CHAPTER 7

Positive Term Series

One very important goal in mathematics is computation. The number π, for example, has been computed to thousands of decimals. How is this done? It turns out that there are some remarkable formulas that can be used to approximate π. For example, the following formula, which comes from the theory of Fourier series, can be used to approximate $\pi^2/6$ and, thus, π.

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \dots \tag{1}$$

This formula means that we may approximate $\pi^2/6$ as accurately as desired by summing sufficiently many of the terms on the right of the equality. Let s_n be the sum of the first n terms on the right. Thus, for example, using 4 terms:

$$\frac{\pi^2}{6} \approx s_4 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} = 1.423 \tag{2}$$

Using 6 terms produces

$$\frac{\pi^2}{6} \approx s_6 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = 1.4911$$

Ten terms produces

$$\frac{\pi^2}{6} \approx s_{10} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{10^2} = 1.5497$$

Using 100 terms (and a computer) we find that

$$\frac{\pi^2}{6} \approx s_{100} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{100^2} = 1.6349 \tag{3}$$

The approximations we just produced are essentially useless unless we can determine their accuracy. We could, of course, simply compare them with the value of $\pi^2/6$ computed using, say, a calculator. However, our goal is to understand how calculators and computers can compute numbers such as $\pi^2/6$. Hence *we must assume that* *we have no means of computing the answer other than using the series; we must determine the accuracy without first knowing the value of* $\pi^2/6$.

For this we use geometry. The term $1/n^2$ is the length of the line segment drawn vertically from n on the x-axis to the curve $y = 1/x^2$ as indicated in Figure 1.

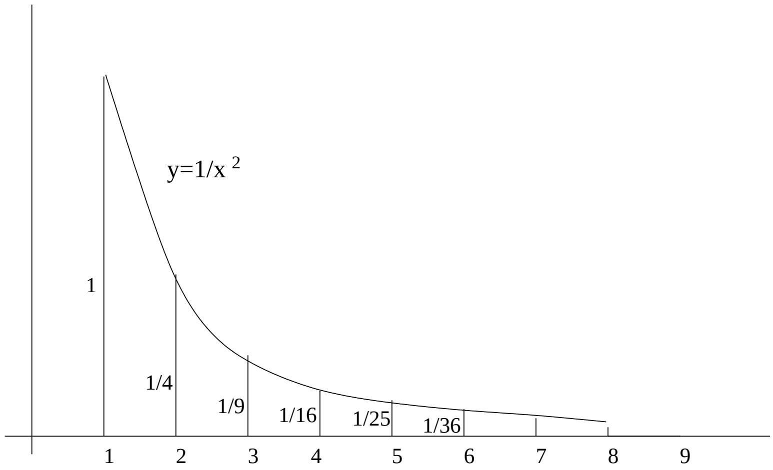

FIGURE 1

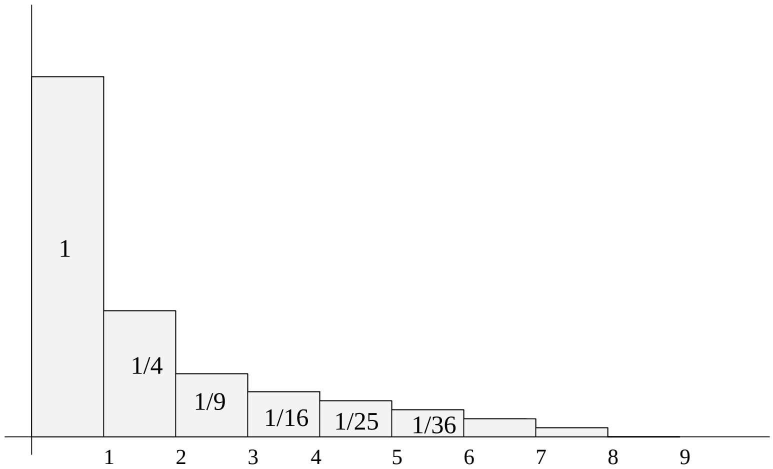

FIGURE 2

We use each of these line segments as the *right* edge of a rectangle of width 1 as in Figure 2.

According to formula (1), the sum of the areas of all of the rectangles is $\pi^2/6$. The approximation (2) is the sum of the areas of the first 4 rectangles. Thus, $\pi^2/6 - s_4$ is the sum of the areas of the rectangles over the interval $[4, \infty)$. (See Figure 3.)

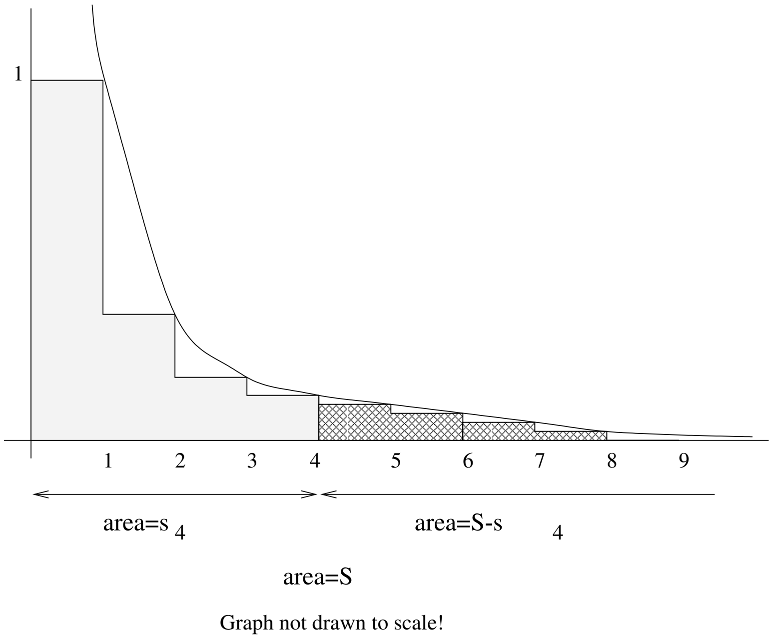

FIGURE 3

Since $y = 1/x^2$ decreases on $x > 0$, each rectangle lies entirely under the graph. Hence

$$\begin{aligned} \frac{\pi^2}{6} - s_4 &< \int_4^\infty \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\Big|_4^\infty = \frac{1}{4} = .25 \end{aligned} \tag{4}$$

Thus, we can guarantee that our approximation (2) is accurate within within $\pm.25$

For n terms,

$$\begin{aligned} 0 < \frac{\pi^2}{6} - s_n &< \int_n^\infty \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\Big|_n^\infty = \frac{1}{n} \end{aligned} \tag{5}$$

Thus, for n terms, the error is at most $1/n$. Thus the approximation (3) is accurate to within $\pm 1/100$. Indeed, our calculator tells us that

$$\pi^2/6 \approx 1.644934068$$

which is within $\pm 10^{-2}$ of (3).

To guarantee 10^{-8} accuracy, we need 10^8 terms. Even for a computer, computing this many terms is out of the question. This tells us that this series is not practical for computing π to high degrees of accuracy.

As mentioned previously, there are other formulas for π. One that you will analyze in the exercises is

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{n^4} + \ldots \tag{6}$$

This series converges considerably more rapidly than the preceding series because $1/n^4$ tends to zero much faster than $1/n^2$.

In general, if a_n is any sequence, we define **the sequence of partial sums** s_n by

$$s_n = a_1 + \cdots + a_n = \sum_1^n a_k$$

We then define the infinite sum of the a_n by

$$\sum_1^\infty a_k = \lim_{n\to\infty} s_n = s \tag{7}$$

provided this limit exists. When analyzing a summation $\sum a_k$, when we refer to "s_n" and "s" we mean the above defined expressions.

More generally, in the summation

$$\sum_{n_o}^\infty a_k$$

we define

$$s_n = \sum_{n_o}^n a_k$$

$$\sum_{n_o}^\infty a_k = \lim_{n\to\infty} s_n.$$

Thus *the "n" in s_n denotes the final index of summation*, not the number of terms being summed. Hence, for example, in the summation

$$\sum_{n=0}^{\infty} 2^n$$

we have

$$s_3 = 2^0 + 2^1 + 2^2 + 2^3.$$

Remark: When we refer to "$\sum_1^\infty a_n$", we are referring to s_n, not a_n. Thus, for example the *sequence*

$$\frac{n+1}{n}$$

converges since its limit is 1. However, the *series*

$$\sum_1^\infty \frac{n+1}{n}$$

diverges since

$$\begin{aligned} s_n &= \frac{1+1}{1} + \frac{2+1}{2} + \frac{3+1}{3} + \cdots + \frac{n+1}{n} \\ &> 1 + 1 + 1 + \cdots + 1 = n \end{aligned}$$

which goes to ∞. This example demonstrates a general principle:

PROPOSITION 1. *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_1^\infty a_n$ *cannot converge.*

Proof We note that

$$s_{n+1} - s_n = a_{n+1}.$$

If follows from Exercise 36 in Chapter 4 that

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} s_{n+1} - \lim_{n\to\infty} s_n \\ &= s - s = 0 \end{aligned}$$

proving our proposition.

The method we used to analyze the series (1) was based on the following facts:

(1) $a_n \geq 0$
(2) There is a *decreasing* function $f(x)$ such that $a_n = f(n)$.
(3) $s = \sum_1^\infty a_n$ exists.

In this case, the same geometrical argument shows:

THEOREM 1. *Suppose f, a_n and s are as described in (a)-(c) above. Then*

$$s - s_n \leq \int_n^\infty f(x)\,dx$$

It is important that in Theorem 1, f be decreasing to guarantee that the rectangles all lie below the graph of f.

In the preceding example, the convergence of series (1) was given. We can prove the convergence using the Bounded Increasing Property.

EXAMPLE 1. Prove the convergence of the series (1).

Solution: We note that

$$\begin{aligned} s_{n+1} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \\ &= s_n + \frac{1}{(n+1)^2} \end{aligned}$$

It follows that $s_{n+1} > s_n$ so the s_n form an increasing sequence. From the Bounded Increasing Property, either $\lim_{n\to\infty} s_n = \infty$ or $\lim_{n\to\infty} s_n$ exists. We will prove that the limit is not ∞ by showing that $s_n < 2$ for all n.

First Attempt: From Figure 3,

$$\begin{aligned} s_n &< \int_0^n \frac{1}{x^2}\,dx \\ &= -\frac{1}{x}\Big|_0^n = -\frac{1}{n} + \frac{1}{0} \end{aligned}$$

But $1/0$ is nonsense. The problem is that in Figure 3, the area of the region above the first rectangle and below the graph is infinite.

Second Attempt: To avoid the problem with the first rectangle, we begin our sum with $n = 2$, Then, from Figure 3,

$$\begin{aligned}\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} &< \int_1^n \frac{1}{x^2}\,dx \\ &< \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1\end{aligned}$$

Hence

$$s_n = 1 + (\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}) < 1 + 1 = 2$$

as claimed, proving convergence.

In this section we only consider "positive term series", meaning that $a_n > 0$ for all n. Such series are special because then s_n is an increasing sequence. Specifically

$$s_{n+1} = s_n + a_{n+1} > s_n$$

since $a_{n+1} > 0$. It follows that if a positive term series does not sum to ∞, it must converge. The student should be aware that non-positive term series have many other ways of diverging. (See Exercise 1 below.)

The following theorem generalizes the computation done in Example 1. The proof is just a repeat of the explanation of Example 1 and will be omitted.

THEOREM 2. *Suppose $a_n > 0$ for all n and $f(x)$ is an integrable, decreasing function on $[0, \infty)$ such that $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $s = \sum_1^\infty a_n$ exists if there is a k such that*

$$\int_k^\infty f(x)\,dx < \infty$$

An important consequence is the following theorem, which is left as an exercise.

THEOREM 3. *The following series converges for all $p > 1$.*

$$\sum_1^\infty \frac{1}{n^p} \tag{8}$$

Here is another example of using integrals to prove convergence and approximate the sum.

EXAMPLE 2. Prove that the following series converges. Compute s to $\pm .001$

$$s = \sum_{1}^{\infty} \frac{1}{n^2 + 1} \tag{9}$$

Solution: We interpret s as the sum of the areas of rectangles under the graph of $f(x) = 1/(x^2 + 1)$ as shown in Figure 4.

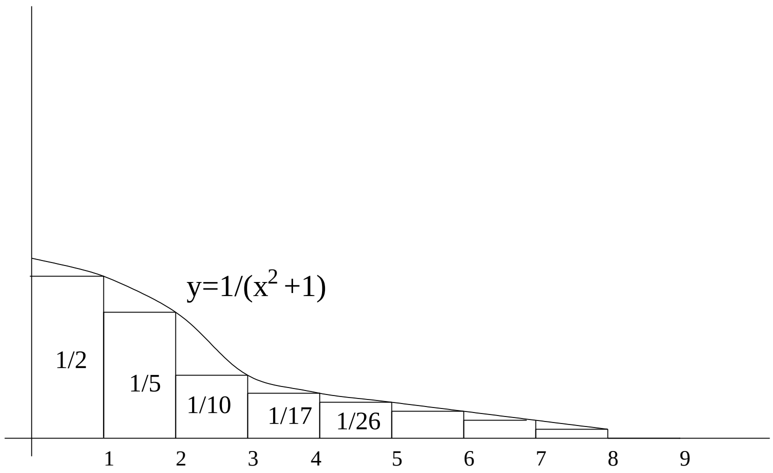

FIGURE 4

To apply Theorem 2, we need to know that $f(x)$ is a decreasing function. In this case this is true since $y = 1 + x^2$ gets larger as x grows, which implies that $1/(1 + x^2)$ gets smaller as x grows.

Comparing areas, we see that

$$s_n \leq \int_0^{\infty} \frac{1}{x^2 + 1}\, dx = \arctan x \Big|_0^{\infty} = \frac{\pi}{2}$$

since $\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$. It follows that $\lim_{n \to \infty} s_n \neq \infty$, proving that the limit exists.

To determine how many terms of the series are required to approximate s to within $\pm .001$ we use Theorem 1:

$$s - s_n \leq \int_n^{\infty} \frac{1}{x^2 + 1}\, dx = \arctan x \Big|_n^{\infty} = \frac{\pi}{2} - \arctan n$$

Hence, we want

$$\begin{aligned}\frac{\pi}{2} - \arctan n &< .001\\ \frac{\pi}{2} - .001 &< \arctan n\\ \tan(\frac{\pi}{2} - .001) &< n\\ 999.999 &< n\end{aligned}$$

(Note that applying the tangent function to the inequality is justified since both it and the arctangent function are increasing.) Hence 1000 terms suffice. We compute (using a computer) that

$$s \approx \frac{1}{1^2+1} + \frac{1}{2^2+1} + \cdots + \frac{1}{(1000)^2+1} = 1.074$$

Hence, $s = 1.074 \pm .001$.

In general, the rate at which the terms tend to zero determines how fast the series converges. In fact, it is possible for the sum to diverge if the terms go to 0 too slowly, as the next example shows.

EXAMPLE 3. Prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty \tag{10}$$

Solution: Let us compute a few of the partial sums

$$\begin{aligned}s_1 &= 1\\ s_2 &= 1 + \frac{1}{2} = 1.5\\ s_3 &= 1 + \frac{1}{2} + \frac{1}{3} = 1.888\ldots\\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 2.083333333\end{aligned}$$

Thus, if the limit of s_n exists, it is greater than 2.

In fact, experimental evidence suggests that we can make the sum as large as desired by summing sufficiently many terms. For example

$$\begin{aligned}s_{11} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{11} \approx 3.019877345 > 3\\ s_{35} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{35} \approx 4.146781419 > 4\end{aligned}$$

Thus, we guess that the sum in (10) just keeps getting larger and larger. To prove this, we interpret $1/n$ as the length of a line segment drawn vertically from n on the x-axis to the curve $y = 1/x$, producing a figure similar to Figure 1. Now, however, we interpret this line segment as the *left* edge of rectangle of width 1. These rectangles now lie *over* the curve as in Figure 5. Then s_n is a sum of areas of the first n rectangles. Comparison of areas shows that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) \tag{11}$$

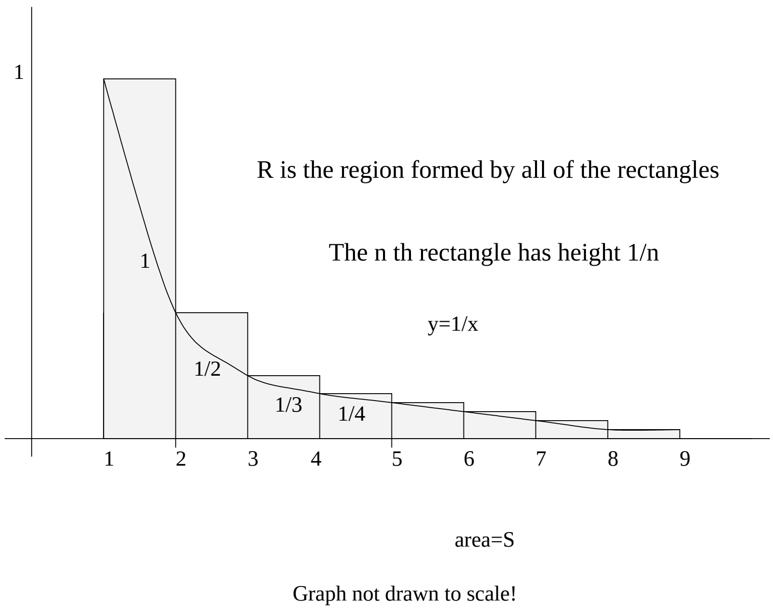

Figure 5

It follows that s_n may be made larger than any given number M. Specifically, $s_n > M$ will hold if $\ln(n+1) > M$ which is equivalent with

$$n + 1 > e^M$$

We see, therefore, that the sum (10) tends to ∞, despite the fact that the terms being summed tend to zero. Intuitively, this says that a lot of very little things can still total to something big.

The reader should note that in Example 1, the number of terms necessary for s_n to rise above 1000 is astronomically large. This series tends to ∞ at a very slow rate.

The method we used to analyze Example 1 also applies general series:

THEOREM 4. *Suppose f, a_n and s are as described in (a)-(c) above Theorem 1. Then*

$$s_n \geq \int_1^{n+1} f(x)\,dx$$

Using the proceeding theorem, we can prove that **the series in Theorem 3 diverges if $p \leq 1$.** Hence, for this type series, $p = 1$ is the line between divergence and convergence.

Theorems 1, 2, and 4 are, together, referred to as the "integral test for convergence." In practice, use of the integral test is complicated by the facts that (i) proving that the function $f(x)$ is decreasing can be difficult and (ii) explicitly evaluation the integral of $f(x)$ might be impossible. Fortunately, there are other techniques.

EXAMPLE 4. Determine whether or not the following sum converges and prove your answer.

$$s = \sum_1^\infty \frac{n}{n^3 + n\ln n + 5} \tag{12}$$

Solution: The fastest growing term in the denominator is n^3. Hence, the sum should behave like $\sum_1^\infty 1/n^2$ which converges from Theorem 3. In fact, from the discussion at the beginning of this section, it converges to $\pi^2/6$.

To prove our answer, we note that

$$\frac{n}{n^3 + n\ln n + 5} < \frac{n}{n^3} = \frac{1}{n^2}$$

Hence

$$\frac{1}{1^3 + 1\ln 1 + 5} + \frac{2}{2^3 + 2\ln 2 + 5} + \cdots + \frac{n}{n^3 + n\ln n + 5}$$
$$< \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < \frac{\pi^2}{6}$$

Since $s_n < \pi^2/6$, $\lim_{n\to\infty} s_n \neq \infty$, proving convergence.

Remark: In the above example, we did not really need to know the actual value of $\sum_1^\infty 1/n^2$. All we required was that it not be infinity. The work done in Example 4 illustrates the **comparison test for convergence:**

THEOREM 5. *Suppose that $0 \leq a_n \leq b_n$ for all n. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty b_n$ converges.*

As stressed above, it is important to know how many terms we must sum to obtain a given accuracy of approximation. When using the comparison test, the rule for determining this information is simple:

THEOREM 6. *Suppose that in Theorem 5, the sum of the first N b_n approximates $\sum_1^\infty b_n$ to within $\pm\epsilon$. Then the same will be true for a_n: i.e. the sum of the first N a_n will approximate $\sum_1^\infty a_n$ to within $\pm\epsilon$.*

Thus, in Example 4 we proved convergence by comparison with $\sum_1^\infty 1/n^2$. We saw using formula (5) that summing 100 terms of this series approximates $\pi^2/6$ to within $\pm.001$. Hence, the sum of the first 100 terms of the series in Example 4 will approximate s to within $\pm.001$. Thus

$$\begin{aligned} s = {} & \frac{1}{1^3 + 1\ln 1 + 5} + \frac{2}{2^3 + 2\ln 2 + 5} + \dots \\ & + \frac{100}{100^3 + 100\ln(100) + 5} = .647 \pm .001 \end{aligned}$$

Proof (of Theorem 6)

Looking, for example, at Figure 2, we see that the error in approximating s by s_k is

$$s - s_k = a_{k+1} + a_{k+2} + \dots$$

Similarly, letting the nth partial sum of the b_n be t_n and $t = \sum_1^\infty b_n$, we see that

$$t - t_k = b_{k+1} + b_{k+2} + \dots$$

From $a_n < b_n$ we see that

$$s - s_k < t - t_k$$

Hence, $t - t_k < \epsilon$ implies that $s - s_k < \epsilon$ which proves Theorem 6.

It is also possible to use the comparison test to prove that a series diverges.

EXAMPLE 5. Does the following sum converge?

$$s = \sum_1^n 1/\sqrt{n}$$

Solution: We note that for all natural numbers n, $1/\sqrt{n} \geq 1/n$. Hence

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

From the work done in Example 2, this latter quantity tend to ∞ as n tends to infinity. Thus, the sum in this example diverges.

Thus, *in Theorem 5, if* $\sum a_n$ *diverges, then* $\sum b_n$ *will also diverge.*

Occasionally, one can find a formula for s_n. In the exercises, you will prove the following:

$$(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1} \tag{13}$$

which, for $x \neq 1$, yields

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \tag{14}$$

EXAMPLE 6. The following sum diverges to ∞. What is the first value of n such that $s_n > 10^{10}$?

$$s = 1 + 2 + 2^2 + \cdots + 2^n + \ldots$$

Solution: From formula (14), with $x = 2$, we see

$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1 \tag{15}$$

Hence, we can find the desired n by the following sequence of inequalities:

$$\begin{aligned} 2^{n+1} - 1 &> 10^{10} \\ (n+1)\ln 2 &> \ln(10^{10} + 1) \\ n + 1 &> \frac{\ln(10^{10} - 1)}{\ln 2} \approx 33.22 \end{aligned}$$

Hence the first n such that $s_n > 10^{10}$ is $n = 33$.

The next example uses formula (14) in a convergent series.

EXAMPLE 7. Find the value of the following sum and prove (using ϵ) that your answer is correct.

$$s = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} + \ldots$$

Solution: From formula (14) with $x = \frac{1}{3}$ we see that

$$\begin{aligned} s_n &= 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n \\ &= \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - 1/3} \\ &= \frac{3}{2} - \frac{3}{2}\left(\frac{1}{3^{n+1}}\right) \end{aligned}$$

Thus the limit is $3/2 = 1.5$.

If we want to approximate the limit to within $\pm\epsilon$, we need

$$\begin{aligned} |\left[\frac{3}{2} - \frac{3}{2}\left(\frac{1}{3^{n+1}}\right)\right] - \frac{3}{2}| &< \epsilon \\ \frac{3}{2}\left(\frac{1}{3^{n+1}}\right) &< \epsilon \end{aligned}$$

which will be true if

$$n > -\frac{\ln \epsilon}{\ln 3} - 1$$

Since an appropriate N exists for all $\epsilon > 0$, the limit is proved.

Examples 6 and 7 demonstrate a general theorem.

THEOREM 7. *Let x be a real number. Then the series on the right side of the following equality converges if, and only if, $|x| < 1$. Furthermore, when it converges, it converges to the stated value.*

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \ldots \tag{16}$$

Theorem 7 gives us another class of series to compare with: those of exponential growth or decay.

EXAMPLE 8. Determine the convergence or divergence of the following series and prove your answer.

$$s = \sum_{1}^{\infty} \frac{n^2}{4^n}$$

Solution: We think that 4^n grows so much faster than n^2 that the series should converge. Specifically, there is an N such that

$$n^2 < 3^n \quad \text{for all } n > N. \tag{17}$$

In fact, the preceding inequality is equivalent with

$$2 \ln n < n \ln 3$$

which, from Proposition 2 in Chapter 3, with $a = (\ln 3)/2$, is valid for $n > 4/a^2 = 13.2$. Hence, for $n > 13$,

$$\frac{n^2}{4^n} < \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

Hence, for $n > 13$, using Theorem 7,

$$\begin{aligned} s_n &= s_{13} + \frac{14^2}{4^{14}} + \frac{15^2}{4^{15}} + \cdots + \frac{n^2}{4^n} \\ &< s_{13} + \left(\frac{3}{4}\right)^{14} + \left(\frac{3}{4}\right)^{15} + \cdots + \left(\frac{3}{4}\right)^n \\ &< s_{13} + \sum_{0}^{\infty} \left(\frac{3}{4}\right)^n = s_{13} + \frac{1}{1 - \frac{3}{4}} = s_{13} + 4 \end{aligned}$$

It follows that s_n is an increasing sequence which cannot tend to infinity; hence converges.

Remark: We didn't really need to find an N for which (17) holds; all we needed was the existence of such an N. Our argument would have been the same, except that we would have used "N" in every place that "13" occured. In fact, for any $N \in \mathbb{N}$ and any sequence a_n, we can write

$$\sum_{1}^{\infty} a_n = \sum_{1}^{N} a_n + \sum_{N}^{\infty} a_n$$

The issue of convergence doesn't arrise for the middle sum since it is finite. Hence **the sum on the left converges if and only if that on the right does.** We use these ideas in the next few examples.

EXAMPLE 9. Determine the convergence or divergence of the following series.

$$s = \sum_{1}^{\infty} \frac{1}{(\ln n)^2}$$

Solution: Since $\ln n$ grows more slowly than any power of n there is a N such that $\ln n < \sqrt{n}$ for $n > N$. Hence

$$\begin{aligned}\sum_{1}^{\infty} \frac{1}{(\ln n)^2} &= s_N + \sum_{N+1}^{\infty} \frac{1}{(\ln n)^2} \\ &> s_N + \sum_{N+1}^{\infty} \frac{1}{n}\end{aligned}$$

This latter sum tends to ∞ from Example 3 so our original sum diverges.

EXAMPLE 10. Determine the convergence or divergence of the following series.

$$\sum_{1}^{\infty} \frac{\ln n}{n^3 - n + 2}$$

Solution: Our thinking is that the fastest growing term in the denominator is n^3 while the numerator grows more slowly than n. Since $n/n^3 = 1/n^2$ which has a finite sum, the series should converge. For the proof, we note that there are positive constants C and N such that

$$Cn^3 < n^3 - n + 2 \quad \text{for } n > N$$

Hence

$$\begin{aligned}\frac{\ln n}{n^3 - n + 2} &< \frac{\ln n}{Cn^3} \\ &< \frac{n}{Cn^3} = \frac{1}{Cn^2}\end{aligned}$$

Hence,

$$\sum_{N+1}^{\infty} \frac{\ln n}{n^3 - n + 2} < \sum_{N+1}^{\infty} \frac{1}{Cn^2} = \frac{1}{C} \sum_{N+1}^{\infty} \frac{1}{n^2}$$

which converges from Theorem 3.

EXAMPLE 11. Determine the convergence or divergence of the following series.

$$\sum_{1}^{\infty} \frac{5^n}{4^n + n^2 + 1}$$

Solution: Our thinking is that the fastest growing term in the denominator is 4^n. Hence, the terms being summed grow like

$$\frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$$

which tends to ∞. Thus, the terms being summed don't even tend to zero, making convergence impossible. For the proof, since exponential growth is faster than power growth, there is an N such that for $n > N$

$$n^2 + 1 < 4^n$$

Hence for $n > N$,

$$\frac{5^n}{4^n + n^2 + 1} > \frac{5^n}{4^n + 4^n} = \frac{1}{2}\left(\frac{5}{4}\right)^n$$

which tends to ∞, proving divergence.

Exercises

(1) For the following series compute s_n for $n = 1, 2, \ldots, 5$. Do the series seem to be converging? What theorem from this section of the notes does this exercise illustrate?

(a) $\quad \dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} \cdots + \dfrac{n}{n+1} + \ldots$

(b) $\quad -1 + 1 + (-1) + \cdots + (-1)^n + \ldots$

(c) $\quad \dfrac{1}{2} + \dfrac{-2}{3} + \dfrac{3}{4} \cdots + \dfrac{(-1)^{n+1}n}{n+1} + \ldots$

(2) How many terms of the series (6) on page 100 does it take to approximate $\pi^4/90$ to $\pm 10^{-2}$? Compute this approximation and compare with the answer your calculator gives for $\pi^4/90$. *Hint:* Repeat the argument done for $\pi^2/6$ on page 97.

(3) Determine the convergence or divergence of the following series. Prove your answers.
   (a) $\sum_1^\infty \frac{1}{\sqrt{1+n}}$
   (b) $\sum_1^\infty \frac{1}{\sqrt{1+n^2}}$
   (c) $\sum_1^\infty \frac{1}{\sqrt{1+n^3}}$
   (d) $\sum_1^\infty \frac{n}{\sqrt{1+n^3}}$
   (e) $\sum_1^\infty \frac{n^2}{\sqrt{1+n^2}}$
   (f) $\sum_1^\infty \frac{2\sqrt{n}}{3n^2+1}$
   (g) $\sum_1^\infty \frac{n^{11}+3n^5+7}{\sqrt{n^{25}+1}}$
   (h) $\sum_1^\infty \frac{1}{n\sqrt{3n-1}}$
   (i) $\sum_1^\infty \frac{\ln n}{n}$
   (j) $\sum_1^\infty \frac{n^2+n+5}{n^4-\ln n+1}$
   (k) $\sum_1^\infty \frac{n}{2^n}$
   (l) $\sum_1^\infty ne^{-n}$
   (m) $\sum_1^\infty ne^{-n^2}$
   (n) $\sum_1^\infty \frac{\ln n}{n2^n}$
   (o) $\sum_1^\infty \frac{1+\cos n}{n^2}$
   (p) $\sum_1^\infty \frac{3^{2n}}{n!}$

(4) For each of the infinite sums $\sum a_n$ from the preceding exercise, find all $x > 0$ for which $\sum a_n x^n$ converges. We solve (a) as an example:

Solution: (To (a))

$$\sum_1^\infty \frac{nx^n}{2^n} = \sum_1^\infty n\left(\frac{x}{2}\right)^n$$

If $\frac{x}{2} > 1$ then $\left(\frac{x}{2}\right)^n$ exhibits exponential growth in n. Thus, the sum cannot converge since the terms don't tend to 0.

If $0 < \frac{x}{2} < 1$, then $\left(\frac{2}{x}\right)^n$ exhibits exponential growth in n. Hence, there is an N such that for all $n > N$

$$\left(\frac{2}{x}\right)^n > n^3$$

Thus, for $n > N$,

$$n\left(\frac{x}{2}\right)^n < \frac{n}{n^3} = \frac{1}{n^2}$$

Hence, the infinite sum converges. The sum clearly diverges for $x = 2$, so the set of positive x for which the sum converges is exactly $(0, 2)$.

(5) Assume that it is given that $y = f(x)$ is decreasing on $[3, \infty)$. (See Figure 5 below for a possible graph of f.). Let $a_n = f(n)$ and $s = \sum_1^\infty a_n$.

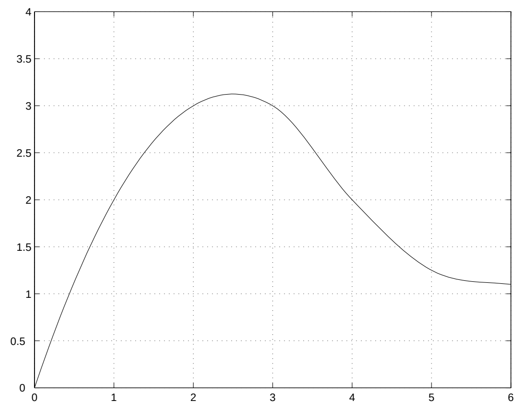

FIGURE 5

(a) Find a specific value of n and m such that the following inequality is guaranteed to hold. Choose n as small as possible and m as large as possible, consistent with the information provided. Justify your answer with a diagram.

$$s - s_n < \int_m^\infty f(x)\,dx$$

(b) Find a specific value of n and m such that the following inequality is guaranteed to hold. Choose m as small as possible and n as large as possible, consistent with the information provided. Justify your answer with a diagram.

$$\int_m^\infty f(x)\,dx < a_n + a_{n+1} + \ldots$$

(c) Find a specific value of n and m such that the following inequality is guaranteed to hold. Choose n as small as possible and m as large as possible, consistent with the information provided. Justify your answer with a diagram.

$$\int_m^{\infty} f(x)\,dx < s - s_n$$

(6) Assume that it is given that $y = f(x)$ is is increasing on $[0, 2.5]$ decreasing on $[2.5, \infty)$. (See Figure 5 above for a possible graph of f.) Let $a_n = f(n)$ and $s = \sum_1^\infty a_n$.

(a) Find a specific value of n, a and b such that the following inequality is guaranteed to hold. Choose both n and b as large as possible and a as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$a_1 + a_2 + \cdots + a_n > \int_a^b f(x)\,dx$$

(b) Find a specific value of n, a and b such that the following inequality is guaranteed to hold. Choose both n and a as large as possible and b as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$a_1 + a_2 + \cdots + a_n < \int_a^b f(x)\,dx$$

(7) Assume that it is given that $y = f(x)$ is increasing on $[0, 5]$ and decreasing on $[5, \infty)$. (See Figure 6 below for a possible graph of f.) Let $a_n = f(n)$ and $s = \sum_1^\infty a_n$.

(a) Find a specific value of n, a and b such that the following inequality is guaranteed to hold. Choose both n and a as large as possible and b as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$a_1 + a_2 + \cdots + a_n < \int_a^b f(x)\,dx$$

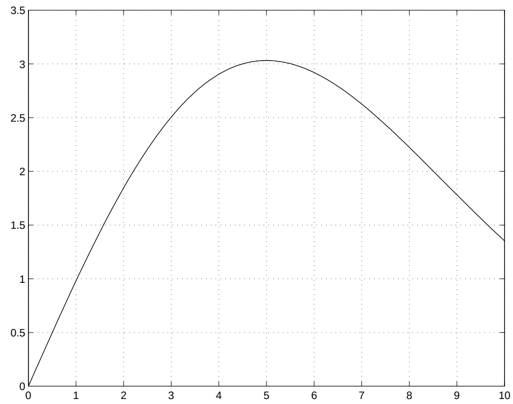

FIGURE 6

(b) Find a specific value of n and a such that the following inequality is guaranteed to hold. Choose a and n as small as possible, consistent with the information provided. Justify your answer with a diagram.

$$s - s_n < \int_a^{\infty} f(x)\,dx$$

(8) Let

$$s = \sum_1^{\infty} \frac{3n^2}{2^n + 1}$$

(a) Find values of k and m such that the following inequality is satisfied. Justify with a diagram. Choose k and m as small as possible, consistent with your diagram.

$$\int_k^{\infty} \frac{3x^2}{2^x + 1}\,dx > a_m + a_{m+1} + \ldots$$

(b) Let k and m be as in part (a) and let $n \geq k$. Find p (depending on n) such that the following inequality holds. Choose p as large as is consistent with the diagram in (a). Justify your answer in terms of your diagram.

$$\int_k^{p} \frac{3x^2}{2^x + 1}\,dx > a_m + a_{m+1} + \cdots + a_n$$

(9) Which of the following series converge and which diverge? If convergent, approximate the sum within $\pm 10^{-3}$. If divergent, determine a value of N so that $s_n > 100$ for all $n \geq N$.

(a) $\sum_2^\infty \frac{3n^2}{(n^3+1)^4}$ Reason as in Example (2) on page 104. Theorem 1 on page 102 is since in this exercise the summand decreases for $n \geq 2$.

(b) $\sum_1^\infty \frac{3n^2 \cos^2 n}{(n^3+1)^4}$

(c) $\sum_1^\infty \frac{1}{\sqrt{2n+5}}$ Reason as in Example 3

(10) In Exercise 9, part (c), prove using M that $\lim_{n\to\infty} s_n = \infty$.

(11) Let

$$s = \sum_1^\infty \frac{n^2 + \sqrt{n} + 1}{n^4 + 3n + 7}$$

(a) Prove that this series converges.

(b) Write a sum which computes s to within $\pm 10^{-3}$.

(12) Let

$$s = \sum_1^\infty \frac{n}{e^{n^2/50}}$$

(a) Find a specific value of n and m such that the following inequality is guaranteed to hold where $f(x) = xe^{-x^2/50}$. Justify your answer with a diagram. *Hint:* Use a graphing calculator to graph $f(x)$. You may assume that $f(x)$ is decreasing where the calculator so indicates.

$$s - s_n < \int_m^\infty f(x)\,dx$$

(b) Find a specific value of n and m such that the following inequality is guaranteed to hold. Justify your answer with a diagram.

$$\int_m^\infty f(x)\,dx < a_n + a_{n+1} + \dots$$

(c) Does this series converge? If so prove it and approximate its limit to within $\pm 10^{-3}$. If not find a value of n for which $s_n > 1000$ for all $n \geq N$.

(d) Prove convergence of the following series and determine how many terms are necessary to approximate its sum to within $\pm 10^{-3}$. *Hint:* Use Theorem 7.

$$s = \sum_{1}^{\infty} \frac{n}{4^{n^2/50} + n + 1}$$

(13) For which p does

$$\sum_{2}^{\infty} \frac{1}{n^p \ln n}$$

converge? *Hint:* For $p \neq 1$, consider rates of growth. For $p = 1$, use Theorem 4. The integral may be evaluated with the substitution $u = \ln x$.

(14) Does the following sum converge? If so, compute its value to within $\pm 10^{-2}$. If not, find a value of n such that $s_n > 1000$.

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \ldots$$

(15) Prove the convergence of the following sum and compute its value to within $\pm 10^{-2}$.

$$\sum_{1}^{\infty} \frac{1}{n^3 + 3n + 1}$$

(16) Diverge or converge? Prove your answer.

$$\text{(a)} \qquad \sum_{1}^{\infty} \frac{n^3 - n^2 + \ln n + 1}{n^6 + n^5 + \ln n + \sqrt{n}}$$

$$\text{(b)} \qquad \sum_{1}^{\infty} \frac{(n \ln n)(15)^n}{n!}$$

(17) Prove the convergence of the following sum and compute its value to within $\pm 10^{-3}$.

$$s = \sum_{1}^{\infty} \frac{\sin^2 n}{2^n}$$

(18) Use Theorem 4 to determine how many terms of the series from Example 5 are required for the sum to exceed 100.

(19) Find all values of x for which the following series converges. Prove your answer using rates of growth.

$$\sum_{1}^{\infty} \frac{nx^n}{3^n \ln n}$$

(20) Let $p > 1$. Use Theorem 2 to prove Theorem 3. Suppose that $p = 1.001$. How many terms will approximate the sum to within $\pm 10^{-1}$?

(21) Let $p < 1$. Use Theorem 4 to prove divergence of the series (8). Suppose that $p = .99$. How many terms will make the sum larger than 10?

(22) Let $p > 1$. Use Theorem 2 to prove convergence of the following series. Suppose that $p = 1.001$. How many terms will approximate the sum to within $\pm 10^{-1}$?

$$\sum_{2}^{\infty} \frac{1}{n(\ln n)^p}$$

(23) Use Theorem 4 to prove that the series in Exercise 22 diverges for $p < 1$. How many terms will make the sum larger than 10?

(24) The following exercise outlines a different proof of the divergence of

$$\sum_{1}^{\infty} \frac{1}{n}$$

For the proof, let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
 (a) Explain why $s_4 > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{2}{4}$.
 (b) Explain why $s_8 > 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8}$.
 (c) From similar lines of reasoning, how big will s_{16} be?
 (d) Find a value of n such that $s_n > 1000$.
 (e) Prove that $\lim_{n\to\infty} s_n = \infty$.

(25) For the following series
 (a) Compute s_1, s_2, s_3 and s_{20}. *Hint:* Use formula (14).
 (b) Find a value of n such that $3 - s_n < 10^{-100}$.
 (c) Prove, using ϵ, that $\lim_{n\to\infty} s_n = 3$.

$$s = \sum_{0}^{\infty} \left(\frac{2}{3}\right)^n .$$

(26) For the following series

(a) Compute s_1, s_2, s_3 and s_{20}. *Hint:* Use formula (14).
(b) Find a value of n such that $s_n > 10^{100}$.
(c) Prove, using M, that $\lim_{n\to\infty} s_n = \infty$.

$$s = \sum_0^\infty \left(\frac{3}{2}\right)^n.$$

(27) Here is a proof of formula (14) for the $n = 2$ case:

$$\begin{aligned}(1-x)(1+x+x^2) &= (1-x)(1+x) + (1-x)x^2 \\ &= (1-x^2) + (x^2 - x^3) = 1 - x^3\end{aligned}$$

Next we use the $n = 2$ case to do $n = 3$:

$$\begin{aligned}(1-x)(1+x+x^2+x^3) &= (1-x)(1+x+x^2) + (1-x)x^3 \\ &= (1-x^3) + (x^3 - x^4) = 1 - x^4\end{aligned}$$

Next we use the $n = 3$ case to do $n = 4$:

$$\begin{aligned}(1-x)(1+x+x^2+x^3+x^4) &= (1-x)(1+x+x^2+x^3) + (1-x)x^4 \\ &= (1-x^4) + (x^4 - x^5) = 1 - x^5\end{aligned}$$

(a) Use a similar argument to do the $n = 5$ and $n = 6$ cases.
(b) Use mathematical induction to prove the formula in general.

(28) Let

$$s_n = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)\cdot(2n+1)}.$$

(a) Use mathematical induction to prove that $s_n = \frac{n}{2n+1}$.
(b) Compute

$$\sum_1^\infty \frac{1}{(2n-1)(2n+1)}$$

(29) Let

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

(a) Show that $a_1 = 1$, $a_2 \approx .8069$, and $a_3 \approx .7347$. Compute a_4, a_5 and a_6. You should find that the values appear to steadily decrease.

(b) Prove that

$$\frac{1}{n+1} < \ln(n+1) - \ln n$$

Hint: Use a single rectangle below the curve to bound

$$\int_n^{n+1} \frac{1}{x} dx$$

(c) Show that that for all n,

$$a_{n+1} - a_n = \frac{1}{n+1} - (\ln(n+1) - \ln n)$$

Note that it now follows from (b) that $a_{n+1} - a_n < 0$; hence the a_n decrease.

(d) Use formula (11) to prove that $a_n > 0$.

It now follows from the Bounded Decreasing Property that $\gamma = \lim_{n\to\infty} a_n$ exists and is less than .69. Hence, for large n

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln n + \gamma$$