CHAPTER 4

# Limits of Sequences

In this section we study limits of sequences. As a preliminary definition, we might define a sequence to be a function whose domain consists of the set $\mathbb{N}$ natural numbers. Thus, when we refer to "the sequence"

$$a_n = n/(n^2 - 3)$$

we are implicitly stating that we will consider this expression only for $n = 1, 2, 3, \ldots$. Hence, the preceding equality does define a sequence, despite the fact that the denominator is zero if $n = \sqrt{3}$.

At times it is more convenient to consider the sequence as beginning with values of $n$ other than 1. For example, $n = 0$ is a common choice. Or if we were to study the following expression, we might want to begin with $n = 6$ to avoid dividing by 0.

$$a_n = \frac{n}{n-5}$$

At times, one even might want to begin with a negative number. Thus, we modify our original definition as follows. Recall that the set $\mathbb{Z}$ of integers is the set consisting of 0 together with $\pm n$ where $n \in \mathbb{N}$. Hence

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

Definition 1. *A sequence is a function whose domain consists of all $n \in \mathbb{Z}$, $n \geq a$.*

We will, however, adopt the convention that unless otherwise indicated, the domain of all sequences is considered to be the set of natural numbers.

We will discuss limits in terms of approximation theory. In science, when we make a statement such as "the value $A$ is $4.7 \pm .01$," we mean that although we do not know the value of $A$ exactly, we are certain that $A$ is between $4.7 - .01$ and $4.7 + .01$. When discussing approximations, **we will always assume that the error term is**

**positive** since negative values are accounted for by the "$\pm$." Thus, we will never say $A = 4.7 \pm (-.01)$ as this is the same as $A = 4.7 \pm .01$

In general, for $\epsilon > 0$, we define

$$A = B \pm \epsilon$$

to mean[1]

$$B - \epsilon < A < B + \epsilon$$

which is the same as either of the following two equivalent statements

$$-\epsilon < A - B < \epsilon$$
$$|A - B| < \epsilon.$$

To relate this to limits, consider the sequence

$$a_n = \frac{n}{n+1}$$

Below are a few values of $a_n$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 6 | 11 | 16 | 21 | 26 | 31 | 36 | 41 | 100 | 1000 |
| $a_n$ | .857 | .917 | .941 | .955 | .963 | .969 | .973 | .976 | .990 | .9990 |

It appears that for large values of $n$, $a_n$ is approximately equal to 1. For example,

$$|a_{16} - 1| = |.917 - 1| = .083 < .1$$

Hence

$$a_{16} = 1 \pm .1$$

Similarly,

$$|a_{100} - 1| = |.990 - 1| = .01 < .02$$
$$|a_{1000} - 1| = |.9990 - 1| = .001 < .002$$

so $a_{100} = 1 \pm .02$ and $a_{1000} = 1 \pm .002$.

The next example illustrates that $a_n$ is a very close approximation to 1 for **all** sufficiently large values of $n$.

EXAMPLE 1. Find a value of $N$ such that

$$\frac{n}{n+1} = 1 \pm 10^{-5}$$

for $n > N$. Prove your answer.

[1]Typically, $A = B \pm \epsilon$ means $B - \epsilon \leq A \leq B + \epsilon$. Our convention excludes the possibility of equality.

**Scratch work:** *We want*

$$\begin{aligned} |\frac{n}{n+1} - 1| &< 10^{-5} \\ |\frac{1}{n+1}| &< 10^{-5} \end{aligned}$$

*Since the term inside the absolute value is positive, this is equivalent with*

$$\begin{aligned} \frac{1}{n+1} &< 10^{-5} \\ n+1 &> 10^{5} \\ n &> 10^{5} - 1 = 99,999 \end{aligned}$$

*For our formal solution, we state a value of $N$ and prove that it works by reversing the above sequence of arguments.*

**Solution:** Let $N = 99,999$ and assume $n > N$. Then

$$\begin{aligned} n &> 10^{5} - 1 \\ n+1 &> 10^{5} \\ \frac{1}{n+1} &< 10^{-5} \\ |\frac{1}{n+1}| &< 10^{-5} \\ |\frac{n}{n+1} - 1| &< 10^{-5} \end{aligned}$$

showing that

$$\frac{n}{n+1} = 1 \pm 10^{-5}$$

as desired.

What if we want

$$\frac{n}{n+1} = 1 \pm 10^{-10}?$$

The argument from Example 1 shows that this will hold for all $n > 10^{10} - 1$. More generally, for any $\epsilon > 0$,

$$\frac{n}{n+1} = 1 \pm \epsilon$$

holds for $n > \frac{1}{e} - 1$.

*This shows that $n/(n+1)$ will approximate 1 to any desired degree of accuracy for* **all** *sufficiently large $n$. How large $n$ must be will, of course, depend on the accuracy desired.*

Example 1 illustrates a general principle: if $a_n$ is a sequence such that $\lim_{n\to\infty} a_n = L$, then $a_n$ will approximate $L$ as closely as desired for all sufficiently large $n$. In fact, an informal definition of limit might read, " $\lim_{n\to\infty} a_n = L$ provided $a_n$ gets arbitrarily close to $L$ as $n$ gets large." This means exactly that $a_n$ approximates $L$ as closely as desired. Thus, we adopt the following statement as our "official" definition of limit for sequences:

Definition 2. *Let $a_n$ be some sequence of numbers and let $L$ be a number. We say that $\lim_{n\to\infty} a_n = L$ provided that for every number $\epsilon > 0$, there is a number $N$ such that*

$$|a_n - L| < \epsilon \tag{1}$$

*for all $n > N$.*

**Remark** Occasionally students feel that it is not really correct to say that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

because $n/(n+1)$ never actually equals 1. They would prefer to say that the limit is "very, very close to" 1. This is not correct. The limit refers to the number that is being approximated; not the numbers doing the approximation. There is one, and only one, number that the terms $n/(n+1)$ approximate better and better as $n$ gets larger and larger, namely 1. Actually, the fact that a convergent sequence can approximate only one number is an important property of limits. We will leave the proof of this fact as an exercise (Exercise 31).

Proposition 1. *A sequence can have only one limit: if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.*

In a number of exercises, you will be asked to prove statements such as $\lim_{n\to\infty} a_n = L$ for some specific sequence $a_n$ and some number $L$ "using $\epsilon$" i.e. directly from the definition. The formal solution to such a problem will typically involve the following steps:

(1) Assume that a value of $\epsilon > 0$ is given.
(2) State an appropriate value of $N$. It will be given by some expression involving $\epsilon$.

(3) Prove that the stated value of $N$ really works i.e. assume that $n > N$ and prove that $|a_n - L| < \epsilon$.

EXAMPLE 2. Find

$$\lim_{n\to\infty} \frac{n^4}{n^5 + 7}$$

Find a number $N$ such that $\frac{n^4}{n^5+7}$ approximates the limit with error less than .001 for all $n > N$. Prove, using $\epsilon$, that your limit is correct.

**Scratch work:** *Since $n^5+7$ grows like $n^5$ our fraction grows (actually decays) like $n^4/n^5 = 1/n$. Hence, we guess that the limit is 0.*

*To prove our guess we we must show that for any given $\epsilon > 0$, $\frac{n^4}{n^5+7} = 0 \pm \epsilon$ for all sufficiently large $n$. i.e.*

$$\begin{aligned} \left|\frac{n^4}{n^5+7} - 0\right| &< \epsilon \\ \frac{n^4}{n^5+7} &< \epsilon \end{aligned} \tag{2}$$

*But*

$$\frac{n^4}{n^5+7} < \frac{n^4}{n^5} = \frac{1}{n}$$

*Thus, formula (2) will be valid if $1/n < \epsilon$, i.e. $n > 1/\epsilon$. In particular, we obtain $\pm.001$ accuracy for $n > 1000$.*

**Formal Solution:** For $n \in \mathbb{N}$

$$\begin{aligned} n^5 &< n^5 + 7 \\ \frac{1}{n^5+7} &< \frac{1}{n^5} \\ \frac{n^4}{n^5+7} &< \frac{n^4}{n^5} = \frac{1}{n} \end{aligned}$$

Now, let $\epsilon > 0$ be given and let $N = 1/\epsilon$. Assume that $n > N$ Then

$$\begin{aligned} n &> \frac{1}{\epsilon} \\ \frac{1}{n} &< \epsilon \\ \frac{n^4}{n^5+7} < \frac{1}{n} &< \epsilon \\ \left|\frac{n^4}{n^5+7} - 0\right| &< \epsilon \end{aligned}$$

This fulfills the requirements for the definition of the limit.

The first part of the problem is solved by letting $\epsilon = .001$, in which case $N = 1000$.

**Remark:** In our solution to Example 2, we used the observation that for $n > 1,000$

$$\frac{n^4}{n^5+7} < \frac{1}{n} < \ .001$$

In general, if we are trying to show that some sequence $a_n$ can be made less than $\epsilon$, we often try to find some relatively simple quantity "?" (which will depend on $n$) such that

$$a_n <? < \epsilon$$

for all sufficiently large $n$.

EXAMPLE 3. Find the following limit and prove your answer using $\epsilon$.

$$\lim_{n\to\infty} \frac{2n^3}{n^3+5n+1}$$

**Scratch Work:** *Considering only the fastest growing terms suggests that our fraction grows like* $2n^3/n^3 = 2$. *Hence we guess the limit to be 2.*

*For the proof, we must show that for any given* $\epsilon$ *there is an* $N$ *such that for all* $n > N$,

$$\begin{aligned} \left|\frac{2n^3}{n^3+5n+1} - 2\right| &< \epsilon \\ \left|\frac{-10n-2}{n^3+5n+1}\right| &< \epsilon \qquad (3) \\ \frac{10n+2}{n^3+5n+1} &< \epsilon \end{aligned}$$

*(We omitted the absolute values in the last line since the quantities involved are clearly positive.)*

*We seek a simple quantity "?" such that, for large enough* $n$,

$$\frac{10n+2}{n^3+5n+1} <? < \epsilon.$$

*Making either the denominator smaller or the numerator larger increases the size of a fraction. Hence, for* $n > 2$,

$$\frac{10n+2}{n^3+5n+1} < \frac{10n+n}{n^3+5n+1} < \frac{11n}{n^3} = \frac{11}{n^2}.$$

*Hence, inequality (3) holds if* $n > 2$ *and*

$$\begin{aligned} \frac{11}{n^2} &< \epsilon \\ \frac{n^2}{11} &> \frac{1}{\epsilon} \\ n &> \sqrt{11/\epsilon} \end{aligned}$$

**Formal Solution:** Let $\epsilon > 0$ be given and let $N$ be the larger of $\sqrt{11/\epsilon}$ and 2. Assume that $n > N$. Then from the "scratch work",

$$\left|\frac{2n^3}{n^3+5n+1} - 2\right| < \frac{11}{n^2} < \epsilon$$

Thus the requirements for the definition of the limit are fulfilled.

**Remark:** The proof in Example 3 is typical of many limit proofs. Often to prove that $\lim_{n\to\infty} a_n = L$ we:

(1) Compute, and simplify $|a_n - L|$ by, for example, putting terms over common denominator and/or factoring.

(2) Eliminate the absolute value by determining the sign of $a_n - L$ for large $n$.
(3) Use rates of growth to estimate the growth of $a_n - L$ so as to show that it may be made less than $\epsilon$

A common mistake is to forget to subtract the limit before estimating the rate of growth. For example,

$$\frac{2n^3}{n^3 + 5n + 1} < \frac{2n^3}{n^3} = 2$$

All this tells us is that the limit, if it exists, is at most 2.

Often the techniques from Chapter 3 play a role, as in the next example.

EXAMPLE 4. Find the following limit and prove your answer using $\epsilon$.

$$\lim_{n\to\infty} \frac{2n^3 - 6n^2}{n^3 - 3n^2 + 5n - 1}$$

**Scratch Work:** *Considering only the fastest growing terms suggests that our fraction grows like $2n^3/n^3 = 2$. Hence we guess the limit to be 2.*

*For the proof, we must show that we can approximate 2 to within $\pm\epsilon$ for any $\epsilon > 0$. This means*

$$\begin{gathered}
\left|\frac{2n^3 - 6n^2}{n^3 - 3n^2 + 5n - 1} - 2\right| < \epsilon \\
\left|\frac{2n^3 - 6n^2 - 2(n^3 - 3n^2 + 5n - 1)}{n^3 - 3n^2 + 5n - 1}\right| < \epsilon \qquad (4) \\
\left|\frac{-10n + 2}{n^3 - 3n^2 + 5n - 1}\right| < \epsilon
\end{gathered}$$

*We seek a simple quantity "?" such that, for large enough $n$,*

$$\left|\frac{-10n + 2}{n^3 - 3n^2 + 5n - 1}\right| <? < \epsilon.$$

*We can make our fraction larger by making the denominator smaller. Specifically, we find positive numbers $C$ and $N_o$ such that*

$$Cn^3 < n^3 - 3n^2 + 5n - 1$$

*for all $n > N_o$.*

*We choose $N_o$ so that both of the following hold for $n > N_o$:*

$$\begin{aligned} 3n^2 &< \frac{1}{3}n^3 \\ 1 &< \frac{1}{3}n^3 \end{aligned}$$

*The first inequality is valid for $n > 9$ and the second for $n > 3^{1/3} \approx 1.44$. For $n > 9$, both are valid and*

$$\begin{aligned} n^3 - 3n^2 + 5n - 1 &> n^3 - 3n^2 - 1 \\ &> n^3 - \frac{1}{3}n^3 - \frac{1}{3}n^3 = \frac{1}{3}n^3 \end{aligned}$$

*Hence we may choose $N_o = 9$ and $C = 1/3$.*

*In particular the denominator is positive for such $n$. Furthermore, for $n \in \mathbb{N}$, $-10n + 2 < 0$. Hence, for $n > 6$,*

$$\begin{aligned} \left|\frac{-10n+2}{n^3 - 3n^2 + 5n - 1}\right| &= \frac{10n - 2}{n^3 - 3n^2 + 5n - 1} \\ &< \frac{10n}{n^3/3} \\ &< \frac{30n}{n^3} = \frac{30}{n^2} \end{aligned} \tag{5}$$

*Hence, inequality (4) holds if*

$$\begin{aligned} \frac{30}{n^2} &< \epsilon \\ \frac{n^2}{30} &> \frac{1}{\epsilon} \\ n &> \sqrt{30/\epsilon} \end{aligned}$$

**Formal Solution:** We should first prove inequality (5). However, since this is not the main point of the problem, we will allow ourselves (and the student) to use the "scratch work" as the proof.

Now let $\epsilon > 0$ be given and let $N$ be the larger of $\sqrt{30/\epsilon}$ and 9. Assume that $n > N$. Then from the "scratch work",

$$\left|\frac{2n^3 - 6n^2}{n^3 - 3n^2 + 5n - 1} - 2\right| = \frac{10n - 2}{n^3 - 3n^2 + 5n - 1} < \frac{10n}{n^3/3} < \epsilon$$

Thus the requirements for the definition of the limit are fulfilled.

What if we try to prove something which is false?

EXAMPLE 5. Attempt an $\epsilon$-proof of the following (incorrect) limit statement. Describe carefully where your proof breaks down.

$$\lim_{n\to\infty} \frac{2n}{n+1} = 1$$

**Solution:** Let $\epsilon > 0$ be given. If the limit statement is true there is an $N$ such that the following holds for $n > N$:

$$\begin{aligned} \left|\frac{2n}{n+1} - 1\right| &< \epsilon \\ \left|\frac{n-1}{n+1}\right| &< \epsilon \end{aligned}$$

For $n \in \mathbb{N}$ the fraction is positive and the absolute value may be dropped:

$$\begin{aligned} \frac{n-1}{n+1} &< \epsilon \\ n-1 &< \epsilon(n+1) \\ n-1 &< \epsilon n + \epsilon \\ (1-\epsilon)n &< 1+\epsilon \end{aligned}$$

If $\epsilon < 1$, we may divide by $1-\epsilon$:

$$n < \frac{1+\epsilon}{1-\epsilon}. \tag{6}$$

But this inequality says that we achieve the desired accuracy only if $n$ is *sufficiently small.* Hence, for large $n$, $2n/(n+1)$ does not approximate 1 to the desired accuracy for *all* large $n$, showing that the limit is not 1.

If you see a difference of two square roots, either in a numerator or in a denominator, it is usually "wise to rationalize"-i.e. multiply both numerator and denominator by the sum of the square roots.

EXAMPLE 6. Find the following limit and prove your answer using $\epsilon$.

$$\lim_{n\to\infty} (\sqrt{n} - \sqrt{n+1}).$$

**Scratch Work:** *To find the limit, we write*

$$\begin{aligned}\sqrt{n}-\sqrt{n+1} &= (\sqrt{n}-\sqrt{n+1})\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}\\ &= \frac{(\sqrt{n})^2-(\sqrt{n+1})^2}{\sqrt{n}+\sqrt{n+1}}\\ &= \frac{-1}{\sqrt{n}+\sqrt{n+1}}\end{aligned}$$

*As* $n \to \infty$ *this will tend to zero. Furthermore,*

$$\left|\sqrt{n}-\sqrt{n+1}-0\right| = \frac{1}{\sqrt{n}+\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

*This will be less than* $\epsilon$ *if* $\sqrt{n} > 1/\epsilon$ *i.e.* $n > 1/\epsilon^2$.

**Formal Solution:** We claim that the limit is 0. To prove this, let $\epsilon > 0$ be given and set $N = 1/\epsilon^2$. Suppose that $n > N$. Then from the "scratch work"

$$\left|\sqrt{n}-\sqrt{n+1}-0\right| < \frac{1}{\sqrt{n}} < \epsilon.$$

Thus, the conditions for the definition of limit are fulfilled.

EXAMPLE 7. Guess the following limit and prove your answer using $\epsilon$.

$$\lim_{n\to\infty} \frac{n^2}{2n^2+3n+n\ln n}.$$

**Scratch Work:** *The denominator grows like* $2n^2$ *since both* $3n$ *and* $n\ln n$ *grow more slowly. Hence, the fraction behaves like* $n^2/(2n^2) = 1/2$*. Hence, the limit should be* $1/2$.

*For the proof, we compute*

$$\begin{aligned}\left|\frac{n^2}{2n^2+3n+n\ln n}-\frac{1}{2}\right| &= \left|\frac{-3n-n\ln n}{2(2n^2+3n+n\ln n)}\right|\\ &= \frac{3n+n\ln n}{2(2n^2+3n+n\ln n)}\\ &< \frac{3n+n\ln n}{4n^2}\end{aligned}$$

*Both $3n$ and $n \ln n$ grow more slowly than $n^{3/2}$. Specifically*

$$n \ln n < n^{3/2} \tag{7}$$

*is true if $\ln n < n^{1/2}$ which, as the reader may show, holds for $n > (16)^2$. (See Example 5 of Chapter 3.) Also*

$$3n < n^{3/2} \tag{8}$$

*holds for $n > 9$. Hence, for $n > (16)^2$,*

$$\begin{aligned} \frac{3n + n \ln n}{4n^2} &< \frac{n^{3/2} + n^{3/2}}{4n^2} \\ &= \frac{1}{2n^{1/2}} \end{aligned} \tag{9}$$

*This will be less than $\epsilon$ if $n^{1/2} > 1/(2\epsilon)$ i.e. $n > 1/(4\epsilon^2)$. Thus, we may choose $N$ to be any number larger than both $1/(4\epsilon^2)$ and $(16)^2$ .*

**Formal Solution:** From the argument in the "scratch work" (7), (8), and (9) all hold for $n > (16)^2$.

Now, let $\epsilon > 0$ be given and let $N$ be the larger of $(16)^2$ and $1/(4\epsilon^2)$ and let $n > N$. Then

$$\begin{aligned} n &> 1/(4\epsilon^2) \\ n^{1/2} &> 1/(2\epsilon) \\ \frac{1}{2n^{1/2}} &< \epsilon \end{aligned}$$

Furthermore, from the scratch work

$$\begin{aligned} \left|\frac{n^2}{2n^2 + 3n + n \ln n} - \frac{1}{2}\right| &= \frac{3n + n \ln n}{2(2n^2 + 3n + n \ln n)} \\ &< \frac{1}{2n^{1/2}} < \epsilon \end{aligned}$$

Thus, the conditions for the definition of limit are fulfilled.

Not all sequences have a limit. A sequence which has a limit **converges** and one which does not **diverges.** The following example illustrates one particular type of divergence–*going to infinity.*

EXAMPLE 8. I have a bank account that pays 5% interest per year, credited on Dec. 31. I deposit \$1 on Jan. 1 of year 0 and make no further deposits. On Jan. 1 of year 1, I have $1 + (.05)1 = 1.05$ dollars on deposit. On Jan. 1 of year 2, I have

$$P_2 = 1.05 + (.05)(1.05) = (1.05)(1.05) = (1.05)^2$$

dollars. After $n$ years, I will have

$$P_n = (1.05)^n \tag{10}$$

dollars.

(1) How many years does it take for my balance to be over \$100? \$10,000? \$100,000? \$1,000,000?
(2) Prove that for any positive number $M$ there is an $n$ such that $P_n > M$. Thus, there is no limit to the amount of money I will eventually have in my account, as long as I leave it on deposit long enough.

**Solution (1)** To have more than \$100, we need

$$\begin{aligned}(1.05)^n &> 100\\ \ln\left((1.05)^n\right) &> \ln(100)\\ n\ln(1.05) &> \ln(100) \qquad (11)\\ n &> \frac{\ln(100)}{\ln(1.05)} = 40.99\end{aligned}$$

Hence, it would take 41 years. Similarly, to get:

$$\begin{aligned}\$10,000 &\quad \text{we need } n > \frac{\ln(10,000)}{\ln(1.05)} \approx 81.98 \text{ years}\\ \$100,000 &\quad \text{we need } n > \frac{\ln(100,000)}{\ln(1.05)} \approx 102.47 \text{ years}\\ \$1,000,000 &\quad \text{we need } n > \frac{\ln(1,000,000)}{\ln(1.05)} \approx 122.97 \text{ years}\end{aligned}$$

Hence, after only 123 years our heirs will be millionaires!

**Solution (2)** For the proof we do the work from part (1) in the general case. Suppose we want $\$M$. Let $n$ be a natural number such that

$$n > \frac{\ln M}{\ln(1.05)}.$$

Then, reversing the steps in (11):

$$\begin{aligned} n &> \frac{\ln M}{\ln(1.05)} \\ n\ln(1.05) &> \ln M \\ \ln\left((1.05)^n\right) &> \ln M \\ (1.05)^n &> M \end{aligned}$$

Example 8 demonstrates what we mean by $\lim_{n\to\infty} a_n = \infty$. The balance in our account grows without bound; for any value $M$ we will eventually have more than $\$M$ in the account as long as we wait sufficiently long. Certainly, however, no matter how large, the balance always much closer to 0 than to infinity! The “official” definition of tending to infinity is:

Definition 3. *Let $a_n$ be a sequence. We say that $\lim_{n\to\infty} a_n = \infty$ if for all $M > 0$ there is an $N$ such that*

$$a_n > M$$

for all $n > N$.

We stress that **infinity is not a limit: going to infinity is a special type of divergence.**

Proving that $\lim_{n\to\infty} a_n = \infty$ involves assuming that a constant $M > 0$ is given and showing that for sufficiently large $N$, $a_n > M$. Often we find some simple quantity “?” such that

$$a_n >? > M$$

as in the following example.

Example 9. Prove, using $M$, that

$$\lim_{n\to\infty} \frac{n^4}{2n^2 + 3n + n\ln n} = \infty.$$

**Scratch Work:** *Let $M > 0$ be given. We seek a quantity “?” such that*

$$\frac{n^4}{2n^2 + 3n + n\ln n} >? > M$$

*for all sufficiently large $n$. Making the denominator larger makes the fraction smaller. Since the fastest growing term in the denominator is $n^2$, we seek constants $C$ and $N_o$ such that*

$$2n^2 + 3n + n\ln n < Cn^2$$

*for all* $n > N_o$. *Since, for* $n > 1$, $n < n^2$ *and* $\ln n < n$, *we see*

$$2n^2 + 3n + n\ln n < 2n^2 + 3n^2 + n^2 = 6n^2. \tag{12}$$

*Hence*

$$\frac{n^4}{2n^2 + 3n + n\ln n} > \frac{n^4}{6n^2} = \frac{n^2}{6}$$

*This is* $> M$ *provided* $n > \sqrt{6M}$. *We also need* $n > 1$ *for the validity of inequality (12).*

**Formal Solution:** Let $M > 0$ be given and let $N$ be the larger of 1 and $\sqrt{6M}$. Let $n \in \mathbb{N}$ satisfy $n > \sqrt{6M}$. Then, from the "scratch work"

$$\begin{aligned} \frac{n^4}{2n^2 + 3n + n\ln n} &> \frac{n^4}{6n^2} = \frac{n^2}{6} \\ &> \frac{(\sqrt{6M})^2}{6} = M. \end{aligned}$$

Thus, the conditions for the definition of $\lim_{n\to\infty} a_n = \infty$ are fulfilled.

A sequence can also diverge by tending to $-\infty$. We could define this notion in a manner similar to our definition of tending to $\infty$. However, the simplest definition is just:

Definition 4. *Let* $a_n$ *be a sequence. We say that* $\lim_{n\to\infty} a_n = -\infty$ *if* $\lim_{n\to\infty} -a_n = \infty$.

The following example demonstrates how to prove general theorems using the definition of limit.

Example 10. Suppose that $\lim_{n\to\infty} a_n = 2$. Prove, using $\epsilon$, that $\lim_{n\to\infty} a_n^2 = 4$.

**Scratch Work:** *Let* $\epsilon > 0$ *be given. We want to show that for all sufficiently large* $n$,

$$|a_n^2 - 4| < \epsilon$$

$$|a_n - 2|\,|a_n + 2| < \epsilon$$

*Now* $a_n - 2$ *can be made as small as we wish while, for large* $n$, $a_n + 2$ *should be near 4. More precisely, we may find an* $N$ *so that* $a_n = 2 \pm 1$ *for* $n > N$. *Hence,*

$$1 < a_n < 3$$

$$3 < a_n + 2 < 5$$

*Thus, for $n > N$,*

$$|a_n - 2|\,|a_n + 2| < 5|a_n - 2|$$

*This is less than $\epsilon$ if $|a_n - 2| < \epsilon/5$ which will be true for all sufficiently large $N$.*

**Formal Proof:** Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = 2$, there is an $N$ such that for $n > N$

$$|a_n - 2| < \epsilon/5$$

Furthermore there is an $N_o$ such that for $n > N_o$

$$\begin{aligned} |a_n - 2| &< 1 \\ -1 < a_n - 2 &< 1 \\ 1 < a_n + 2 &< 5 \\ |a_n + 2| &< 5 \end{aligned}$$

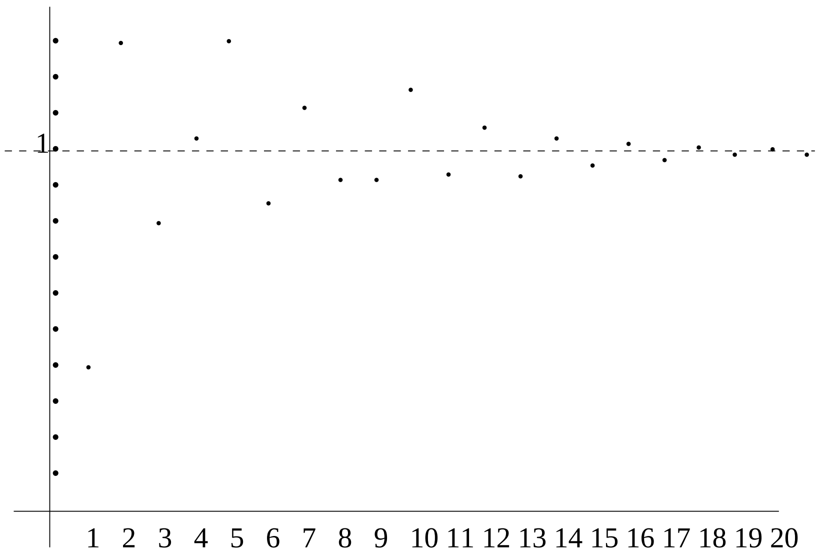
Let $N_1$ be the larger of $N$ and $N_o$. Then for $n > N_1$,

$$|a_n^2 - 4| = |a_n - 2|\,|a_n + 2| < (\epsilon/5)5 = \epsilon$$

Hence the requirements for the definition of limit are fulfilled.

### Exercises

(1) The figure below shows the first 20 terms of a sequence $a_n$ which, apparently, is converging to 1. Assume that the terms not shown are closer to one than any of the shown terms. What (approximately) is the first $n$ where we become confident that all of the rest of the terms are within $\pm.2$ of 1? $\pm.1$? $\pm.05$? $\pm.025$? (Note that the 'tick marks' on the vertical axis are .1 apart.) Explain in your own words, how this problem demonstrates the general definition of limit of a sequence.

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(2) For each sequence below, find the limit and prove your answer using $\epsilon$.
   (a) $\lim_{n\to\infty}(1 - 1/n^2)$
   (b) $\lim_{n\to\infty} \frac{3n^2}{n^2+1}$
   (c) $\lim_{n\to\infty}(\sqrt{n^2+1} - \sqrt{n^2+2})$ (See Example 5)
   (d) $\lim_{n\to\infty}(\sqrt{n^2+1} - n)$
   (e) $\lim_{n\to\infty} \frac{2n}{n^2+n-5}$.
   (f) $\lim_{n\to\infty} \frac{2n \ln n}{n^2+n-5}$. (See Example 6)
   (g) $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n$ *Hint:* Use logs.
   (h) $\lim_{n\to\infty} \frac{2^n}{n!}$. *Hint:* See Exercise 8, Chapter 3.
   (i) $\lim_{n\to\infty} \frac{1+(-1)^n}{n} + 7$
   (j) $\lim_{n\to\infty} \frac{n^3}{2n^3-7}$
   (k) $\lim_{n\to\infty} \frac{2n}{2^n+n-5}$.
   (l) $\lim_{n\to\infty} \frac{2^n}{3(2^n)+n-5}$.
   (m) $\lim_{n\to\infty} \frac{\sin n}{n^2+1}$
   (n) $\lim_{n\to\infty} \frac{(-1)^n \sin n}{n^2+1}$
   (o) $\lim_{n\to\infty} \frac{n^3}{2n^3+n\ln n+1}$

(p) $\lim_{n\to\infty} \frac{n^3}{2n^3-n^2+2}$
(q) $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$ *Hint:* Put $a_n - L$ over a common denominator and then rationalize the numerator.
(r) $\lim_{n\to\infty} \frac{\sqrt{4n+1}}{n}$
(s) $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}}$
(t) $\lim_{n\to\infty} \frac{5n^3-10}{n^3+n^2-1}$

(3) In Example 7 on page 63, we used $\ln n < \sqrt{n}$ for $n > (16)^2$. Why would it not work to use $lnn < n$ instead?
(4) Prove, using $\epsilon$, that for all $k > 0$, $\lim_{n\to\infty} \frac{1}{n^k} = 0$.
(5) Prove that the answer to part (b) of Exercise 2 is not 4. Reason as in Example 4.
(6) The statements $\lim_{x\to\infty} f(x) = L$ and $\lim_{n\to\infty} a_n = L$ mean slightly different things. In the first statement, $f(x)$ is a function that is defined for all sufficiently large real numbers $x$ and the limit considers all such $x$. In the second statement, $a_n$ is a sequence, implying that $n$ assumes only integral values. As a demonstration of this, explain, using a graph of $f(x) = \cos(2\pi x)$, why the first limit below exists and the second does not. What is the value of the first limit?

$$\lim_{n\to\infty} \cos(2\pi n)$$
$$\lim_{x\to\infty} \cos(2\pi x)$$

**Remark:** We define $\lim_{x\to\infty} f(x) = L$ as follows

Definition 5. *Let $f(x)$ be a function that is defined for all real numbers $x > 0$ and let $L$ be a number. We say that $\lim_{x\to\infty} f(x) = L$ provided that for every number $\epsilon > 0$, there is a real number $N$ such that*

$$|f(x) - L| < \epsilon$$

*for all $x > N$.*

(7) Use the preceding definition to prove the following limit statements.

(a) $$\lim_{x\to\infty} \frac{x}{x+1} = 1$$
(b) $$\lim_{x\to\infty} \frac{2x^5}{x^5-7} = 2$$

(8) Let $f$ be a function that is defined for all positive real numbers $x$. Suppose that $\lim_{x\to\infty} f(x) = L$. Prove (using $\epsilon$) that $\lim_{n\to\infty} f(n) = L$ where "$f(n)$" denotes the sequence $a_n$ defined by $a_n = f(n)$ for all $n \in \mathbb{N}$.

(9) We say that $a_n$ is *bounded from above* if there is a number $M$ such that $a_n < M$ for all $n$, in which case $M$ is referred to as an *upper bound* for $a_n$. The following sequence of arguments proves that if $L = \lim_{n\to\infty} a_n$ exists, then $a_n$ is bounded from above. Give a reason (or proof) for each statement.

There is a number $N$ such that $L - 1 < a_n < L + 1$ for all $n > N$.

There is a number $M_o$ such that $a_n < M_o$ for all $1 \le n \le N$, $n \in \mathbb{N}$.

Let $M$ be the larger of $L + 1$ and $M_o$. Then $M$ is an upper bound for $a_n$.

(10) Let $a_n$ be a convergent sequence. Prove that $a_n$ is bounded from below. Use a similar argument to that sketched in the preceding exercise.

(11) Each of the following sequences $a_n$ tends to $\infty$. Demonstrate this by finding a value of $N$ such that for all $n > N$ (i) $a_n > 100$, (ii) $a_n > 1000$ and (iii) $a_n > 1,000,000$. Finally, prove using $M$ that $\lim_{n\to\infty} a_n = \infty$.
(a) $2^n$.
(b) $\frac{2^n}{\ln n}$. *Hint:* For sufficiently large $n$, $\ln n < n < (1.5)^n$.
(c) $\ln n$.
(d) $\frac{n^3}{n-1}$.
(e) $\frac{n^5}{n^2-n-\ln n}$.
(f) $\frac{n^3}{n+1}$.
(g) $\frac{n^5}{n^2+n+1}$.

(12) Using logs, prove the following:
(a) For $a > 1$, $\lim_{n\to\infty} a^n = \infty$.
(b) For $0 < a < 1$, $\lim_{n\to\infty} a^n = 0$.

(13) Let $x$ be a positive number. We note that

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$$

Hence

$$\begin{aligned}(1 + x)^3 &= (1 + x)(1 + x)^2 \\ &> (1 + x)(1 + 2x) = 1 + 3x + 2x^2 > 1 + 3x\end{aligned}$$

Similarly

$$\begin{aligned}(1+x)^4 &= (1+x)(1+x)^3 \\ &> (1+x)(1+3x) = 1+4x+3x^2 > 1+4x\end{aligned}$$

(a) Use the fact that $(1+x)^4 > 1+4x$ to prove that $(1+x)^5 > 1+5x$.
(b) Use the fact that $(1+x)^5 > 1+5x$ to prove that $(1+x)^6 > 1+6x$.
(c) Suppose that we have succeeded in proving that $(1+x)^n > 1+nx$ for some value of $n$. *Use this* to prove that $(1+x)^{n+1} > 1+(n+1)x$.
*Warning:* You cannot simply replace $n$ by $n+1$ since our assumption is only that you have proved the inequality for *some* $n$, not *all* $n$. Instead, you should use reasoning similar to that used in (a).
(d) Explain how it follows from (b) that $(1+x)^n > 1+nx$ for all $n$.
(e) Use (c) to prove that for all $a > 1$, $\lim_{n\to\infty} a^n = \infty$.

**Remark:** The proof sketched in (b) is a *Mathematical Induction* argument. Mathematical Induction is a method of proving an infinite number of statements, *one at a time*. We first prove the first satement. We then *use the first* to prove the second, *use the second* to prove the third, *use the third* to prove the fourth, *etc.* To prove *all* of the statements we need to prove that this process can be continued indefinitely. In practice this means assuming that we have proved the theorem for some $n$ (but not yet for $n+1$). We then *use the* $n^{th}$ *case* to prove the $(n+1)^{st}$ case.

(14) Suppose that $\lim_{n\to\infty} a_n = 7$. Use $\epsilon$ to prove that
(a) $\lim_{n\to\infty} a_n^2 = 49$
(b) $\lim_{n\to\infty} \frac{1}{a_n+1} = \frac{1}{8}$
(c) $\lim_{n\to\infty} \sqrt{a_n+2} = 3$ (*Hint*: Rationalize)
(d) $\lim_{n\to\infty} \frac{a_n}{a_n+7} = \frac{1}{2}$

(15) Suppose that $\lim_{n\to\infty} a_n = -\infty$. Prove that for each $M < 0$, there is an $N$ such that $a_n < M$ for all $n > N$. *Hint:* In the text we defined $\lim_{n\to\infty} a_n = -\infty$ by $\lim_{n\to\infty} -a_n = \infty$.

(16) Prove the converse of the preceding exercise: i.e. Suppose that for all $M < 0$, there is an $N$ such that $a_n < M$ for all $n > N$. Prove that $\lim_{n\to\infty} a_n = -\infty$.
(17) Prove, using $\epsilon$, that $\lim_{n\to\infty} a_n = L$ if and only if $\lim_{n\to\infty}(a_n - L) = 0$, i.e. first prove that if $\lim_{n\to\infty}(a_n - L) = 0$ then $\lim_{n\to\infty} a_n = L$. Then prove that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty}(a_n - L) = 0$.
(18) Suppose that $\lim_{n\to\infty} a_n = L$. Prove (using $\epsilon$) that $\lim_{n\to\infty}(-a_n) = -L$.
(19) Suppose that $\lim_{n\to\infty} |a_n| = 0$. Does it follow that $\lim_{n\to\infty} a_n = 0$? Prove your answer.
(20) Suppose that $\lim_{n\to\infty} a_n = 0$. Does it follow that $\lim_{n\to\infty} |a_n| = 0$? Prove your answer.
(21) Suppose that $\lim_{n\to\infty} |a_n| = 1$. Does it follow that $\lim_{n\to\infty} a_n$ exists? Prove your answer.
(22) Suppose that $\lim_{n\to\infty} a_n = L$. Prove, using $\epsilon$, that $\lim_{n\to\infty} 5a_n = 5L$.
(23) If $C$ is any constant, then $\lim_{n\to\infty} C = C$–i.e. if $a_n = C$ for all $n$, then $\lim_{n\to\infty} a_n = C$.
(24) Suppose that $\lim_{n\to\infty} a_n = L$. Prove (using $\epsilon$) that for any constant $C$, $\lim_{n\to\infty} Ca_n = CL$.
(25) Suppose that for all $n$, $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Prove, using $\epsilon$, that then $\lim_{n\to\infty} b_n = L$. *Hint:* $a_n - L \leq b_n - L \leq c_n - L$.
(26) Suppose that $\lim_{n\to\infty} a_n = L$ where $L > 0$. Prove that there is an $N$ such that $a_n > 0$ for all $n > N$. *Hint:* How close to $L$ must $a_n$ be to guarantee that $a_n > 0$. What tells you that can get this close?
(27) (a) The following statement is false. Find an example that demonstrates its falsehood. "If $a_n < 0$ for all $n$ and $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n < 0$."
(b) Prove that if $a_n < 0$ for all $n$ and $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n \leq 0$. *Hint:* This is almost immediate from the result of Exercise 26. Explain.
(28) Suppose that $\lim_{n\to\infty} a_n = L$ where $L < 0$. Prove that there is an $N$ such that $a_n < 0$ for all $n > N$.
(29) Use Exercise 28 to prove that if $a_n > 0$ for all $n$ and $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n \geq 0$.

(30) Suppose that $a_n > L$ for all $n$. Prove that $\lim_{n\to\infty} a_n \geq L$. *Hint:* The result of Exercises 17 and 29 might help.

(31) Prove Proposition 1. *Hint:* For your proof, assume that $L \neq M$. Let $\epsilon = |L - M|$.

(32) Suppose that $a_n$ and $b_n$ are sequences where $\lim_{n\to\infty} a_n = 0$. Suppose also that and $|b_n| < 1$ for all $n$. Prove, using $\epsilon$, that $\lim_{n\to\infty} a_n b_n = 0$.

(33) Let $a_n$ be a sequence of nonnegative real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove, using $\epsilon$, that $\lim_{n\to\infty} \sqrt{a_n} = 0$.

(34) Let $a_n$ be a convergent sequence of nonnegative real numbers. Prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$.

*Hint:* Let $L = \lim_{n\to\infty} a_n$. You may assume $L > 0$ as the $L = 0$ case was done in the preceding exercise. Rationalize $|\sqrt{a_n} - \sqrt{L}|$ and note that $\sqrt{a_n} + \sqrt{L} \geq \sqrt{L}$.

(35) Suppose that $\lim_{n\to\infty} a_n = 1$. Prove, using $\epsilon$, that $\lim_{n\to\infty} 1/a_n = 1$. *Hint:* First prove that there is an $N$ such that $.5 \leq a_n \leq 1.5$ for all $n > N$. Then simplify $|1/a_n - 1|$.

(36) Let $\lim_{n\to\infty} a_n = L$ . Prove that $\lim_{n\to\infty} a_{n+1} = L$.

(37) Let $a_n$ be such that $\lim_{n\to\infty} a_n = \infty$. Prove, using $\epsilon$, that $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

(38) Is the converse of Exercise 37 true? That is, if $\lim_{n\to\infty} 1/a_n = 0$, does it follow that $\lim_{n\to\infty} a_n = \infty$. If not, under what circumstances would it follow? Prove your answers.

(39) Most mathematical theorems can be expressed in the form "If P, then Q" where P and Q are statements. (A *statement* is a sentence that has the potential of being either true or false.) For example, the converse to the statement in Exercise 37 is the statement in Exercise 38.

Below are some true statements. For each statement, formulate the converse statement and state whether or it is true. If false, give a counter example.

(a) If $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} ca_n$ exists for all constants $c$.
(b) If $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} b_n$ exists, then $\lim_{n\to\infty}(a_n + b_n)$ exists.
(c) If $\lim_{n\to\infty} a_n$ exists, then $a_n$ is bounded.
(d) If $\lim_{n\to\infty} a_n = \infty$, then $a_n$ is unbounded.
(e) If $\lim_{n\to\infty} a_n = \infty$, then $a_n$ is not bounded from above.
(f) Suppose that $\lim_{n\to\infty} a_n = b$. Then $\lim_{n\to\infty}(a_n - b) = 0$.