CHAPTER 9

Irrational Numbers

We are familiar with the equality

$$\frac{1}{3} = .3333333\ldots$$

This is really a statement about an infinite series. Explicitly, it states

$$\begin{aligned}\frac{1}{3} &= \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n} + \ldots \\ &= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{n-1}} + \ldots\right) \\ &= \frac{3}{10}\left(\frac{1}{1 - 1/10}\right) = \frac{1}{3}\end{aligned}$$

where we used the formula for the sum of the geometric series in the last equality.

In general, if a_n is a sequence of digits (i.e. a_n is a sequence of integers between 0 and 9) then we define

$$.a_1a_2a_3\cdots = \sum_1^\infty \frac{a_n}{10^n}$$

This series converges. In fact, for each n

$$s_n = .a_1a_2\ldots a_n < 1$$

Hence, $\lim_{n\to\infty} a_n$ exists from the Bounded-Increasing Theorem. Hence **every infinite decimal you can write represents some specific number.** Thus, for example,

$$a = .1010010001000010000010000001000000010000000001000\ldots$$

represents a real number, where each 1 is followed by one more 0 than the previous 1.

Conversely, every real number is representable as an infinite decimal:

THEOREM 1. *Let x be a positive real number. Then there is a natural number n and sequence of digits a_k such that*

$$x = n.a_1a_2\dots a_k\dots$$

Proof There is a non-negative integer n such that

$$n \le x < n+1.$$

This n is the integer part of the decimal expansion of x. Next, we divide the interval $[n, n+1]$ into 10 subintervals, each of length 1/10. Since x belongs to one of these subintervals, there is an integer $0 \le a_1 \le 9$ such that

$$n.a_1 \le x < n.a_1 + \frac{1}{10}$$

Next, by dividing the interval $[n.a_1, n.a_1 + \frac{1}{10}]$ into 10 subintervals, each of length 1/100, we find that there is an integer a_2 between 0 and 9 such that

$$n.a_1a_2 \le x < n.a_1a_2 + \frac{1}{100}$$

Continuing, we produce a sequence of digits a_n such that

$$n.a_1a_2\dots a_n \le x < n.a_1a_2\dots a_n + \frac{1}{10^n}$$

To prove that this sequence converges to x, let $\epsilon > 0$ be given. Then

$$0 \le x - n.a_1a_2\dots a_n < \frac{1}{10^n}$$

$$|x - n.a_1a_2\dots a_n| < \frac{1}{10^n}$$

This will be less than ϵ provided $10^{-n} < \epsilon$ which is true if $n > N$ where $N = -\ln\epsilon/\ln 10$, proving our theorem.

A constant theme in our studies has been that, as much as possible, we should only use the axioms or their consequences in our proofs. The preceding proof used the fact that for every positive real number x, there is a natural number n such that $x < n+1$. This is a version of what is usually referred to as the Archimedean Property. Since it is not one of our axioms, we should either prove it or assume it as another axiom. Remarkably, it is a consequence of the Least Upper Bound Axiom. We have, in fact, already used this property several times without comment. For example, in our solution to Example 1 on page 84, we stated at one point "if $n > \frac{1}{\epsilon} - 1$, then

$1-\epsilon < \frac{n}{n+1}$ showing that $1-\epsilon$ is not an upper bound for S." It is the Archimedean Property that guarantees the existence of such n. We leave it as an exercise (Exercise 16 on page 150 below) to explain how this property follows from the GLB axiom.

Theorem 2. *For all numbers M there is a non-negative integer n such that*

$$n \le M < n+1.$$

A decimal expansion is **finite** if eventually all of the a_n are zero. Thus, $1 = 1.0$ and 2.74 are finite. Every non-zero number that has a finite expansion is also has an infinite expansion. We claim, for example, that

$$1.0 = .99999\ldots$$

In fact,

$$\begin{aligned} .9999\ldots &= \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \ldots \\ &= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{n-1}} + \ldots\right) \\ &= \frac{9}{10}\left(\frac{1}{1-1/10}\right) = 1 \end{aligned}$$

where we used the formula for the geometric series in the last equality.

Remark: Many students find the equality

$$1 = .99999\ldots \tag{1}$$

confusing. They argue that no matter how many 9's we write after the decimal, we will never reach 1. Hence, the best we can say is that

$$1 \approx .9999\ldots$$

This is akin to saying that we should write

$$1 \approx \lim_{n\to\infty} \frac{n}{n+1}$$

instead of

$$1 = \lim_{n\to\infty} \frac{n}{n+1}$$

since the values of $n/(n+1)$ never actually equal 1. The explanation is that the limit refers to the number being approximated and not to the numbers that are doing the approximating. The preceding equality means that 1 is the unique number that the fractions $n/(n+$

1) approximate as n gets large. Similarly, the equality (1) means that 1 is the unique number that we approximate as we write more and more 9's after the decimal.

Dividing both sides of equation (1) by powers of 10 shows that:

$$.1 = .09999\dots$$
$$.01 = .009999\dots$$

and so forth. Hence

$$\begin{aligned} 2.74 &= 2.73 + .01 \\ &= 2.73 + .0099999\dots \quad = 2.7399999\dots \end{aligned}$$

This example illustrates the following general principle: *every number which has a finite decimal representation will also have an infinite decimal representation.* It is a theorem (which we will not prove) that **the numbers that have finite expansions are the only ones which can have two expansions.** For all other numbers the numbers a_n are unique.

In general, when we write a bar over a sequence of digits, we mean that the sequence repeats forever. For example

$$2.7\overline{36} = 2.73636363636\dots.$$

We say that such an expansion is a **repeating** expansion.

EXAMPLE 1. Express $2.7\overline{36}$ as a fraction.

Solution: We note that

$$\begin{aligned} 2.73636363636\dots &= 2.7 + .03636363636\dots \\ &= 2.7 + \frac{1}{10}.3636363636\dots \end{aligned}$$

Furthermore,

$$\begin{aligned} .\overline{36} &= \frac{36}{100} + \frac{36}{100^2} + \dots + \frac{36}{100^n} + \dots \\ &= \frac{36}{100}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \dots + \frac{1}{100^{n-1}} + \dots\right) \\ &= \frac{36}{100}\,\frac{1}{(99/100)} = \frac{4}{11} \end{aligned}$$

Hence, our answer is

$$\frac{27}{10} + \frac{1}{10}\frac{4}{11} = \frac{301}{110}$$

Recall that a number x is said to be **rational** if $x = p/q$ where p and q are integers with $q \neq 0$. For example $3/7$, $4 = 4/1$, and $-5/3$ are all rational. A number that is not rational is **irrational**. Examples include π, e and $\sqrt{2}$.

The general repeating decimal may be converted into a fraction using the same technique as in the preceding example. The result will be rational due to the observation that if x is rational, then

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

is also rational. Hence **every repeating expansion represents a rational number.** Conversely, **the expansion of a rational number must repeat.** (The exercises explore why.)

Remark: Students often describe irrational numbers as "those numbers whose decimal expansions go on forever." This is incorrect. The expansion $2.736363636\ldots$ "goes on forever" and yet represents the rational number: $\frac{4}{11}$. The proper statement is that an irrational number is a number whose decimal expansion does not repeat. It should be noted that numbers with finite expansions do repeat. For example $2.7 = 2.7\overline{0}$.

One very important irrational number is $\sqrt{2}$. Here is a proof of its irrationality.

THEOREM 3. *The number $\sqrt{2}$ is irrational.*

Proof If $\sqrt{2}$ were rational, then there would be integers p and q such that $\sqrt{2} = p/q$. We assume that the fraction p/q is in lowest terms so that p and q have no common factors.

Note that

$$2 = (\sqrt{2})^2 = \frac{p^2}{q^2}$$

Hence

$$p^2 = 2q^2.$$

This makes p^2 even. Hence p must be even, since the square of an odd number would be odd. Thus $p = 2k$ where k is an integer. Substituting:

$$4k^2 = 2q^2$$
$$2k^2 = q^2.$$

Thus, q must be even also. But this contradicts the assumption that p and q have no common factors, proving our theorem.

Remark: The irrationality of $\sqrt{2}$ was discovered by the early Greek mathematicians who proved that the length of the hypotenuse of an isosoles right triangle was an irrational multiple of the side lengths. This actually precipitated a crisis in Greek mathematics. Much of Greek mathematics, including the theorems about parallel lines cutting transversals, was based on an assumption (called commensurability) which was equivalent to the statement that only rational numbers existed. When this was found to be wrong, all of mathematics seemed to be coming apart at the seams. For many Greek mathematicians, this was more than a loss of a career. It was also a failure of their religion, since a number of them felt that mathematics was the medium through which God spoke to them. It is reported that some even committed suicide! Eventually, the crisis was resolved by Eudoxus who said, in essence, that between any two real numbers there is a rational number. With this additional axiom, they were able to correct the proofs of their theorems.

Once we know one irrational number, we can produce as many as we want.

EXAMPLE 2. Prove that $x = \frac{2}{3} + \frac{4}{5}\sqrt{2}$ is irrational.

Solution: We work by contradiction, showing that assuming that x is rational leads to nonsense. Specifically, suppose that there are integers p and q such that

$$\frac{p}{q} = \frac{2}{3} + \frac{4}{5}\sqrt{2}$$

We solve the above equality for $\sqrt{2}$:

$$\begin{aligned} \frac{p}{q} - \frac{2}{3} &= \frac{4}{5}\sqrt{2} \\ \frac{5}{4}\,\frac{3p-2q}{3q} &= \sqrt{2} \\ \frac{15p-10q}{12q} &= \sqrt{2} \end{aligned}$$

Since p and q are integers, $15p - 10q$ and $12q$ are both integers. Thus, $\sqrt{2}$ is a ratio of integers, which is nonsense since $\sqrt{2}$ is irrational.

The argument just described generalizes to the following result which you will prove in the exercises:

PROPOSITION 1. *Let Z be an irrational number and let x and y be rational numbers with $x \neq 0$. Then $xZ + y$ is irrational.*

There are so many irrational numbers that *there is an irrational number between every pair of rational numbers.*

THEOREM 4. *Let x and y be rational numbers with $x < y$. Then there is are an infinite number of irrational numbers z satisfying $x < z < y$.*

Proof Since $\lim_{n\to\infty} \sqrt{2}/n = 0$, there is an N such that

$$\frac{\sqrt{2}}{n} < y - x$$

for all $n > N$. For such n

$$x < \frac{\sqrt{2}}{n} + x < y$$

From Theorem 1, $\frac{\sqrt{2}}{n} + x$ is irrational. This proves the theorem.

It is also true that between any two irrational numbers there are an infinite number of rational numbers.

THEOREM 5. *Let x and y be irrational numbers with $x < y$. Then there is are an infinite number of rational numbers z satisfying $x < z < y$.*

Proof We may write

$$y = M + .a_1a_2a_3\dots$$

where M is an integer and, of course, $.a_1a_2a_3\dots$ is a decimal. Then

$$y = \lim_{n\to\infty}(M + .a_1a_2\dots a_n).$$

Since $y > x$, there is an N such that for all $n > N$

$$M + .a_1a_2\dots a_n > x$$

each of the numbers on the left side of the inequality is rational and lies between x and y, proving our theorem.

There are an infinite number of both irrational and of rational numbers. However, there is a very real sense in which the set of irrationals is vastly larger than the set of rationals. Below, we have begun a list which will eventually include all all rational numbers in the interval $(0, 1)$:

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{1}{4} \quad \frac{3}{4} \quad \frac{1}{5} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{4}{5} \quad \frac{1}{6} \quad \frac{5}{6} \quad \cdots$$

The pattern here is that we list fractions by increasing value of the denominator. For a given value of denominator, we go from smallest to largest, omitting fractions which are not in reduced form. Let us think of the elements of this sequence as being expressed as decimals where we use the finite expansion whenever we have a choice:

$$\begin{aligned} a_1 &= .33333333\ldots \\ a_2 &= .66666666\ldots \\ a_3 &= .50000000\ldots \\ a_4 &= .75000000\ldots \\ a_5 &= .20000000\ldots \\ &\cdots \end{aligned} \tag{2}$$

On the other hand, listing the irrational numbers in $(0, 1)$ is, we claim, quite impossible. To see this, let us imagine that we have somehow managed to list all irrationals in this interval. Our list might look something like:

$$\begin{aligned} b_1 &= .31415027\ldots \\ b_2 &= .14936815\ldots \\ b_3 &= .22719664\ldots \\ b_4 &= .97652234\ldots \\ b_5 &= .62718891\ldots \\ &\vdots \end{aligned}$$

We imagine the decimal expansions extending out to infinity and the list extending down the page to infinity. We claim that no matter what the specific numbers in the list, there will always be some irrational number r which is not in the list. To see this, look at the first digit of the first number in the list. In our case, it is a three. We choose some number between 0 and 9, other than 3, and make it be the first digit of r. Lets choose 4, so $r = .4+$. This insures that $r \neq b_1$. Next, we look at the second digit of the second number on the list: 4. We change it, declaring, say, $r = .47+$. This guarantees that r is also not equal to b_2. We continue this way, at each step choosing the n^{th} digit of r to be some some number between 0 and 9 which differs from the n^{th} digit of b_n. For the list above, r might look like $r = .47647+$. It is clear that in this manner we produce a number r which appears nowhere on our list.

We also must be careful that the number r we produce is not rational, since we claimed that there is an *irrational* number not on the list. For this, all we need do is to guarantee that r does not appear on the list of rationals on page 140. This is easily accomplished; we simply select the nth digit of r so that it also differs from the n^{th} digit of a_n. We also want to avoid selecting a 9. This is to avoid the problem of non-uniqueness of representation of rational numbers. Specifically, the number $.27999\ldots$ appears on the list of rationals as 2.28. If we were to choose $r = .279999\ldots$, then r appears on the list of rational numbers, even though its expansion is different from all of the listed expansions. Once this is done, r appears on neither list; hence our list has excluded at least one irrational. In fact, by making different selections of the digits, we prove that our list excludes an infinite number of irrationals.

Sets of numbers which may be listed are called **countable**. Those which cannot are called **uncountable**. There is a very close relationship between these notions and the notion of counting. Consider, for example, what happens if you ask a kindergartner how many objects there are in the set $S = \{+, *, \times\}$. She might first hold up one finger, then another and finally a third. She would then say 'T'ree'. What she has done is set up a correspondence between the elements of the set S and the first three fingers on her hand:

$$\begin{aligned} \text{First Finger} &\to + \\ \text{Second Finger} &\to * \\ \text{Third Finger} &\to \times \end{aligned}$$

Her correspondence is one-to-one; each finger corresponds to a different symbol. Her correspondence is 'onto'; she hasn't left out any symbols. For her, a set has three objects if she can find correspondence between the first three fingers on her hand and the objects of the set which is both one-to-one and onto.

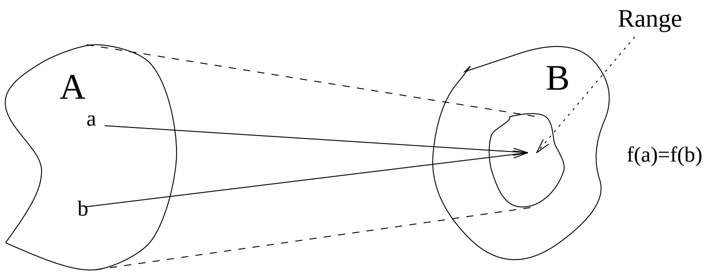
When we listed the rational numbers in the interval $(0, 1)$, we set up a correspondence between the natural numbers and the rationals:

$$\begin{aligned}
1 &\to \frac{1}{2}\\
2 &\to \frac{1}{3}\\
3 &\to \frac{2}{3}\\
4 &\to \frac{1}{4}\\
5 &\to \frac{3}{4}\\
6 &\to \frac{1}{5}\\
7 &\to \frac{2}{5}\\
8 &\to \frac{3}{5}\\
&\cdots
\end{aligned}$$

It is as if we had a hand with an infinite number of fingers (one for each natural number) and we were using our infinite number of fingers to count the rationals in $(0, 1)$. Notice that each of our 'fingers' corresponds to a different rational number. (This is why we omitted non-reduced fractions from our list.) Thus our correspondence is one-to-one. Each rational in the interval eventually gets counted. Thus our correspondence is onto. In the case of the real numbers, our infinite number of fingers was not enough to get the job done. No matter how we try, we always have uncounted real numbers. It is as if we asked our kindergartner to count a set with six objects. She might reply "I can't do it. Don't got enough fingers on my hand."

Let us raise the level of the discussion a little. What, mathematically, do we mean by a "correspondence?" We know that a sequence is just a function whose domain is the set of natural numbers. Thus, our listing of the rational numbers is just a function whose domain is the natural numbers and whose range is contained in the set of rational numbers.

In general, a function f from a set A to a set B is a correspondence between points of A and points of B with the property that to each point a of A there corresponds a well defined point $f(a)$ of B. In elementary calculus, the set B is almost always a set of numbers, although, in general, it can be any set. The set A is called the domain of the function. The range of f is the set of points $f(a)$. It is often denoted by the symbol $f(A)$. The function is said to map A *onto* B if $B = f(A)$. In counting, this means that each point of B gets counted. The function is said to be one-to-one if for all y in the range of f, there is only one x in the domain such that $f(x) = y$–i.e. if $f(a) = f(b)$, then $a = b$. In terms of counting, this means that each point of B is counted only once. The function f is said to be a **one-to-one correspondence** if it is both one-to-one and onto.



Not one-to-one, not onto.

Thus, we arrive at the following definition:

Definition 1. *An infinite set B is* **countable** *if there is a one-to-one function f whose domain is the set $\mathbb{N}$ of natural numbers and whose range is B. All finite sets are also countable.*

More generally, if A and B are two sets, then we say that A has the same **cardinality** *as B if there is a one-to-one function f whose domain is A and whose range is B.*

Example 3. Prove that the intervals $A = (-1, 1)$ and $B = (1, 5)$ have the same cardinality by finding an explicit one-to-one correspondence. Prove your answer.

Solution: We hope for a correspondence of the form $y = ax + b$. We assume that the points correspond somewhat as in the diagram below.

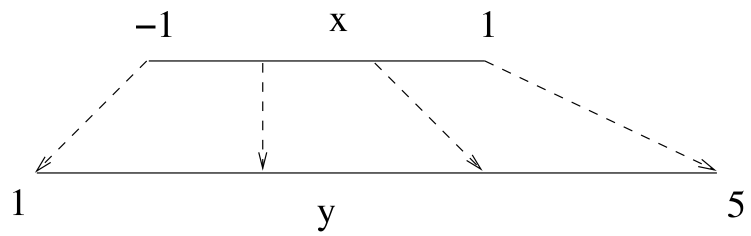


FIGURE 1. Corresponding intervals

Then $f(-1) = 1$ and $f(1) = 5$. Hence

$$\begin{aligned} 1 &= a(-1) + b \\ 5 &= a + b \end{aligned}$$

Solving this pair of equations yields $a = 2$ and $b = 3$; hence the correspondence is

$$y = 2x + 3 \tag{3}$$

Proof:

Into: We first show that $f(x)$ maps A into B. Assume $x \in A$. Then

$$\begin{aligned} -1 &< x < 1 \\ -2 &< 2x < 2 \\ 1 &< 2x + 3 < 5 \\ 1 &< y < 5 \end{aligned}$$

Hence f maps $(-1, 1)$ *into* $(-1, 5)$.

Onto:

Reversing the preceding argument shows that if $-1 < y < 5$ then $x \in (-2, 1)$. Hence f maps $(-2, 1)$ *onto* $(-1, 5)$. *3 pt.*

One-to-one: Suppose that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 2x_1 + 3 &= 2x_1 + 3 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

showing one-to-one.

In solving the preceding example we could have use B as the domain of our correspondence and A as the range. Hence, instead of corresponding x to y in Figure 1, we correspond y to x. We obtain a formula defining such a correspondence by solving equation (3) on page 144 for x:

$$x = \frac{1}{2}(y-3). \tag{4}$$

In general, if $f : A \mapsto B$ is one-to-one and onto, then for every $y \in B$, there is one, and only one, $x \in A$ such that $f(y) = x$. We write $x = f^{-1}(y)$. The function $f^{-1} : B \mapsto A$ is the *inverse* of $f(x)$. Thus, formula (4) says that the function $g(y) = \frac{1}{2}(y-3)$ is the inverse of the function $f(x) = 2x + 3$.

Remark. Two functions f and g are said to be equal if they have the same domain, the same range, and $f(x) = g(x)$ for all x in the domain. The particular symbols we use to denote the elements of the domain and range have no intrinsic meaning. Thus, the following formulas both define the inverse to the function defined by formula (3) on page 144, provided that it is understood that the domainand range are, respectively, $(1, 5)$ and $(-1, 1)$:

$$\begin{aligned} g(y) &= \frac{1}{2}(y-3) \\ g(x) &= \frac{1}{2}(x-3). \end{aligned}$$

The following proposition tells us that if A has the same cardinality as B, then B has the same cardinality as A. Its proof is an exercise. (Exercise 24 on page 138.)

Proposition 2. *Suppose that $f : A \mapsto B$ is one-to-one and onto. Then $f^{-1} : B \mapsto A$ is one-to-one and onto.*

Remarkably, when we get to infinite sets, this notion of size becomes very unintuitive. For example, it is easy to define a one-to-one correspondence between the natural numbers and the even natural numbers. We simply define $f(n) = 2n$. Hence,

$$\begin{aligned} 1 &\to 2 \\ 2 &\to 4 \\ 3 &\to 6 \\ 4 &\to 8 \\ &\dots \end{aligned}$$

Thus, the set of all natural numbers has the same cardinality as the set of all even natural numbers, despite the fact that the even numbers are a proper subset of the set of all natural numbers!

This causes many people immense difficulty. They absolutely refuse to accept that the set of even natural numbers could be the same size as the set of all natural numbers. They are not questioning our mathematics; the fact that we have a one-to-one function whose domain is $\mathbb{N}$ and whose range is the even natural numbers is clear. What they are objecting to is our *interpretation* of our mathematics. They are telling us that they cannot accept any notion of size which allows a set S to have the same size as one of its proper subsets. This is O.K. If the term 'size' bothers you, call it 'one-to-one correspondence'.

Remark: The notion of cardinality was introduced by the mathematician Georg Cantor in 1874. At the time, it was quite controversial. Eventually, however, the notion was found to be very useful and gained wide acceptance.

Exercises

(1) Use an infinite series to express the following as fractions:

$$\begin{aligned} &(a) && .\overline{68} \\ &(b) && 2.74\overline{68} \\ &(c) && .\overline{7324} \end{aligned}$$

(2) Let $x = .\overline{68}$. Use the decimal expansion to explain why $100x = 68 + x$. Use this information to express x as a fraction.

(3) Express $2.74\overline{68}$ as a fraction using the technique of Exercise 2.

(4) Express $47.56\overline{789}$ as a fraction using the technique of Exercise 2.
(5) Use the technique of Exercise 2 to prove that $.\overline{9} = 1$.
(6) If we compute 4/13 on a hand held calculator with an 8 digit display, we get

$$\frac{4}{13} \approx .30769231$$

which certainly does not seem to repeat. Our calculator is not capable of displaying enough digits for us to see the repetition. There is, however, a cleaver 'trick' for getting our calculator to produce as many digits as we want. We first multiply the above equation by 10000000 getting

$$\frac{40000000}{13} \approx 3076923.1. \tag{5}$$

The decimals for this number are the same as those for 4/13, just shifted over seven places.

Formula (5) tells us that 13 goes into 40000000, 3076923 times with some remainder. To find the remainder, we compute (using our calculator!)

$$40000000 - 13 \cdot 3076923 = 1.$$

Hence

$$40000000/13 = 3076923 + 1/13.$$

Now, comes the main point: 1/13 represents the part of formula (5) which is after the decimal point. The decimal expansion of 1/13 will yield the 8 digits of the expansion of 4/13 after .3076923. Using our calculator we find

$$\frac{1}{13} = .07692308. \tag{6}$$

Hence

$$4/13 \approx .307692307692308.$$

It seems clear that what we are obtaining is $.\overline{307692}$. Just to be sure, though, let's compute another batch of digits. We multiply equation (6) by 10000000 getting

$$\frac{10000000}{13} = 769230.8.$$

The remainder is

$$10000000 - 13 \cdot 769230 = 10$$

Hence

$$\frac{100000000}{13} = 76923076 + \frac{10}{13}.$$

The decimal expansion for 10/13 is .76923077 which yields the next 8 digits of 1/13; hence the next 8 digits for 4/13. Thus

$$4/13 \approx .3076923076923076923077.$$

This certainly seems to confirm the repetition. Of course, if we wish to be *absolutely* certain, we could use the technique of Exercise 2 to express this repeating decimal as a fraction and see if we really get 4/13.

Finally, we are ready for the problem!

(a) Use the calculator technique described above to compute the decimal expansion of 7/17 until it starts repeating.
(b) Using long division, compute the decimal expansion of 3/7 until it repeats. Explain how you know that it will keep repeating. Explain why the repetition cycle in $p/7$ is at most 7 for any natural number p.
(c) More generally, explain why the repetition cycle in a/n is at most n where a and n are natural numbers.

(7) Prove that (a) the square of an even number is even and (b) the square of an odd number is odd. *Hint:* An integer n is even if $n = 2k$ for some integer k. It is odd if $n = 2k + 1$ for some integer k.

(8) The concepts of even and odd group all integers into two classes: those of the form $2k$ and those of the form $2k + 1$. We can also divide the set of integers into 3 classes: those of the form $3k$ (class 0), those of the form $3k + 1$ (class 1) and those of the form $3k + 2$ (class 2).
(a) Determine the class of each of the following: 0,1,2,3,4,5,6,7,8,9,10,11.
(b) Prove that the square of a class 0 number is class 0.
(c) Prove that the square of a class 1 number is class 1.
(d) Prove that the square of a class 2 number is class 1.

Note that there are no numbers whose square is class 2. Hence a class 2 number can never be a perfect square.

(9) This exercise in a continuation of Exercise 8.
  (a) Prove that if p in an integer and p^2 is class 0, then p is class 0. (*Hint:* Explain why p cannot be class 1 or 2.) It follows that $p = 3k$ for some integer k.
  (b) Prove that $\sqrt{3}$ is irrational. *Hint:* Repeat the proof of Theorem 3, using class instead of even and odd.

(10) Let x and y be rational numbers. Prove that the following numbers are all rational. You may assume that the product and sum of two integers is an integer. (In part (a) we also assume $y \neq 0$ to avoid division by 0.) We prove (a) as an example.

$$\begin{array}{ll} (a) & x/y \\ (b) & x+y \\ (c) & x-y \\ (d) & xy \end{array}$$

Proof of (a) Since x and y are rational, there are integers a, b, c, and d such that $x = a/b$ and $y = c/d$. Then

$$x/y = \frac{a/b}{c/d} = \frac{a}{b}\,\frac{d}{c} = \frac{ad}{bc}$$

This represents a rational number since both ad and bc are integers.

(11) Using the technique of Example 2 on page 138, prove that the following numbers are irrational. **You may not use Proposition 1.** You may, however, assume that π and $\sqrt{3}$ are irrational. You may also assume that the sum, product, and quotient of any two rational numbers is rational.
  (a) $\frac{4}{5}\pi$
  (b) $\frac{4}{9} - \frac{17}{11}\sqrt{3}$
  (c) $\frac{1}{3-5\sqrt{2}}$
  (d) $y+x\sqrt{2}$ where x and y are rational numbers with $x \neq 0$.
  (e) $y + xZ$ where x and y are rational numbers, $x \neq 0$ and Z is an irrational number.
  (f) $\sqrt{2} + \sqrt{3}$ *Hint* Let $x = \sqrt{2} + \sqrt{3}$ and assume that x is rational. Ust the rationality of x^2 to prove that $\sqrt{6}$

is rational. Hence both x and $x\sqrt{6} = 2\sqrt{3} + 3\sqrt{2}$ are rational. So?

(12) Only one of the following "theorems" is true. Prove the true theorem and find an example showing the falsehood of each of the others.
  (a) The sum of two irrational numbers is irrational.
  (b) The product of two irrational numbers is irrational.
  (c) If $x+y$ is irrational, then either x or y must be irrational.

(13) (a) Find a rational number between $\sqrt{101}$ and $\sqrt{102}$. Express your answer as a fraction. (*Hint:* Use the decimal expansions.)
  (b) Find a rational number between $\ln 101$ and $\ln 102$. Express your answer as a fraction.

(14) Find an explicit irrational number between $\frac{517}{578}$ and $\frac{519}{577}$. *Hint:* Look at the proof of Theorem 4.

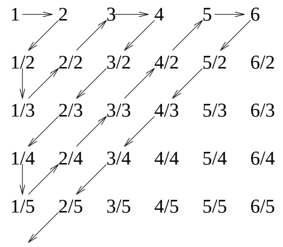
(15) Find an explicit irrational number between $\frac{3166}{4799}$ and $\frac{3165}{4797}$. *Hint:* Look at the proof of Theorem 4.

(16) (a) Prove that for each $M \in \mathbb{R}$, $M \geq 0$, there is an $m \in \mathbb{N}$ such that $m \geq M$. *Hint:* Assume that this is false. Explain how it follows that the set of natural numbers is bounded from above. Let $s = \sup \mathbb{N}$. Then there is a natural number n satisfying $s - .5 < n \leq s$. (Explain!) What does this say about $n + 1$?
  (b) Let $x = \inf\{n \in \mathbb{N} \mid n \geq M\}$. Then there is an $n \in \mathbb{N}$ such that $x \leq n < x+.5$. (Why?) Hence $n-1 < x$. How does it follow that $n - 1 < M$. How does Theorem 2 on page 135 follow?

(17) Find a one-to-one correspondence between the set of natural numbers and the set of multiples of 3.

(18) Describe a listing of the integers. This shows that the set of integers has the same cardinality as the set of natural numbers.

(19) The picture below describes a method of listing *all* positive rational numbers. We simply follow the indicated path, listing each rational number in the order we hit it. Let us call the resulting sequence a_n. Thus, $a_1 = 1$, $a_2 = 2$, $a_3 = 1/2$, $a_4 = 1/3$, etc.



Counting the rationals

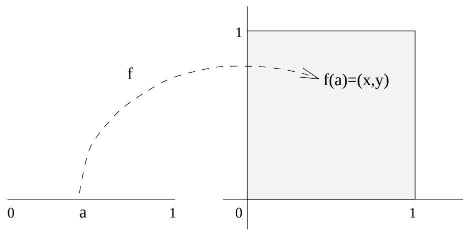
(a) What is the first n such that $a_n = 5/3$?
(b) This is not a one-to-one correspondence. Demonstrate this by finding values of n and m with $n \neq m$ such that $a_n = a_m = 2/3$.
(c) We may obtain a one-to-one correspondence by omitting all non-reduced fractions. What is the 14^{th} fraction in this reduced list?
(d) Describe a listing of the set of all rational numbers, including negatives and 0.

(20) The Hotel Infinity has an infinite number of rooms numbered 1,2,3,... All rooms are occupied. A guest comes in and asks for a room. You respond, "No problem. We'll just move each guest over one room, so that the guest in room 1, moves to room 2, the one in room 2 moves to room 3, etc., leaving room 1 free for you."
(a) Next a bus with 10 people shows up. How do you accommodate them?
(b) A bus with an infinite (but countable) number of guests shows up. How do you accommodate them?
(c) Now (HELP!) an infinite (but countable) number of buses, each with an infinite number (but countable) of guests shows up. You can still accommodate all of them. How?

(21) You might think that large intervals contain more points than small intervals. Not so. Find an explicit one-to-one correspondence between the given intervals.
(a) $(1, 2)$ and $(3, 7)$. (Hint: Try a function of the form $f(x) = ax + b$.)
(b) $(0, 1)$ and $(1, \infty)$
(c) $(0, 1)$ and $(0, \infty)$
(d) $(-\infty, \infty)$ and $(-1, 1)$.

(22) Find a one-to-one correspondence between $[0, 1]$ and $(0, 1)$. *Hint:* Using lists similar to that on p. 140, describe a one-to-one correspondence between the rational numbers in $[0, 1]$ and those in $(0, 1)$. What is left?

(23) Well surely there are more points in a square than in an interval. NOT SO! Here is a function f whose domain is the interval $[0, 1]$ and whose range is the square $S = \{(x, y) \,|0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.



f maps the interval onto the square!

Let $a \in [0, 1]$. Write a as a decimal as $a = .a_1a_2a_3 \ldots$. We shall stipulate that, for the purposes of computing our function, if a has two expansions, then we shall always choose the expansion with an infinite number of 9's. Thus, if $a = .27$, we choose the expansion $a = .26\overline{9}$.

We define

$$f(a) = (x, y)$$

where

$$x = .a_1a_3a_5 \ldots a_{2n+1} \ldots$$

and

$$y = .a_2a_4a_6 \ldots a_{2n} \ldots$$

(Here, (x, y) is a *point* in $\mathbb{R}^2$, not an open interval in $\mathbb{R}$!) Thus, for example

$$\begin{aligned} f(7/55) &= f(.1272727\ldots) \\ &= (.1777\ldots, .222\ldots) \\ &= (8/45, 2/9) \\ f(.27) &= f(.26999\ldots) \\ &= (.2999, .6999) \\ &= (.3, .7) = (3/10, 7/10) \end{aligned}$$

(a) Compute $f(1/n)$ for $1 \leq n \leq 6$. Express the answer in fractional form. Remember to use the expansion with an infinite number of 9's if you have a choice!

(b) Find a values of a and b such that $f(a) = (1/3, 1/4)$ and $f(b) = (1, 1/3)$. Remember to use the expansion with an infinite number of 9's if you have a choice!

(c) Prove that the range of f is the whole square.

(24) Prove Proposition 2 on page 145.