Solns – Ma 26100M – Exam 2

1. Use Lagrange multipliers to find the absolute maximum and minimum of $f(x,y) = xy$ subject to the constraint $\underbrace{x^2 + 2y^2}_{g(x,y)} = 4$.

$f_x = \lambda g_x \quad \Rightarrow \quad y = \lambda \cdot 2x$ ①

$f_y = \lambda g_y \qquad\quad x = \lambda \cdot 4y$ ②

$x^2 + 2y^2 = 4 \qquad x^2 + 2y^2 = 4$ ③

If $x \neq 0$ & $y \neq 0$: ① & ② $\Rightarrow \lambda = \frac{y}{2x} = \frac{x}{4y}$

So $4y^2 = 2x^2 \Rightarrow 2y^2 = x^2$

$\Rightarrow \boxed{x = \pm\sqrt{2}\, y}$

Plug into ③:

$4 = x^2 + 2y^2 = 2y^2 + 2y^2 = 4y^2$

So $y^2 = 1$ & $y = \pm 1$

Pts: $(\sqrt{2}, 1)$, $(-\sqrt{2}, 1)$, $(\sqrt{2}, -1)$, $(-\sqrt{2}, -1)$

Value of f: $\sqrt{2}$ $\quad -\sqrt{2}$ $\quad -\sqrt{2}$ $\quad \sqrt{2}$

If $x = 0$: ① $\Rightarrow y = 0$ & this is impossible by ③

If $y = 0$: ② $\Rightarrow x = 0$ & this is impossible

So abs. max of $f = \sqrt{2}$

abs. min. of $f = -\sqrt{2}$

2. Find all critical points of f and classify them as local maxima, local minima, saddle points, or none of these, where

$$f(x,y) = x^3 + y^2 - 3x^2 + 10y + 6.$$

$f_x = 3x^2 - 6x = 3x(x-2) = 0$

$f_y = 2y + 10 = 2(y+5) = 0$

$\Rightarrow \boxed{x=0 \text{ or } x=2}$ and $y = -5$

$\boxed{\text{crit. pts. } (0,-5),\ (2,-5)}$

$f_{xx} = 6x - 6,\ f_{xy} = 0,\ f_{yy} = 2$

$D = f_{xx} f_{yy} - (f_{xy})^2 = (6x-6)2 - 0^2$

$= 12x - 12$

$D|_{(0,-5)} = -6 \cdot 2 = -12 < 0 \Rightarrow$

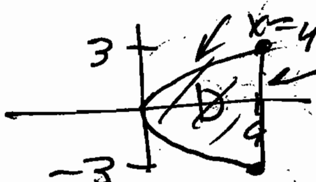
saddle pt @ $(0,-5)$

$D|_{(2,-5)} = 6 \cdot 2 = 12 > 0$

Also $f_{yy} = 2 > 0$ so have local minimum @ $(2,-5)$

3. Set up <u>but do not evaluate</u> a double integral that gives the volume of the solid under the graph of $f(x, y) = x + y$, and above the region in the plane bounded by $x = y^2$ and $x = 9$.

should be $3+x+y$



$$\text{vol} = \iint_D f\,dA = \int_{-3}^{3}\int_{y^2}^{9} (3+x+y)\,dx\,dy$$

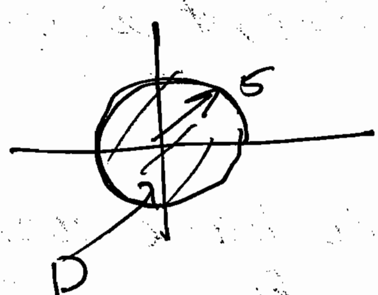
intersection of $x = y^2 = 9$

$\Rightarrow y^2 = 9$

$y = \pm 3$

$$\left(\underline{\text{OR}} \int_0^9 \int_{-\sqrt{x}}^{\sqrt{x}} (3+x+y)\,dy\,dx\right)$$

4. Find the area of the part of the surface $z = \underbrace{xy}_{f(x,y)}$ that lies within the cylinder $x^2 + y^2 = 25$.



$$\text{surface area} = \iint_D \underbrace{\sqrt{1+f_x^2+f_y^2}\,dA}_{=dS}$$

$$= \iint_D \sqrt{1+y^2+x^2}\,dA$$

$f_x = y$

$f_y = x$

polar coords $= \int_0^{2\pi} \underbrace{\int_0^5 \sqrt{1+r^2}\, r\,dr}\, d\theta$

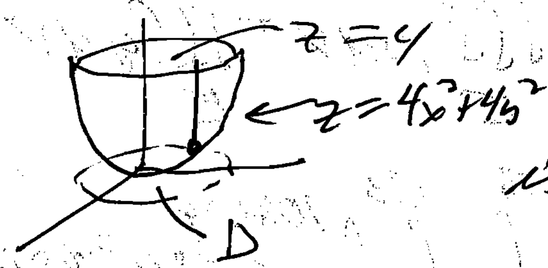
$$= \frac{(1+r^2)^{3/2}}{3}\Big|_0^5 = \frac{26^{3/2}-1}{3}$$

$$= 2\pi \frac{(26^{3/2}-1)}{3}$$

5. Evaluate the triple integral

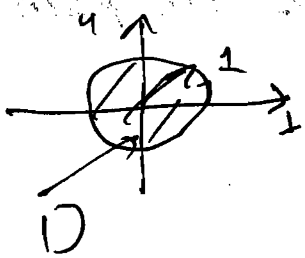
$$\iiint_E 8z\,dv$$

where E is bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = 4$.



Intersection of $z = 4x^2+4y^2$ $= 4$

is $x^2+y^2=1$ so D = unit circle



$$\iiint_E 8z\,dV = \iint_D \underbrace{\left(\int_{4x^2+4y^2}^{4} 8z\,dz \right)}_{= 4z^2 \Big|_{z=4x^2+4y^2}^{z=4} = 64(1-[x^2+y^2]^2)} dA$$

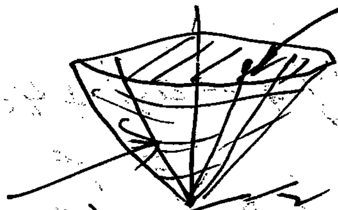
$$= \iint_D 64(1-[x^2+y^2]^2)\,dA$$

polar coords $= \int_0^{2\pi}\int_0^1 64(1-r^4)\,r\,dr\,d\theta$

$$= \int_0^{2\pi}\underbrace{\int_0^1 64(r-r^5)\,dr}_{=64\left(\frac{r^2}{2}-\frac{r^6}{6}\right)\Big|_0^1 = \frac{64}{3}}\,d\theta$$

$$= \int_0^{2\pi} \frac{64}{3}\,d\theta = \frac{128\pi}{3}$$

6. Let E be the solid region between the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and the plane $z = 10$. If the mass density of a material in E is $\rho(x,y) = y$, set up (but do not evaluate) a triple integral in spherical coordinates for the mass.



$$\text{mass} = \iiint_E \rho\, dV = \iiint_E y\, dV$$

Spherical coords

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{10\sec\phi} \rho^3 \sin^2\phi \sin\theta \, d\rho\, d\phi\, d\theta$$

$y = \rho\sin\phi\sin\theta$

$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$\boxed{z=10} \Leftrightarrow \rho\cos\phi = 10$

so $\boxed{\rho = \frac{10}{\cos\phi} = 10\sec\phi}$

cone: $z = \frac{r}{\sqrt{3}} \Leftrightarrow \tan\phi = \frac{\rho\sin\phi}{\rho\cos\phi}$

$= \frac{r}{z} = \sqrt{3}$

$\Rightarrow \phi = \frac{\pi}{3}$

E: $0 \le \rho \le 10\sec\phi$,

$0 \le \phi \le \pi/3$,

$0 \le \theta \le 2\pi$

7. Evaluate the line integral

$$\int_C \vec{F}\cdot d\vec{r} \quad \text{where} \quad \vec{F}(x,y) = x^2y^3\vec{\imath} - y\sqrt{x}\,\vec{\jmath}$$

and C is given by $\vec{r}(t) = t^2\vec{\imath} - t^3\vec{\jmath}$, $0 \le t \le 1$.

$$= \int_0^1 \vec{F}(x(t),y(t)) \cdot \frac{d\vec{r}}{dt}\,dt$$

$$\vec{F}(x(t),y(t)) = F(t^2,-t^3) = \langle (t^2)^2(-t^3)^3, -(-t^3)\sqrt{t^2}\rangle$$

$$= \langle -t^{13}, t^4\rangle$$

$$\frac{d\vec{r}}{dt} = \langle 2t, -3t^2\rangle$$

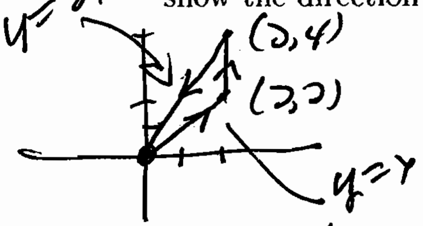
$$= \int_0^1 [-2t^{14} - 3t^6]\,dt$$

$$= \left[\frac{-2t^{15}}{15} - \frac{3t^7}{7}\right]\Big|_0^1 = \frac{-2}{15} - \frac{3}{7}$$

8. Use Green's theorem to compute

$$\int_C xy^2dx + 3x^2ydy$$

where C is the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$. Draw a graph of C and show the direction of C you are using in your answer.



C: direction counterclockwise

$$= \oint_C \underbrace{xy^2}_{P}\,dx + \underbrace{3x^2y}_{Q}\,dy$$

Green's Thm

$$= \iint_D (Q_x - P_y)\,dA = \iint_D (6xy - 2xy)\,dA$$

$$= \iint_D 4xy\,dA = \int_0^2 \int_x^{2x} (4xy)\,dy\,dx$$

$$= 2xy^2\Big|_{y=x}^{y=2x} = 6x^3$$

$$= \int_0^2 6x^3\,dx = \frac{6x^4}{4}\Big|_0^2 = 24$$