MODIFIED HILBERT AXIOMS

Note: We will *assume* elementary facts from set theory involving set membership, unions, and intersections.

Undefined terms: Point, line.

Incidence Axiom 0 A line is a set of points.

Definition A point P is said to be *on* a line ℓ if it is an element of ℓ.

Incidence Axiom 1 If P and Q are two different points then there is exactly one line ℓ with both P and Q on it.

Incidence Axiom 2 For every line ℓ there are at least 2 points on it.

Incidence Axiom 3 For every line ℓ there is at least one point not on it.

Undefined term: $A * B * C$ (read this as "B is between A and C.")

Betweenness Axiom 1 If $A * B * C$ is true then A, B, and C are three different points all on the same line, and $C * B * A$ is also true.

Betweenness Axiom 2 If B and D are two different points, then there are points A, C and E lying on line $\overleftrightarrow{BD}$ such that the statements $A * B * D$, $B * C * D$, and $B * D * E$ are all true.

Betweenness Axiom 3 If A, B, and C are three different points on a line ℓ then exactly one of the statements $A * B * C$, $A * C * B$, and $B * A * C$ is true.

Definition Given two different points A and B we define the *segment* AB to be the set consisting of A, B, and all points C for which $A * C * B$ is true.

Definition Let ℓ be any line, and let A and B be any points that do not lie on ℓ. We say that "A and B are *on opposite sides of* ℓ" if the segment AB intersects ℓ. Otherwise we say that "A and B are *on the same side of* ℓ."

Betweenness Axiom 4 Let ℓ be any line and A, B and C any three points not on ℓ.

- **(i)** If A and B are on the same side of ℓ, and B and C are also on the same side of ℓ, then A and C are on the same side of ℓ.

(ii) If A and B are on opposite sides of ℓ, and B and C are also on opposite sides of ℓ, then A and C are on the same side of ℓ.

Definition If A and B are two different points then the ray $\overrightarrow{AB}$ is the set consisting of A and B and all points C for which either $A * C * B$ or $A * B * C$ is true.

Definition If A, B and C are three points which are not all on the same line then $\angle ABC$ is the union of the two sets $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Undefined terms: Congruence of segments and of angles.

Congruence Axiom 1 If C is a point on ray $\overrightarrow{AB}$ with $AB \cong AC$ then $B = C$.

Congruence Axiom 2 Every segment is congruent to itself. If $AB \cong CD$ then $CD \cong AB$. If $AB \cong CD$ and $CD \cong EF$ then $AB \cong EF$.

Congruence Axiom 3 Suppose that the statements $A * B * C$, $A' * B' * C'$ are both true.

(i) If $AB \cong A'B'$ and $BC \cong B'C'$ then $AC \cong A'C'$.

(ii) If $AB \cong A'B'$ and $AC \cong A'C'$ then $BC \cong B'C'$.

Congruence Axiom 4 Given $\angle ABC$, if D is a point on the same side of $\overleftrightarrow{AB}$ as C, and if $\angle ABC \cong \angle ABD$, then D is on $\overrightarrow{BC}$.

Congruence Axiom 5 Every angle is congruent to itself. If $\angle AB \cong \angle CD$ then $\angle CD \cong \angle AB$. If $\angle AB \cong \angle CD$ and $\angle CD \cong \angle EF$ then $\angle AB \cong \angle EF$.

Definition $\angle ABC$ is a *right* angle if there is a point D on line $\overleftrightarrow{AB}$ with $A * B * D$ and $\angle ABC \cong \angle DBC$.

Congruence Axiom 6 If two angles are both right angles, then they are congruent.

Definition If A, B, and C are three points not all on the same line, then $\triangle ABC$ is the union of the segments AB, AC, and BC.

Definition $\triangle ABC \cong \triangle DEF$ means that all of the following statements are true: $AB \cong DE$, $AC \cong DF$, $BC \cong EF$, $\angle ABC \cong \angle DEF$, $\angle ACB \cong \angle DFE$, and $\angle BAC \cong \angle EDF$.

Congruence Axiom 7 If $AB \cong DE$, $AC \cong DF$, and $\angle BAC \cong \angle EDF$ then $\triangle ABC \cong \triangle DEF$.

Definition Given two different points A and B, the *circle with center A passing through B* is the set of all points C with $AC \cong AB$. We will write $B(A)$ for this circle.

Definition A point D is *inside* circle $B(A)$ if either $D = A$ or there is a point C on the circle with $A * D * C$. A point D is *outside* circle $B(A)$ if there is a point C on the circle with $A * C * D$.

Intersection Axiom 1 If D is inside circle $B(A)$ then any ray starting at D intersects the circle at exactly one point.

Intersection Axiom 2 If a circle has one point inside and one point outside another circle, then the two circles intersect.

Definition Two lines are parallel if they do not intersect.

Intersection Axiom 3 If two lines are parallel then any line which intersects one must also intersect the other.