

CHAPTER 6

Max, Min, Sup, Inf

We would like to begin by asking for the maximum of the function $f(x) = (\sin x)/x$. An approximate graph is indicated below. Looking at the graph, it is clear that $f(x) \leq 1$ for all x in the domain of f . Furthermore, 1 is the smallest number which is greater than all of f 's values.

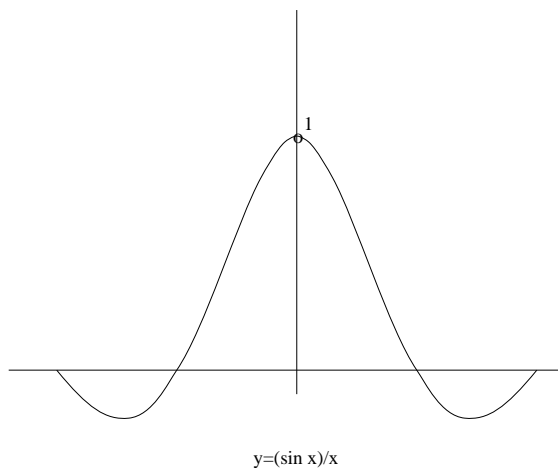


Figure 1

Loosely speaking, one might say that 1 is the ‘maximum value’ of $f(x)$. The problem is that one is not a value of $f(x)$ at all. There is no x in the domain of f such that $f(x) = 1$. In this situation, we use the word ‘supremum’ instead of the word ‘maximum’. The distinction between these two concepts is described in the following definition.

DEFINITION 1. *Let S be a set of real numbers. An upper bound for S is a number B such that $x \leq B$ for all $x \in S$. The supremum, if it exists, (“sup”, “LUB,” “least upper bound”) of S is the smallest*

upper bound for S . An upper bound which actually belongs to the set is called a maximum.

Proving that a certain number M is the LUB of a set S is often done in two steps:

- (1) Prove that M is an upper bound for S —i.e. show that $M \geq s$ for all $s \in S$.
- (2) Prove that M is the *least* upper bound for S . Often this is done by assuming that there is an $\epsilon > 0$ such that $M - \epsilon$ is also an upper bound for S . One then exhibits an element $s \in S$ with $s > M - \epsilon$, showing that $M - \epsilon$ is not an upper bound.

EXAMPLE 1. Find the least upper bound for the following set and prove that your answer is correct.

$$S = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}$$

Solution: We note that every element of S is less than 1 since

$$\frac{n}{n+1} < 1$$

We claim that the least upper bound is 1. Assume that 1 is not the least upper bound. Then there is an $\epsilon > 0$ such that $1 - \epsilon$ is also an upper bound. However, we claim that there is a natural number n such that

$$1 - \epsilon < \frac{n}{n+1}.$$

This inequality is equivalent with the following sequence of inequalities

$$\begin{aligned} 1 - \frac{n}{n+1} &< \epsilon \\ \frac{1}{n+1} &< \epsilon \\ \frac{1}{\epsilon} &< n+1 \\ \frac{1}{\epsilon} - 1 &< n. \end{aligned}$$

Reversing the above sequence of inequalities shows that if $n > \frac{1}{\epsilon} - 1$, then $1 - \epsilon < \frac{n}{n+1}$ showing that $1 - \epsilon$ is not an upper bound for S . This verifies our answer.

If a set has a maximum, then the maximum will also be a supremum:

PROPOSITION 1. *Suppose that B is an upper bound for a set S and that $B \in S$. Then $B = \sup S$.*

Proof Let $\epsilon > 0$ be given. Then $B - \epsilon$ cannot be an upper bound for S since $B \in S$ and $B > B - \epsilon$, showing that B is indeed the *least* upper bound.

EXAMPLE 2. Find the least upper bound for the following set and prove that your answer is correct.

$$T = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}.$$

Solution: From the work done in Example 1, 1 is an upper bound for S . Since $1 \in S$, $1 = \sup S$.

EXAMPLE 3. Find the max, min, sup, and inf of the following set and prove your answer.

$$S = \left\{\frac{2n+1}{n+1} \mid n \in \mathbb{N}\right\}.$$

Solution: We write the first few terms of S :

$$S = \left\{\frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \frac{11}{6}, \dots\right\}.$$

The smallest term seems to be $\frac{3}{2}$ and there seems to be no largest term, although all of the terms seem to be less than 2. Since $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2$ we conjecture that:

(a) There is no maximum, (b) $\sup S = 2$, and (c) $\min S = \inf S = \frac{3}{2}$.

Proof

Sup

We must first show that 2 is an upper bound—i.e.

$$\begin{aligned}\frac{2n+1}{n+1} &< 2 \\ 2n+1 &< 2n+2 \\ 1 &< 2\end{aligned}$$

which is always true. Reversing the above argument shows that 2 is an upper bound.

Next we show that 2 is the *least* upper bound. If 2 is not the LUB, there is an $\epsilon > 0$ such $2 - \epsilon$ is an upper bound. However, we claim that there are $n \in \mathbb{N}$ such that

$$2 - \epsilon < \frac{2n+1}{n+1}$$

showing that $2 - \epsilon$ is not an upper bound.

To prove our claim, note that the above inequality is equivalent with

$$\begin{aligned}-\epsilon &< \frac{2n+1}{n+1} - 2 \\ \epsilon &> \frac{1}{n+1} \\ n &> \frac{1}{\epsilon} - 1\end{aligned}$$

Since there exist $n \in \mathbb{N}$ satisfying the above inequality, our claim is proved. Hence 2 is the LUB.

Next we show that 2 is not a maximum. This means showing that there is no n such that

$$2 = \frac{2n+1}{n+1}.$$

This however is equivalent with

$$\begin{aligned}2(n+1) &= 2n+1 \\ 2n+2 &= 2n+1 \\ 2 &= 1\end{aligned}$$

which is certainly false.

Next we prove that $\min S = \frac{3}{2}$. We first note that $\frac{3}{2} \in S$ since

$$\frac{3}{2} = \frac{2 \cdot 1 + 1}{1 + 1}.$$

Hence, it suffices to show that $\frac{3}{2}$ is a lower bound which we do as follows:

$$\begin{aligned}\frac{2n+1}{n+1} &\geq \frac{3}{2} \\ 2(2n+1) &\geq 3(n+1) \\ n &\geq 1\end{aligned}$$

which is true for all $n \in \mathbb{N}$. Reversing the above argument shows that $\frac{3}{2}$ is a lower bound. \square

The central question in this section is “Does every non-empty set of numbers have a sup?” The simple answer is no—the set \mathbb{N} of natural numbers does not have a sup because it is not bounded from above. O.K.—we change the question: “Does every set of numbers which is bounded from above have a sup?” The answer, it turns out, depends upon what we mean by the word “number”. If we mean “rational number” then our answer is NO!

Recall that the set of integers is the set of positive and natural numbers, together with 0. I.e.

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots, \pm n, \dots \mid n \in \mathbb{N}\}.$$

The set of rational numbers is the set

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Thus, for example, $\frac{2}{3}$ and $-\frac{9}{7}$ are elements of \mathbb{Q} . In Chapter 9 (Theorem 2) we prove that $\sqrt{2}$ is not rational.

Now, let S be the set of all positive *rational* numbers r such that $r^2 < 2$. Since the square root function is increasing on the set of positive real numbers,

$$S = \{0 < r < \sqrt{2} \mid r \in \mathbb{Q}\}.$$

Clearly, $\sqrt{2}$ is an upper bound for S . It is also a limit of values from S . In fact, we know that

$$\sqrt{2} = 1.414213562 + .$$

Each of the numbers 1.4, 1.41, 1.414, 1.4142, etc. is rational and has square less than 2. Their limit is $\sqrt{2}$. Thus, $\sup S = \sqrt{2}$. (See Exercise 6 below.) Since $\sqrt{2}$ is irrational, S is then an example of a set of rational numbers whose *sup* is irrational.

Suppose, however, that we (like the early Greek mathematicians) only knew about rational numbers. We would be forced to say that S

has no sup. The fact that S does not have a sup in \mathbb{Q} can be thought of as saying that the rational numbers do not completely fill up the number line; there is a missing number “directly to the right” of S . The fact that the set \mathbb{R} of all real numbers *does* fill up the line is such a fundamentally important property that we take it as an axiom: the *completeness axiom*. (The reader may recall that in Chapter I, we mentioned that we would eventually need to add an axiom to our list. This is it.) We shall also refer to this axiom as the *Least Upper Bound Axiom*. (LUB Axiom for short.)

Least Upper Bound Axiom: *Every non-empty set of real numbers which is bounded from above has a supremum.*

The observation that the least upper bound axiom is false for \mathbb{Q} tells us something important: *it is not possible to prove the least upper bound axiom using only the axioms stated in Chapters 1 and 2*. This is because the set of rational numbers satisfy all the axioms from Chapters 1 and 2. Thus, if the least upper bound axiom were provable from these axioms, it hold for the rational numbers.

Of course, similar comments apply to minimums:

Definition: *Let S be a set of real numbers. A lower bound for S is a number B such that $B \leq x$ for all $x \in S$. The infimum (“inf”, “GLB,” “greatest lower bound”) of S , if it exists, is the largest lower bound for S . A lower bound which actually belongs to the set is called a minimum.*

Fortunately, once we have the LUB Axiom, we do not need another axiom to guarantee the existence of inf’s. The existence of inf’s is a theorem which we will leave as an exercise. (Of course, we could have let the existence of inf’s be our completeness axiom, in which case the existence of sup’s would be a theorem.)

Greatest Lower Bound Property: *Every non-empty set of real numbers which is bounded from below has a infimum.*

Proving that a certain number M is the GLB of a set S is similar to a LUB proof. It requires:

- (1) Proving that M is a lower bound for S —i.e. proving that $M \leq s$ for all $s \in S$.
- (2) Proving that M is the *greatest* lower bound for S . Often this is done by assuming that there is an $\epsilon > 0$ such that $M + \epsilon$ is a lower bound for S . One then exhibits an element s of S satisfying $s < M + \epsilon$, showing that $M + \epsilon$ is not a lower bound for S .

EXAMPLE 4. Prove that the inf of $S = (1, 5]$ is 1.

Solution: By definition S is the set of x satisfying $1 < x \leq 5$. Hence 1 is a lower bound for S . Suppose that 1 is not the GLB of S . Then there is an $\epsilon > 0$ such that $1 + \epsilon$ is also a lower bound for S . To contradict this, we exhibit $x \in S$ such that $1 < x < 1 + \epsilon$. Since

$$0 < \frac{\epsilon}{2} < \epsilon$$

we see that

$$x = 1 + \frac{\epsilon}{2}$$

satisfies

$$1 < x < 1 + \epsilon.$$

Since $1 + \epsilon$ is (by assumption) a lower bound for S and $5 \in S$, $1 + \epsilon \leq 5$, showing that $x \in (1, 5]$. Thus, $1 + \epsilon$ is not a lower bound, proving that 1 is the greatest lower bound.

EXAMPLE 5. Find upper and lower bounds for $y = f(x)$ for $x \in [-1, 1.5]$ where

$$f(x) = -x^4 + 2x^2 + x$$

Use a graphing calculator to estimate the least upper bound and the greatest lower bound for $f(x)$.

Solution: From the triangle inequality

$$\begin{aligned} |f(x)| &= |-x^4 + 2x^2 + x| \leq |x^4| + |2x^2| + |x| \\ &= |x|^4 + 2|x|^2 + |x| \end{aligned}$$

The last quantity is largest when $|x|$ is largest, which occurs when $|x| = 1.5$. Hence

$$|f(x)| \leq 1.5^4 + 2(1.5)^3 + 1.5 = 13.3125$$

Hence, $M = 14$ is an upper bound and $M = -14$ is a lower bound. As a check, we graph $y = -x^4 + 2x^2 + x$ with $x_{\min} = -.5$, $x_{\max} = 1.5$, $y_{\min} = -15$ and $y_{\max} = 15$, as well as the lines $y = 14$ and $y = -14$. Since the graph lies between the lines, the value of M is acceptable, although considerably larger than necessary. To estimate the least bound, we trace the curve using the trace feature of the calculator, finding that the maximum and minimum y -values are approximately 2.0559 and $-.130$ respectively. These values are (approximately) the least upper bound and the greatest lower bound respectively.

Remark: It is important to note that in the preceding example, the values of the function at the end points of the interval are not bounds for the function because $f(x)$ is not monotonic (i.e. it is neither increasing nor decreasing) over the stated interval. On the other hand, the largest value of $|x|^4 + 2|x|^2 + |x|$ is at the largest endpoint because this function increases as $|x|$ increases.

The next example uses both the triangle inequality and the observation that making the denominator of a fraction smaller increases its value.

EXAMPLE 6. Find a bound for $y = f(x)$ for $x \in [-2, 2]$ where

$$f(x) = \frac{x^3 - 3x + 1}{1 + x^2}$$

Solution: We note that

$$\left| \frac{x^3 - 3x + 1}{1 + x^2} \right| = \frac{|x^3 - 3x + 1|}{1 + x^2}$$

Since $x^2 > 0$, we see

$$\begin{aligned} x^2 + 1 &> 1 \\ \frac{1}{x^2 + 1} &< \frac{1}{1} = 1 \end{aligned}$$

Hence

$$\frac{|x^3 - 3x + 1|}{1 + x^2} \leq |x^3 - 3x + 1| \leq |x^3| + |3x| + 1 \leq 2^3 + 3 \cdot 2 + 1 = 15$$

Thus, our bound is $M = 15$.

In all of the examples considered above, the least upper bound for $f(x)$ is the maximum of $f(x)$. This is always the case if $f(x)$ has a maximum. Similarly, the greatest lower bound is the minimum of $f(x)$ if $f(x)$ has a minimum.

In Chapter 4, we studied sequences which diverge because they tend to infinity. Another class of sequences that have no limit are ones which might be dubbed “wishy-washy.” These are sequences that can’t make up their minds what their limit is, tending simultaneously to several numbers. An example is

$$a_n = \frac{n + (-1)^n n}{n + 1}$$

The first 10 values are listed below.

n	1	2	3	4	5	6	7	8	9	10
a_n	0	1.33	0	1.60	0	1.71	0	1.78	0	1.82

It appears that some values approach (in fact equal) 0 while others approach 2. Indeed, for odd n

$$a_n = \frac{n - n}{n + 1} = 0$$

which tells us that if the limit exists, it must be 0. For even n

$$a_n = \frac{n + n}{n + 1} = \frac{2n}{n + 1}$$

which tends to 2. From Proposition 1, a sequence can have only one limit. Hence, there is no limit.

A related type divergence is what might be referred to as “wandering,” where values wander in a seemingly random manner, never getting close to particular number. An example is

$$a_n = \cos n.$$

The first 10 values are listed below.

n	1	2	3	4	5	6	7	8	9	10
$\cos n$.540	-.416	-.990	-.654	.284	.960	.754	-.146	-.911	-.839

There certainly seems to be no tendency toward any one number.

A sequence a_n is *non-decreasing* if

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n < \cdots$$

The sequence a_n is *increasing* if each of the above inequalities is strict.

For example, $a_n = n^2$ is increasing :

$$1 < 2^2 < 3^2 < 4^2 < \dots$$

The sequence from Example 1 is also increasing:

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \dots$$

The two preceding sequences demonstrate that an increasing sequence can either go to infinity or can converge. However, an increasing sequence cannot wander; once it has exceeded a certain value, it can never return to that value. Hence we arrive at the following theorem:

THEOREM 1 (Bounded Increasing Theorem). *For a non-decreasing sequence a_n , either $\lim_{n \rightarrow \infty} a_n$ exists or $\lim_{n \rightarrow \infty} a_n = \infty$.*

Proof $\lim_{n \rightarrow \infty} a_n = \infty$ implies that for all $M > 0$ there is an N such that $a_n \geq M$ for all $n \geq N$. For a non-decreasing sequence, this is equivalent with the statement that for all M there is an n such that $a_n \geq M$. Hence if $\lim_{n \rightarrow \infty} a_n \neq \infty$, there is an M such that $a_n < M$ for all n . Thus

$$S = \{a_1, a_2, \dots, a_n, \dots\}.$$

is bounded from above. Let $a = \sup S$. Then

$$a_n \leq a$$

for all n . Furthermore, since a is the smallest upper bound, $a - \epsilon$ is not an upper bound for any $\epsilon > 0$. Hence, for all $\epsilon > 0$, there is at least one a_N such that

$$a - \epsilon < a_N \leq a.$$

However, since a_n is increasing and a is an upper bound for S , we see that for all $n \geq N$,

$$a_N \leq a_n \leq a.$$

It follows that for all $n \geq N$,

$$a - \epsilon \leq a_n \leq a + \epsilon$$

proving that a is the limit of a_n , as desired.

The bounded increasing theorem is one of the most important ways of proving that sequences converge. In particular, we will make good use of in Chapters 7 and 8.

There is, of course, nothing special about increasing as opposed to decreasing. It is an immediate consequence of the Bounded-Increasing Theorem that a decreasing sequence which is bounded from below also has a limit. (See Exercise 10 below.)

We have already used repeatedly, without comment, a consequence of the GLB axiom. For example, in our formal solution to Example 2 in Chapter 4, we said "...let $\epsilon > 0$ be given and let $n > 1/\epsilon$." Certainly, we know from experience that there are natural numbers n greater than $1/\epsilon$. However, none of the axioms from Chapters 1 and 2 tell us that such numbers exist. Their existence follows from the following theorem. We leave it as an exercise (Exercise 11 below) to explain how this property follows from the GLB axiom.

THEOREM 2 (Archimedean Property). *For all numbers M there is a natural number n such that $n > M$.*

Exercises

- (1) Compute the sup, inf, max and min (whenever these exist) for the following sets.¹
- (a) $\{1 + 1/n \mid n \in \mathbb{N}\}$
 - (b) $[0, 2)$
 - (c) $\{\frac{n^2+15}{n+1} \mid n \in \mathbb{N}\}$
 - (d) $\{x \mid x \in \mathbb{Q} \text{ and } x^2 < 2\}$
 - (e) $\{y \mid y = x^2 - x + 1 \text{ and } x \in \mathbb{R}\}$
 - (f) $\{x \mid x^2 - 3x + 2 < 0 \text{ and } x \in \mathbb{R}\}$
 - (g) $\{\frac{1}{n} - \frac{1}{m} \mid n, m \in \mathbb{N}\}$
 - (h) $\{1 + \frac{1+(-1)^n}{n} \mid n \in \mathbb{N}\}$
 - (i) $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots\}$. (This is a list of the fractions in the interval $(0, 1)$. The pattern is that we list fractions by increasing value of the denominator. For a given value of denominator, we go from smallest to largest, omitting fractions which are not in reduced form.)
 - (j) $\{n/(1 + n^2) \mid n \in \mathbb{N}\}$
 - (k) $\{3n^2/(1 + 2n^2) \mid n \in \mathbb{N}\}$
 - (l) $\{n/(1 - n^2) \mid n \in \mathbb{N}, n > 1\}$

¹In set theory, the symbol ' \mid ' is read "where." Thus, the set in part (a) is the set of numbers of the form $1 + 1/n$ where n is a natural number."

- (m) $\{(1 - 2n^2)/3n^2 \mid n \in \mathbb{N}\}$
- (n) $\{2n^2/(3n^2 - 1) \mid n \in \mathbb{N}\}$
- (o) $\{3n/\sqrt{1 + n^2} \mid n \in \mathbb{N}\}$
- (p) $\{3/\sqrt{1 + 2n^2} \mid n \in \mathbb{N}\}$
- (q) $\{\frac{n-1}{n^2+5} \mid n \in \mathbb{N}\}$

Hint: In the proof of the upper bound (which is requested in Exercise 2), you will discover that you need to prove $n^2 - 7n + 12 \geq 0$. Factor this polynomial and determine when it can be negative.

- (r) $\{\frac{3n+1}{n+1} \mid n \in \mathbb{N}\}$
- (2) Prove your answers in Exercise 1.
- (3) In Example 5, use a graphing calculator to estimate the sup and the inf for $f(x)$.
- (4) For the following functions $f(x)$, (i) find a number M such that $|f(x)| \leq M$ for all x in the stated interval. (ii) Use a graph to estimate the sup s and inf t for $f(x)$. Sketch your graph on a piece of paper. (iii) As evidence for the statement that s is the LUB, find an x such that $f(x) > s - \epsilon$ for the stated value of ϵ . (iv) As evidence for the statement that t is the GLB, find an x such that $f(x) < t + \epsilon$ for the stated value of ϵ .

(a) $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1 \quad x \in [-1, 1], \quad \epsilon = .1$

(b) $f(x) = \frac{x^3 - 3x^2}{1 + x + x^2} \quad x \in [0, 2], \quad \epsilon = .01$

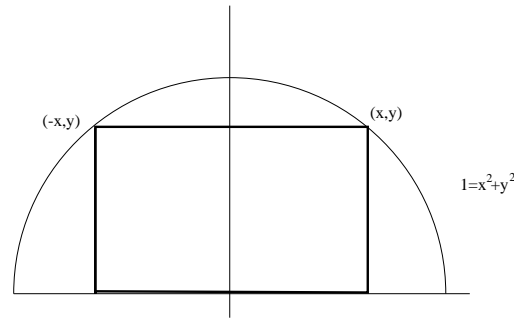
(c) $f(x) = x^3 - 3x^2 + 2x - 1 + 5 \sin x \quad x \in [-1, 2], \quad \epsilon = .05$

(d) $f(x) = \frac{x - 4 \cos x}{x^2 + 5} \quad x \in [2, 4], \quad \epsilon = .02$

(e) $f(x) = x^2 + 5 \sin x \quad x \in [-4, 4], \quad \epsilon = .2$

(f) $f(x) = \frac{x^4 + 1}{x^2 + 2 + \cos x} \quad x \in [0, \pi], \quad \epsilon = .03$

- (5) The figure below shows a rectangle inside the circle $x^2 + y^2 = 1$. One side of the rectangle is formed by the x axis while two vertices lie on the circle. Find the sup and inf of the set of possible areas for the rectangle. Is the sup also a max? Is the inf also a min?



- (6) Suppose that M is an upper bound for the set S . Suppose also that there is a sequence $s_n \in S$ such that $M = \lim_{n \rightarrow \infty} s_n$. Prove that $M = \sup S$.
- (7) Prove the converse of Exercise 6—i.e. suppose that $M = \sup S$. Prove that there is a sequence $a_n \in S$ such that $M = \lim_{n \rightarrow \infty} a_n$. *Hint:* For all $n \in \mathbb{N}$, $M - 1/n$ is not an upper bound.
- (8) Suppose that M is a lower bound for the set S . Suppose also that there is a sequence $s_n \in S$ such that $M = \lim_{n \rightarrow \infty} s_n$. Prove that $M = \inf S$.
- (9) Prove the converse of Exercise 8—i.e. suppose that $M = \inf S$. Prove that there is a sequence $a_n \in S$ such that $M = \lim_{n \rightarrow \infty} a_n$. *Hint:* Modify the argument from Exercise 7.
- (10) A sequence is said to be non-increasing if for all n , $a_{n+1} \leq a_n$. Prove that a non-increasing sequence either tends to $-\infty$ or converges. This is the Bounded Decreasing Theorem. *Hint:* Consider $-a_n$.
- (11) Prove the Archimedean Property. *Hint:* Assume that the Archimedean Property is false. Explain how it follows that the set of natural numbers is bounded from above. Let $s = \sup \mathbb{N}$. Then there is a natural number n satisfying $s - .5 < n \leq s$. (Explain!) What does this say about $n + 1$?
- (12) If S is any set of numbers, we define

$$-S = \{-x \mid x \in S\}$$

Compute the sup and inf of $-S$ for each set S in Exercise 9. How do the sup and inf of S relate to the sup and inf of $-S$?

- (13) This exercise uses the notation from Exercise 12.
- (a) Let S be a set of numbers that is bounded from below by a number B . Show that $-S$ is bounded from above by $-B$.

- (b) Let S be as in part (a). It follows from (a) that $-S$ is bounded from above. The Least Upper Bound Theorem implies that $-S$ has a least upper bound L . Prove that $-L$ is the GLB for S . For this you must explain why (i) $-L$ is a lower bound for S and (ii) why there is no greater lower bound.

Remark It follows from (b) that S has a GLB. Hence the above sequence of arguments proves the GLB property.

- (14) Let $a_1 = 1$. Let a sequence a_n be defined by

$$(1) \quad a_{n+1} = \sqrt{2a_n}.$$

Thus, for example,

$$a_2 = \sqrt{2 \cdot 1} = 1.414$$

$$a_3 = \sqrt{2 \cdot a_2} = 1.6818$$

$$a_4 = \sqrt{2 \cdot a_3} = 1.8340$$

- (a) Compute a_5 , a_6 , and a_7 (as decimals.) You should find that each is less than 2.
 (b) Prove that if $a_n < 2$ then $a_{n+1} < 2$ as well. How does it follow that $a_n < 2$ for all n ?
 (c) Prove that for all n , $a_{n+1} > a_n$.
 (d) Explain how it follows from the Bounded Increasing Theorem that $\lim_{n \rightarrow \infty} a_n = L$ exists.
 (e) Prove that $L = \sqrt{2L}$. What then is the value of L ?
Hint: Take the limit of both sides of (1).
- (15) Let $a_1 = 1$. Let a sequence a_n be defined by

$$(2) \quad a_{n+1} = \sqrt{2 + a_n}.$$

- (a) Compute a_2 , a_3 , and a_4 (as decimals.) You should find that each is less than 2.
 (b) Prove that if $a_n < 2$ then $a_{n+1} < 2$ as well. How does it follow that $a_n < 2$ for all n ?
 (c) Prove that for all n , $a_{n+1} > a_n$.
 (d) Explain how it follows from the Bounded Increasing Theorem that $\lim_{n \rightarrow \infty} a_n = L$ exists.
 (e) Prove that $L = \sqrt{2 + L}$. What then is the value of a ?
Hint: Take the limit of both sides of (2).

- (16) The following exercise develops the “divide and average” method of approximating $r = \sqrt{2}$. From $r^2 = 2$, we see that $r = 2/r$. If r_1 is some approximation to r , then $2/r_1$ will be another. The average

$$r_2 = \frac{1}{2}\left(r_1 + \frac{2}{r_1}\right)$$

will (hopefully) be a better approximation. We repeat this process with r_2 in place of r_1 to produce r_3 . In general, we set

$$(3) \quad r_{n+1} = \frac{1}{2}\left(r_n + \frac{2}{r_n}\right).$$

For example, if $r_1 = 2$, then

$$r_2 = \frac{1}{2}(2 + 2/2) = 1.5$$

$$r_3 = \frac{1}{2}(r_2 + 2/r_2) = 1.416666667$$

$$r_4 = \frac{1}{2}(r_3 + 2/r_3) = 1.414215686$$

- (a) Compute r_5 , r_6 and r_7 .
- (b) Prove that $r_n^2 > 2$. *Hint:* Write this as $r_{n+1}^2 > 2$ and use formula (3) above together with some algebra.
- (c) Prove that for all n , $0 < r_{n+1} \leq r_n$. How does it follow from the Bounded Decreasing Property that $\lim_{n \rightarrow \infty} r_n$ exists.
- (d) Show that $r = \lim_{n \rightarrow \infty} r_n$ satisfies the equation

$$r = \frac{1}{2}\left(r + \frac{2}{r}\right).$$

Use this to prove that $r^2 = 2$. *Hint:* Take the limit of both sides of formula (3).

Remark: This exercise proves the existence of $\sqrt{2}$.