CHAPTER 8

Absolute convergence

The reader should note that all of the techniques demonstrated so far are based on the Bounded Increasing Theorem, which requires that the a_n be non-negative.

Any series with positive and negative terms can be written as the difference of two positive term series. For example,

$$\begin{aligned}
&\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \\
&= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \\
&\quad - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots \right)
\end{aligned} \tag{1}$$

Both of the positive series in the last equality converge since each is less than

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots$$

The convergence of (1) follows.

Notice that the preceding series is just the series on the left of the equality in (1) with all minus signs changed to pluses. The argument just done generalizes to prove the following theorem.

THEOREM 1. *Let a_n be a sequence of real numbers. Then $\sum_1^\infty a_n$ will converge if $\sum_1^\infty |a_n|$ converges.*

Proof Let t_n be the sum of the non-negative a_i for i ranging from 1 to n and let u_n be the sum of the negative a_i for i in the same range. Then

$$s_n = t_n + u_n$$

Note that t_n is increasing since it is a sum of non-negative terms. For similar reasons, $-u_n$ is increasing. (Note that if $a_i < 0$, $-a_i > 0$.) Furthermore, both t_n and $-u_n$ are less than

$$|a_1| + |a_2| + \dots + |a_n| + \dots$$

which is (by hypothesis) finite. It follows from the bounded increasing theorem that both t_n and $-u_n$ converge. Thus $s_n = t_n - (-u_n)$ converges, proving the theorem.

Theorem 1 allows us to apply the notion of rates of convergence to non-positive series.

EXAMPLE 1. Does the following series converge?.

$$\frac{1}{1^3-3} - \frac{\sqrt{2}}{2^3-3} + \frac{\sqrt{3}}{3^3-3} + \cdots + \frac{(-1)^{n+1}\sqrt{n}}{n^3-3} + \dots$$

Solution: Replacing each term by its absolute value yields the series

$$\frac{1}{2} + \frac{\sqrt{2}}{2^3-3} + \frac{\sqrt{3}}{3^3-3} + \cdots + \frac{\sqrt{n}}{n^3-3} + \dots$$

Our thinking is that for large values of n,

$$\frac{\sqrt{n}}{n^3-3} \approx \frac{\sqrt{n}}{n^3} = \frac{1}{n^{2.5}}$$

indicating convergence.

More precisely, there is are positive values of C and N such that

$$Cn^3 < n^3 - 3$$

for all $n > N$. Hence, for such n

$$\frac{\sqrt{n}}{n^3-3} < \frac{\sqrt{n}}{Cn^3} = \frac{1}{Cn^{2.5}}$$

whose sum converges from Theorem 3 in Chapter 6, proving the convergence.

The rate at which the a_n tends to 0 is not, however, the whole story. Consider, for example,

$$s = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \tag{2}$$

On the basis of rates of growth one might expect this series to diverge since the a_n grow like $1/n$ which has infinite sum. Remarkably, however, the sum actually converges. To see why, we compute a few values of s_n

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 - \frac{1}{2} = .5 \\
s_3 &= 1 - \frac{1}{2} + \frac{1}{3} = .833 \\
s_4 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = .583 \\
s_5 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = .783 \\
s_6 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = .617
\end{aligned}$$

We make the following observations:

(1) The values follow a high-lower-higher pattern. Specifically, $s_1 > s_2 < s_3$, $s_3 > s_4 < s_5$ i.e. **each odd numbered sum is greater than the next even numbered sum and each even numbered sum is less than the next odd numbered sum.**
(2) $s_1 > s_3 > s_5$ while $s_2 < s_4 < s_6$ i.e. the **odd numbered sums decrease and the even numbered sums increase.**

In Exercise 7, you will prove that these observations are indeed true for all n. Granted them, we may prove the convergence as follows:

Since the even sums increase, either they have a limit or they tend to ∞. They cannot tend to ∞ since for n even

$$s_n < s_{n-1} < s_1$$

since the odd numbered sums decrease. Hence they converge.

Similarly, the odd term must either have a limit or must tend to $-\infty$. They cannot tend to $-\infty$ since, for n odd,

$$s_n > s_{n+1} > s_2$$

Let

$$\begin{aligned}
E &= \lim_{n\to\infty} s_{2n} \\
O &= \lim_{n\to\infty} s_{2n+1}
\end{aligned}$$

be the limits of the even and odd terms.

Since

$$s_{2n+1} = s_{2n} + \frac{1}{2n+1}$$

we see

$$O - E = \lim_{n\to\infty}(s_{2n+1} - s_{2n}) = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

showing that $E = O$.

It now follows from Exercise 9 that $s = \lim_{n\to\infty} s_n$ exists and equals both E and O, showing convergence.

Remark: Since the odds decrease and the evens increase, we see that for n odd

$$s_n \geq s \geq s_{n+1}$$

Letting $n = 5$ we find, for example,$.617 < s < .783$. In general, since s lies between s_n and s_{n+1},

$$|s - s_n| < |s_{n+1} - s_n| = \frac{1}{n+1}$$

Thus, for s_n to approximate s to within $\pm 10^{-2}$, we require $1/(n+1) < 10^{-2}$ which corresponds to 100 terms. Approximating s to within $\pm 10^{-8}$ would require 100,000,000 terms! It turns out that the exact value of the sum is $\ln 2$.

The argument used to analyze the series (2) generalizes to prove the following theorem, which is left as an exercise.

THEOREM 2. *Suppose that a_n is a positive, decreasing sequence where $\lim_{n\to\infty} a_n = 0$. Then*

$$s = \sum_{1}^{\infty} (-1)^n a_n$$

converges. Furthermore

$$|s - s_n| < a_{n+1}$$

The series in Example 1 converges for relatively simple reasons: a_n tends to 0 fast enough for

$$\sum_{1}^{\infty} |a_n|$$

to be finite. Such series are said to be **absolutely** convergent.

The series (2) is not absolutely convergent since

$$\sum_{1}^{\infty} |a_n| = \sum_{1}^{\infty} \frac{1}{n} = \infty$$

Its convergence is due to the cancellation of the pluses and minuses in the summation. A convergent series which is not absolutely convergent is said to be **conditionally convergent.** Conditional convergence is a particularly unpleasant form of convergence. For example, **the commutative law fails for conditionally convergent series** as the next example shows.

EXAMPLE 2. We commented that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \ldots$$

sums to a number less than 1. ($\ln 2$ to be precise.) Show how to rearrange the terms so as to sum to exactly 2.

Solution: Note that neither the odd nor the even numbered a_n have finite sum:

$$1 + \frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{2n+1} + \ldots = \sum_{1}^{\infty} \frac{1}{2n+1} = \infty$$

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{-1}{2n} - \ldots = -\sum_{1}^{\infty} \frac{1}{2n} = -\infty$$

We begin by summing just enough odd numbered a_n to make the sum ≥ 2. We find

$$1 + \frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{17} = 2.02 \geq 2$$

Next, we subtract the first even numbered a_n

$$1 + \frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{17} - \frac{1}{2} = 1.52$$

which brings us below 2. We knew that this would happen because our odd sum can be at most $1/17$ units above 2 and $a_2 = 1/2 > 1/17$.

Next, we add on just enough additional odd terms to make the sum ≥ 2:

$$1 + \frac{1}{3} + \frac{1}{7} + \cdots + \frac{1}{41} - \frac{1}{2} = 2.004$$

We then subtract a_4 which, as before, brings the sum below 2:

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{41}& \\ -\frac{1}{2}-\frac{1}{4}&=1.754 \end{aligned}$$

We keep going:

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{69}& \\ -\frac{1}{2}-\frac{1}{4}&=2.009 \end{aligned}$$

and

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{69}& \\ -\frac{1}{2}-\frac{1}{4}-\frac{1}{6}&=1.842 \end{aligned}$$

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{95}& \\ -\frac{1}{2}-\frac{1}{4}-\frac{1}{6}&=2.0007 \end{aligned}$$

and

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{95}& \\ -\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}&=1.8757 \end{aligned}$$

In each step of this process, we use many of the odd numbered a_n and one of the even. Thus, we eventually use all of the a_n. At the nth step the difference between the sum and 2 is at most the value of the even numbered term that was subtracted. Since these terms tend to zero, this difference can be made less than any given ϵ, showing that the limit is indeed 2.

There is nothing special about 2. *By rearranging the terms of (2) we can make the sum converge to and value we wish!* There is also nothing special about series (2) other than that it is only conditionally convergent: any conditionally convergent series can be arranged to converge to any desired value!

The alternating series test is often used to answer questions such as the following:

EXAMPLE 3. Find all x for which the following series converges. For which x is the convergence absolute?

$$\sum_0^\infty \frac{x^n}{2^n\sqrt{n}}$$

Solution:

$$|a_n| = \frac{1}{\sqrt{n}}\left(\frac{|x|}{2}\right)^n$$

For $|x| > 2$, $\left(\frac{|x|}{2}\right)^n$ exhibits exponential growth which is so much faster than the decay of $1/\sqrt{n}$ that this should tend to ∞ making convergence impossible. In fact, there is a N such that for $n > N$,

$$\left(\frac{|x|}{2}\right)^n > n$$

showing that for such n, $|a_n| > \sqrt{n}$, making convergence impossible.

For $|x| < 2$,

$$\frac{1}{\sqrt{n}}\left(\frac{|x|}{2}\right)^n < \left(\frac{|x|}{2}\right)^n$$

These terms have a finite sum, showing that our original series converges absolutely for $|x| < 2$.

If $|x| = 2$ then $x = \pm 2$. For $x = 2$ our series becomes

$$\sum_0^\infty \frac{1}{\sqrt{n}}$$

which diverges. For $x = -2$ our series becomes

$$\sum_0^\infty \frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test. Hence, the series converges for $x \in [-2, 2)$.

Exercises

(1) Decide which of the following series (a) converge absolutely (b) converge conditionally or (c) diverge. This exercise is to

be done by "inspection." No proofs or reasons required (for the moment).

(a) $$\sum_1^\infty \frac{(-1)^n \ln n}{n}$$

(b) $$\sum_2^\infty \frac{(-1)^n n}{\ln n}$$

(c) $$\sum_1^\infty \frac{(-1)^n n}{n+1}$$

(d) $$\sum_1^\infty \frac{(-1)^n n}{n^2 + \ln n}$$

(e) $$\sum_1^\infty \frac{(-1)^n n}{n^3 + 1}$$

(f) $$\sum_2^\infty \frac{\cos n}{n^2 \ln n}$$

(g) $$\sum_1^\infty \frac{(-1)^n 3^n}{2^n - n^5 + 1}$$

(h) $$\sum_1^\infty \frac{(-1)^n 2^n}{2^n - n^5 + 1}$$

(i) $$\sum_1^\infty \frac{(-1)^n (1.5)^n}{2^n - n^5 + 1}$$

(j) $$\sum_1^\infty \frac{(-1)^n (n^3 - \sqrt{n})}{n^3 + \ln n}$$

(k) $$\sum_1^\infty \frac{(-1)^n (n^3 - \sqrt{n})}{n^{3.001} + \ln n}$$

(2) Prove the answers given in Exercise 1.
(3) For each of the series $\sum a_n$ in Exercise 1, find all x such that the series $\sum a_n x^n$ converges. For which x is the convergence absolute? Prove your answers.

(4) For each of the series $\sum_1^\infty a_n$ in Exercise 3 of Chapter 7, find all x such that the series $\sum_1^\infty a_n x^n$ converges. For which x is the convergence absolute? Prove your answers.

(5) Create 5 *interesting* examples (readers choice) of series which converge only conditionally. Do not use any of the series from Exercise 1 or from the text.

(6) Create 5 *interesting* examples (readers choice) of alternating series which diverge. Do not use any of the series from exercise 1 or from the text.

(7) For the series (2):
   (a) Prove that for n odd, $s_n < s_{n+1}$ *Hint:* $s_{n+1} = s_n + ?$.
   (b) Prove that for n even, $s_n > s_{n+1}$ *Hint:* $s_{n+1} = s_n + ?$.
   (c) Prove that for n odd, $s_n > s_{n+2}$ *Hint:* $s_{n+2} = s_n + ?$.
   (d) Prove that for n even, $s_n < s_{n+2}$ *Hint:* $s_{n+2} = s_n + ?$.

(8) Suppose that for all n $a_n > 0$ and $a_n > a_{n+1}$. Show that (a)-(d) from Exercise 7 hold for $\sum_1^\infty (-1)^{n+1} a_n$.

(9) Let b_n be a sequence such that $\lim_{n\to\infty} b_{2n}$ and $\lim_{n\to\infty} b_{2n+1}$ both exist and are equal. Prove, using ϵ, that $\lim_{n\to\infty} b_n$ exists.

(10) Repeat the discussion from Example 3 to show how to rearrange the series (2) to sum to 1. Do the first three iterations.

(11) Modify the discussion from Example 3 to show how to rearrange the series (2) to sum to -2. Do the first three iterations. *Hint:* A programmable calculator might help in computing the sums.

(12) Suppose that $\sum_0^\infty a_n x^n$ converges absolutely. Prove that $\sum_0^\infty a_n y^n$ converges absolutely for $|y| < |x|$.

(13) Suppose that $\sum_1^\infty a_n$ converges, but not necessarily absolutely. Prove that $\sum_0^\infty a_n x^n$ converges absolutely for $|x| < 1$. *Hint:* From ? (you fill in ?) in Chapter 7, $\lim_{n\to\infty} a_n = 0$. Hence, there is an N such that for $n > N$, $|a_n| < 1$. (Why?)

(14) Suppose that $\sum_0^\infty a_n x^n$ converges, but not necessarily absolutely, where $x \neq 0$. Prove that $\sum_0^\infty a_n y^n$ converges absolutely for $|y| < |x|$. Hint: Apply the preceding exercise with a_n replaced by $a_n x^n$ and x replaced by y/x.

   Remark: It follows from this exercise that if $\sum_0^\infty a_n x^n$ doesn't converge for all x, then there is a number r (the

radius of convergence) such that the series converges absolutely for $|x| < r$ and diverges for $|x| > r$.

(15) Suppose that a_n is a positive, decreasing sequence such that $\sum_0^\infty (-1)^n a_n$ converges conditionally. For which x does $\sum a_n x^n$ converge? For which x does the sum diverge? Explain.

(16) (a) Suppose that there is an N such that for all $n > N$, $|a_n|^{1/n} < \frac{1}{2}$. Prove that $\sum_1^\infty a_n$ converges absolutely.
(b) Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = 1/4$. How does it follow from (a) that $\sum_1^\infty a_n$ converges absolutely.
(c) Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = c < 1$. Prove that $\sum_1^\infty a_n$ converges absolutely.
(d) Use part (c) to prove that

$$\sum_1^\infty \left(\frac{n^3+3}{3n^3-7}\right)^n$$

converges.
(e) Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = c > 1$. Prove that $\lim_{n\to\infty} |a_n| = \infty$, showing that $\sum_1^\infty a_n$ cannot converge.

Remark: The theorem proved in (c) and (e) is called the **root test.**

(17) (a) Suppose that for all $n \geq 0$, $|a_{n+1}| \leq \frac{1}{2}|a_n|$. Prove that $\sum_0^\infty a_n$ converges absolutely. *Hint:* Show that for all $n \geq 0$, $|a_n| \leq \left(\frac{1}{2}\right)^n |a_0|$.
(b) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}$. Prove that $\sum_0^\infty a_n$ converges absolutely.
(c) Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = c < 1$. Prove that $\sum_0^\infty a_n$ converges absolutely.
(d) Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = c > 1$. Prove that $\lim_{n\to\infty} |a_n| = \infty$, showing that the series cannot converge.

Remark: The theorem proved in (c) and (d) is called the **ratio test.**