Supplementary Problems

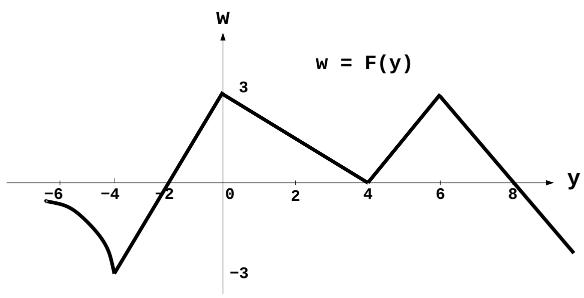
A For what value(s), if any, of A will $y = Axe^{-2x}$ be a solution of the differential equation $2y' + 4y = e^{-2x}$? For what value(s), if any, of B will $y = Be^{-2x}$ be a solution ?

B Using the substitution $u(x) = y + x$, solve the differential equation $\dfrac{dy}{dx} = (y + x)^2$.

C Using the substitution $u(x) = y^3$, solve the differential equation $y^2 \dfrac{dy}{dx} + \dfrac{y^3}{x} = \dfrac{2}{x^2} \quad (x > 0)$.

D Find the explicit solution of the Separable Equation $\dfrac{dy}{dt} = y^2 - 4y, \quad y(0) = 8$. What is the largest open interval containing $t = 0$ for which the solution is defined ?

E The graph of $F(y)$ vs y is as shown:



(a) Find the equilibrium solutions of the autonomous differential equation $\dfrac{dy}{dt} = F(y)$.

(b) Determine the stability of each equilibrium solution.

F Solve the differential equation $\dfrac{ds}{dt} = \dfrac{2ts}{s^2 - t^2}$.

G (a) If $y' = -2y + e^{-t}, \; y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (**eul**) with step sizes of the form $h = \dfrac{1}{n}$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05 .

H (a) If $y' = 2y - 3\,e^{-t}, \; y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (**eul**) with step sizes of the form $h = \dfrac{1}{n}$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05 .

I Approximation methods for differential equations can be used to estimate definite integrals:

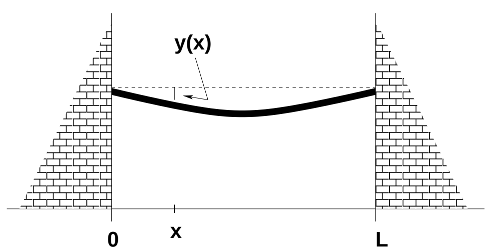
(a) Show that $y(t) = \int_0^t e^{-u^2}\,du$ satisfies the initial value problem $\dfrac{dy}{dt} = e^{-t^2}, \quad y(0) = 0$.

(b) Use the Euler Method (**eul**) with $h = 0.1$ to approximate the integral $\int_0^{1.5} e^{-u^2}\,du$.

J Given that the general solution to $t^2y'' - 4ty' + 4y = 0$ is $y = C_1t + C_2t^4$, solve the following initial value problem:

$$\begin{cases} t^2y'' - 4ty' + 4y = -2t^2 \\ y(1) = 2 \\ y'(1) = 0 \end{cases} .$$

K From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement $y(x)$ satisfies the *Boundary Value Problem* $\begin{cases} y^{(4)} = -P \\ y(0) = y(L) = 0 \\ y'(0) = y'(L) = 0 \end{cases}$, where $P > 0$ is a constant depending on the beam's density and rigidity and L is the distance between supporting walls:



(a) Solve the above boundary value problem when $L = 4$ and $P = 24$.

(b) Show that the maximum displacement occurs at the center of the beam $x = \dfrac{L}{2} = 2$.

L Laplace transforms may be used to find particular solutions to some nonhomogeneous differential equations. Use Laplace fransforms to find a particular solution, $y_p(t)$, of $y''+4y = 20\,e^t$.

Hint : Solve the initial value problem $\begin{cases} y'' + 4y = 20\,e^t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$.

M Tank # 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank # 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank # 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank # 1 into Tank # 2 at the same rate of 5 gal/min. The solution in Tank # 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank # 1 and Tank # 2, respectively, SET UP BUT DO NOT SOLVE an initial value problem describing this system.

N If $\vec{\mathbf{x}}^{(1)}(t)$ and $\vec{\mathbf{x}}^{(2)}(t)$ are linearly independent solutions to the 2×2 system $\vec{\mathbf{x}}\,' = A\,\vec{\mathbf{x}}$, then the matrix $\Phi(t) = \left(\vec{\mathbf{x}}^{(1)}(t),\, \vec{\mathbf{x}}^{(2)}(t)\right)$ is called a **Fundamental Matrix** for the system. Find a Fundamental Matrix $\Phi(t)$ of the system $\vec{\mathbf{x}}\,' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \vec{\mathbf{x}}$.

O Laplace transforms may be used to find solutions to some linear systems of differential equations. Consider the linear system of differential equations: $(*)\ \begin{cases} x' = x + y \\ y' = 4x + y \end{cases}$ with initial conditions $x(0) = 0$ and $y(0) = 2$.

(a) Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transforms of the functions $x(t)$ and $y(t)$, respectively. Take the Laplace transform of each of the differential equations in $(*)$ and solve for $X(s)$ (i.e., eliminate $Y(s)$).

(b) Using the function $X(s)$ from (a), determine $x(t)$.

(c) Use the expression for $x(t)$ and the first equation in $(*)$ to determine $y(t)$.

P Find a particular solution $\vec{\mathbf{x}}_p(t)$ of these nonhomogeneous systems:

(a) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 5\,e^{2t} \\ 3 \end{pmatrix}$

(b) $\vec{\mathbf{x}}\,' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 10\,\cos t \\ 0 \end{pmatrix}$