^5}{x^3 - 3x^2 + 5x - 1} > \frac{x^2}{2}$$

Hence, we may choose $m = 2$, $C = 1/2$, $D = 9$ and $N = 9$.

Remark: The principle that increasing the size of the denominator makes the fraction smaller is only valid if both the numerator and denominator are positive. For example

$$\frac{5}{-7} < \frac{5}{3}$$

despite the fact that $-7 < 3$. Thus, the positivity of the C and D found in Example 2 was crucial in Example 3.

EXAMPLE 5. Find a value of m such that the following function grows like a multiple of x^m. Prove your answer.

$$f(x) = \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1}$$

Solution: We need to find $C > 0$, $D > 0$, m, and N such that

$$Cx^m < \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1} < Dx^m.$$

for all $x > N$.

Solution: Since the denominator grows like a multiple of x^3 and the numerator like a multiple of x^2, the whole fraction should grow like a multiple of x^2/x^3 suggesting that $m = -1$. To prove this we find the C and D's for the numerator and denominator. Specifically, from inequality (9), for $x > 9$,

$$\frac{1}{3}x^3 < x^3 - 3x^2 + 5x - 1 < 6x^3$$

which, upon inversion, becomes.

$$\frac{3}{x^3} > \frac{1}{x^3 - 3x^2 + 5x - 1} > \frac{1}{6x^3} \tag{10}$$

For the numerator we find that for $x > 1$

$$4x^2 > 3x^2 + 1 > 3x^2. \tag{11}$$

Multiplying inequalities (10) and (11) shows that, for $x > 9$,

$$\frac{12x^2}{x^3} > \frac{3x^2 + 1}{x^3 - 3x^2 + 5x - 1} > \frac{3x^2}{6x^3}$$

(The multiplication is allowed since, for $x > 9$, both inequalities involve only positive numbers.) Hence $C = \frac{1}{2}$, $D = 12$, $N = 9$ works.

Logarithmic growth is particularly slow. Recall that

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

Thus, $\ln x$ is the area under the curve $y = 1/x$ between 1 and x. Comparison of areas as in Figure 3 shows that for $x > 1$,

$$\ln x < x \tag{12}$$

The same inequality holds for $1 \geq x > 0$ since in this range, $\ln x \leq 0$. Thus $\ln x$ *grows slower than* x.

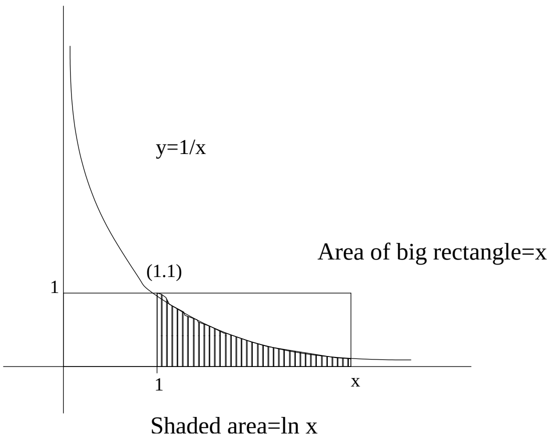

FIGURE 3. $\ln x < x$

We summarize this discussion in the following important theorem:

THEOREM 1. *For all* $x > 0$, $\ln x \leq x$.

Actually, ax dominates $\ln x$ for any $a > 0$.

PROPOSITION 2. *Let* $0 < a$. *Then*

$$\ln x < ax \tag{13}$$

for $x > 4/a^2$.

Proof From Theorem 1, for all $b > 0$,

$$\begin{aligned} \ln(bx) &\leq bx \\ \ln b + \ln x &\leq bx \\ \ln x &\leq bx - \ln b \\ &= bx + \ln \frac{1}{b} \\ &\leq bx + \frac{1}{b} \end{aligned}$$

If $x > \frac{1}{b^2}$, then $bx > \frac{1}{b}$ so

$$\ln x \leq bx + bx = 2bx.$$

Our result follows by letting $b = \frac{a}{2}$. □

It is a consequence of Proposition 2 that *logarithmic growth is slower than power growth, regardless of the power, as long as the power is positive,* as the following example demonstrates.

EXAMPLE 6. Find a value of N such that $\ln x < \frac{x^{1/2}}{3}$ for all $x > N$.

Scratch Work: *To make the powers of x on both sides of the inequality the same, we multiply by $\frac{1}{2}$ and use properties of the logarithm:*

$$\begin{aligned} \ln x &< \frac{x^{1/2}}{3} \\ \frac{1}{2}\ln x &< \frac{x^{1/2}}{6} \\ \ln\left(x^{\frac{1}{2}}\right) &< \frac{x^{1/2}}{6} \end{aligned}$$

According to inequality (13) on page 42

$$\ln y < \frac{y}{6} \textit{ for } y > \frac{4}{(1/6)^2} = 144.$$

Hence, letting $y = x^{1/2}$,

$$\ln\left(x^{\frac{1}{2}}\right) < \frac{x^{1/2}}{6} \textit{ for } x^{1/2} > 144$$

which is the same as $x > (144)^2$. Hence, $N = (144)^2$ works.

Proof: Let $N = (144)^2$. Then, if $x > N$, $x^{1/2} > 144$ so according to Proposition 2 on page 42,

$$\ln\left(x^{\frac{1}{2}}\right) < \frac{x^{1/2}}{6}$$

which, from the scratch work, is equivalent with the desired inequality.

Remark: The value $N = (144)^2 = 20736$ found above is by no means the "best possible" answer. A graphing calculator indicates that in fact $\ln x < \frac{x^{1/2}}{3}$ is true for all $x > 289$. This shows that the value of N given in Proposition 2 can be vastly larger than necessary. This, however, does not matter in determining which is the dominant function.

Exponential growth is faster than power growth, as the next example shows.

EXAMPLE 7. Prove that 2^x dominates x^3.

Scratch work: *We must show that there is an $N > 0$ such that*

$$x^3 < 2^x \tag{14}$$

for all $x > N$. Since $\ln x$ is an increasing function, our inequality is equivalent with

$$\ln x^3 < \ln 2^x$$
$$3 \ln x < x \ln 2$$
$$\ln x < \frac{\ln 2}{3} x$$

which, according to Proposition 2, with $a = \frac{\ln 2}{3}$, is true for

$$x > 36/(\ln 2)^2 \approx 74.93.$$

We cannot, however, use 74.93 as the value of N since this value is only an approximation. If the actual value of $36/(\ln 2)^2$ is slightly greater than 74.93, then it is conceivable that inequality (14) might fail for some $x > 74.93$. However, presuming that our calculator has at least 2 decimal accuracy, we can be certain that, say, $80 > 36/(\ln 2)^2$. Hence, we may use $N = 80$.

Solution: Assume that $x > 80$. Then

$$36/(\ln 2)^2 < x$$

Thus, from Proposition 2,

$$\begin{aligned} \ln x &< \frac{\ln 2}{3} x \\ 3 \ln x &< x \ln 2 \\ \ln x^3 &< \ln 2^x \end{aligned}$$

Since e^x is an increasing function we may continue this sequence of inequalities as follows:

$$\begin{aligned} e^{\ln x^3} &< e^{\ln 2^x} \\ x^3 &< 2^x \end{aligned}$$

as desired.

Remarks: Note that the formal solution required the fact that the exponential function is increasing; not that the logarithm function is increasing, as our "scratch work" had suggested. Typically, if we apply a particular function to both sides of an inequality in the "scratch work," then in the formal solution we will apply the corresponding *inverse* function to both sides of an inequality. Fortunately, it is a general principal that the inverse of an increasing function is increasing. (See Exercise 13 below.) Similar comments apply for decreasing functions. (See Exercise 14 below.)

Example 7 also demonstrates that in rounding computed values of N, we should never "round down." If a particular inequality is known to true for, say, all $x > 4.01$, then it might not be valid for all $x > 4$. It will, however, hold for all $x > 5$.

Another general principle is that if $f(x)$ is dominated by $g(x)$ which is dominated by $h(x)$, then $f(x)$ is dominated by $h(x)$. (See Exercise 11.) We apply this principle in the next example.

Example 8. Find a value of N such that $7x^2 < 2^x$ for all $x > N$.

Scratch work: *If we attempt to solve this using the same idea as in Example 6, we take the log of the inequality obtaining*

$$\begin{aligned}\ln(7x^2) &< \ln 2^x \\ \ln 7 + \ln x^2 &< x \ln 2 \\ \ln 7 + 2\ln x &< x \ln 2\end{aligned}$$

Unfortunately, this cannot be transformed into something to which Proposition 2 applies.

Solution: We reason that $7x^2$ is dominated by x^3 which is dominated by 2^x. Specifically, $7x^2 < x^3$ for $x > 7$ and, from the solution to Example 7, $x^3 < 2^x$ for $x > 80$. Hence, $7x^2 < 2^x$ for $x > 80$.

EXAMPLE 9. Find an $a > 0$ such that the following function grows like a multiple of a^x. Prove your answer.

$$\frac{2^x + x\ln x + x^3}{4^x + x^3 3^x + 1}$$

Solution The fastest growing term in the numerator and denominator are, respectively, 2^x and 4^x. Hence, we expect that the fraction should grow like $2^x/4^x = (1/2)^x$, suggesting that we may use $a = 1/2$. To prove our answer, we must find constants $C > 0$, $D > 0$ and N such that

$$C2^{-x} < \frac{2^x + x\ln x + x^3}{4^x + x^3 3^x + 1} < D2^{-x} \tag{15}$$

for all $x > N$.

We first consider the denominator. Since $x^3 3^x$ and 1 should grow more slowly than 4^x, there should exist an N such that for $x > N$,

$$\begin{aligned}1 &< 4^x \\ x^3 3^x &< 4^x\end{aligned}$$

The first inequality is true for all $x > 0$. Dividing by 3^x and using the fact that $\ln x$ is an increasing function, we see that the second inequality is equivalent with

$$\begin{aligned} x^3 &< \left(\frac{4}{3}\right)^x \\ 3\ln x &< x \ln\frac{4}{3} \\ \ln x &< (\frac{1}{3}\ln\frac{4}{3})x \end{aligned}$$

which, from Proposition 2, holds for $x > 4/((\frac{1}{3}\ln\frac{4}{3})^2 \approx 434.9876$. Hence, for, say, $x > 450$

$$4^x < 4^x + x^3 3^x + 1 < 4^x + 4^x + 4^x = 3(4^x)$$

Inverting, we see

$$\frac{1}{3(4^x)} < \frac{1}{4^x + x^3 3^x + 1} < \frac{1}{4^x} \tag{16}$$

For the numerator, we note that for $x > 1$,

$$\begin{aligned} 2^x + x\ln x + x^3 &< 2^x + x\cdot x + x^3 \\ &= 2^x + x^2 + x^3 \\ &< 2^x + 2x^3 \end{aligned}$$

where we used $\ln x < x$ in the first line.

On the other hand, from Example 7, $x^3 < 2^x$ for $x > 80$. Hence, for such x,

$$2^x + x\ln x + x^3 < 3(2^x). \tag{17}$$

Inequalities (16) and (17) will both hold for $x > 450$. Multiplying these inequalities shows that inequality (15) holds for all $x > 450$ with $C = 1/3$ and $D = 3$, finishing the proof.

The rate of growth of the sum of two functions is determined by the fastest growing term in the sum. The situation is more complicated in the case of products. For example, $x^3 3^x$ dominates either 3^x or x^3. However, since exponential growth is faster than power growth, $x^3 3^x$ grows slower than a^x for any $a > 3$.

Example 10. Prove that $x^3 3^x$ is dominated by $(3.1)^x$.

Solution We must show that there is an N such that for all $x > N$

$$x^3 3^x < (3.1)^x \tag{18}$$

However, since $\ln x$ is an increasing function, this equation is equivalent with:

$$x^3 < \left(\frac{3.1}{3}\right)^x$$
$$3\ln x < x(\ln 3.1 - \ln 3)$$

which, from Proposition 2, holds for

$$x > \frac{36}{(\ln 3 - \ln 3.1)^2} \approx 33482.99.$$

Thus, we can be certain that the stated inequality holds for, say, $x > 40,000$.

The next example is based on this idea.

EXAMPLE 11. Find positive numbers C, D, a, b, and N such that

$$Ca^x < \frac{2^x + x\ln x + x^3}{x^3 3^x + 1} < Db^x$$

for all $x > N$.

Solution From the solution to Example 10, equation (18) holds for $x > 40,000$. Hence, for such x

$$3^x < x^3 3^x + 1 < (3.1)^x + (3.1)^x = 2(3.1)^x.$$

Inverting:

$$\frac{1}{2(3.1)^x} < \frac{1}{x^3 3^x + 1} < \frac{1}{3^x}.$$

Multiplication by inequality (17) (which holds for $x > 450$) yields the estimate

$$\frac{2^x}{2(3.1)^x} < \frac{2^x + x\ln x + x^3}{x^3 3^x + 1} < 3\frac{2^x}{(3^x)}$$
$$\frac{1}{2}\left(\frac{2}{3.1}\right)^x < \frac{2^x + x\ln x + x^3}{x^3 3^x + 1} < 3\left(\frac{2}{3}\right)^x$$

which is true for $x > 40,000$. Thus, we may choose $a = 2/3.1$, $b = 2/3$, $C = 1/2$, $D = 3$, and $N = 40,000$. In place of 3.1 we could, of course use any number strictly greater than 3, although different choices result in different values of N. We cannot use 3.

Exercises

(1) Suppose that $0 < a < b$.
   (a) Prove that if $x > 1$, then $x^a > 1$. *Hint:* Use calculus to prove that, $y = x^a$ is increasing on $(0, \infty)$.
   (b) Prove that if $x > 1$, then $x^b > x^a$. *Hint:* From (a), $x^{b-a} > 1$.

(2) For each pair of functions below (i) determine which is the dominant function, (ii) prove your answer by finding a number N fulfilling the requirements of Definition 1 on page 35. Prove that your value of n really works.

   (a) $3\sqrt{x}, \quad x^{1/3}$
   (b) $8x^2, \quad x^3$
   (c) $8x^2, \quad x^3 + 1$
   (d) $8x^2 - x, \quad x^3$
   (e) $8x^2 + x, \quad x^3$ *Hint:* $x < x^2$ for $x > 1$
   (f) $x^5, \quad x^3 + 2x^2 - x + 1$
   (g) $x^5 - 7x^2 + 2x + 1, \quad x^3 + x + 1$
   (h) $x^3 + 3x^2 + 1, \quad x^3 + x + 1$
   (i) $x^3 + 3x^2 + 1, \quad x^3 + 2x^2 + x + 7$
   (j) $3\sqrt{x}, \quad x - x^{1/3}$
   (k) $x^5 - 7x^2 + 2x + 1, \quad x^4$
   (l) $x^5 - 7x^2 + 2x + 1, \quad (1.1)x^5$
   (m) $x^5 - 7x^2 + 2x + 1, \quad (.99)x^5$

(3) For each function $f(x)$, find a value of m such that $f(x)$ grows like a multiple of $g(x) = x^m$. Prove your answer by finding constants C, D and N fulfilling the requirements of Definition 2 on page 38.
   (a) $f(x) = x^4 + 3x^2 + 1$
   (b) $f(x) = 5x^2 - 3x + 7$
   (c) $f(x) = \frac{5x^2-3x+7}{x^4+3x^2+1}$
   (d) $f(x) = \frac{x^4+3x^2+1}{5x^2-3x+7}$
   (e) $f(x) = x^4 + 3x^2 - 5x + 1$
   (f) $f(x) = x^2 - 3x - \sqrt{x} + 7$
   (g) $f(x) = \frac{x^2-3x-\sqrt{x}+7}{x^4+3x^2-5x+1}$
   (h) $f(x) = \frac{x^4+3x^2-5x+1}{x^2-3x-\sqrt{x}+7}$
   (i) $f(x) = \frac{x^5-4x^2-5}{x^5+14x-3}$
   (j) $f(x) = \frac{x^5+14x-3}{x^5-4x^2-5}$

(4) In each part, find constants $C > 0$, $D > 0$ and N such the stated inequalities are valid for all $x > N$. State in words what each inequality tells you about rates of growth.
  (a) $C\left(\frac{2}{3}\right)^x < \frac{2^x+14x-3}{3^x-4x^2-5} < D\left(\frac{2}{3}\right)^x$
  (b) $C\left(\frac{1.5}{3}\right)^x < \frac{2^x x^2+14x-3}{3^x-4x^2-5} < D\left(\frac{2.5}{3}\right)^x$
  (c) $C\left(\frac{5}{4}\right)^x < \frac{5^x-3x-\sqrt{x}+7}{4^x+3x^2-5x+1} < D\left(\frac{5}{4}\right)^x$
  (d) $C\left(\frac{4.9}{4}\right)^x < \frac{5^x x^3-3x-\sqrt{x}+7}{4^x+3x^2-5x+1} < D\left(\frac{5.1}{4}\right)^x$

(5) Find a value of N such that for $x > N$,
  (a) $x^2 - 3x + 27 < x^2$
  (b) $x^2 - 3x + 27 > .9x^2$
  (c) Explain why there is no N such that $x^2 - 3x + 27 > x^2$ for all $x > N$.

(6) Find a value of N such that for $x > N$,
  (a) $x^3 + 3x^2 - 27x + 1 < (1.01)x^3$
  (b) $x^3 + 3x^2 - 27x + 1 > x^3$
  (c) Explain why there is no N such that $x^3+3x^2-27x+1 < x^3$ for all $x > N$.

(7) For each pair of functions below (i) determine which is the dominant function, (ii) prove your answer by finding a number N fulfilling the requirements of Definition 1 on page 35. Prove that your value of n really works.
  (a) $\ln x, \quad x^{1/4}$
  (b) $\frac{1}{2}\sqrt{x}, \quad \ln x$
  (c) $x, \quad (\ln x)^{\frac{1}{3}}$
  (d) $\sqrt{x}, \quad (\ln x)^5$
  (e) $x^4 \ln x, \quad x^{4.25}$
  (f) $1, \quad \frac{3x^3 \ln x}{x^3-3x^2+5x-1}$
  (g) $3^x, \quad x^7$
  (h) $\sqrt{x}, \quad 1.5^x$
  (i) $(1.1)^x, \quad 5x^2$
  (j) $x^{1/5}, \quad (\ln x)^3$
  (k) $3^x, \quad 5x^3 \ln x$
  (l) $(\ln x)^{10}, \quad x^{1/10}$

(8) (a) Prove that for $n > 3$, $n! > \frac{2}{9}(3^n)$. *Hint* $3! = 3{\cdot}2 = (3^3)\frac{2}{9}$. $5! = 5{\cdot}4{\cdot}3! > 3{\cdot}3{\cdot}3! = (3^5)\frac{2}{9}$. Repeat the same argument in the general case.
  (b) Find an N such that $\frac{2}{9}3^n > 2^n$ for $n > N$. How does it follow that $n!$ dominates 2^n?

(c) Use the reasoning from (a) to find an N such that $n! > \frac{3}{32}4^n$ for $n > N$. Use this to find an N such that $n! > 3^n$ for all $n > N$.

(9) For each pair of functions below (i) determine which is the dominant function, (ii) prove your answer by finding a number N fulfilling the requirements of Definition 1 on page 35. Prove that your value of n really works.

(a) $x^2, \quad e^x$

(b) $x^2 \ln x, \quad (1.5)^x$

(c) $3^x, \quad x^3 2^x$

(d) $\dfrac{x^3}{2^x}, \quad \dfrac{1}{(1.5)^x}$

(e) $\dfrac{1}{(1.5)^x}, \quad \dfrac{x^2 \ln x}{2^x}$

(f) $\dfrac{3^n}{n!}, \quad \dfrac{1}{2^n}$

(g) $\dfrac{3^n + n^2 + 1}{2^n - 5}, \quad \dfrac{3^n}{2^n}$

(h) $\dfrac{n!}{100^n}, \quad (1.1)^n$

(10) Are there positive constants C and N such that

$$Cx < \ln x$$

for all $x > N$. If so, find them. If not, prove that such constants don't exist.

(11) Suppose that $f(x)$, $g(x)$, and $h(x)$ are functions such that $f(x)$ dominates $g(x)$ and $g(x)$ dominates $h(x)$. Use Definition 1 on page 35 to prove that then $f(x)$ dominates $h(x)$.

(12) Let f be an invertible function and let $g = f^{-1}$ be the inverse function. (Hence, $g(f(x)) = x$ for all x in the domain of f.) Prove that $f(g(y)) = y$ for all y in the range of f. *Hint:* Since y is in the range of f, $y = f(x)$ for some x.

(13) Let f be an increasing, invertible function and let $g = f^{-1}$. Prove that g is also increasing. *Hint* Suppose that there are numbers $a < b$ in the domain of g such that $g(a) \geq g(b)$. What do you know about the effect of applying f to inequalities?

(14) State (carefully) a version of Exercise 13 that applies to decreasing functions. Then solve your exercise.

(15) In the notes we used integrals to prove that for $x > 0$, $\ln x < x$.

(a) Prove this by using calculus to find the minimum for the function $f(x) = x - \ln x$. Use the second derivative test to prove that the value you find really is a minimum.

(b) Prove that $\ln x \leq (1/e)x$ for all $x > 0$. Why is this inequality false with "$\leq$" replaced by "$<$"? *Hint:* Use the same idea as you used in (a).

(c) Use the result from (b) to prove that $\ln x < \sqrt{x}$ for all $x > 0$. *Hint:* Apply (b) with x replaced by $x^{1/2}$.

(d) Use the result from (c) to prove that Proposition (2) on page 42 holds with $1/a^2$ in place of $4/a^2$.