11/12/2008 **MA 425** Solutions

(20) 1. These questions require short answers. Always justify your answer with a reason.

(a). If $\int_{-\infty}^{\infty} f(x)\,dx$ exists, must $p.v.\ \int_{-\infty}^{\infty} f(x)\,dx$ exist?

Yes. $\int_{-\infty}^{\infty} dx$ means $\lim_{X,Y\to\infty} \int_{-X}^{Y} f(x)dx$ exists; for pv need $\lim_{X\to\infty} \int_{-X}^{X} f(x)dx$ exists

(b). If γ is a closed curve that does not pass through the origin, is

$$\int_{\gamma} z^{-4}\,dz = 0?$$

Yes! z^{-4} has an antiderivative

(c).Let $f(z)$ be a rational function. To what extent does f have a unique Laurent expansion in powers of z?

If the poles are at $a, b, \ldots$, $|a|<|b|<\cdots$, then there is one expansion in each annulus free of poles.

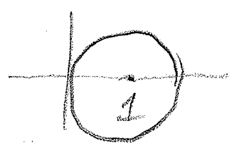
(d). Let f have a pole of order 4 at $z=0$. Could the real part of f always be of only one sign (so either always + or −) in $0<|z|<\eta$ for some $\eta>0$?

No. $f(z) = \frac{a}{z^4}(1+\varepsilon(z))$, $\varepsilon(z)\to 0$. But $\frac{a}{z^4}$ has both signs near $z=0$.

(e). Let f be analytic in $0<|z|<\eta$ for some $\eta>0$. Could e^f have a pole at $z=0$?

No, it will be an essential singularity

(f). Is there a power series in powers of $z-1$ which converges at z=2 and divirges at $z=0$?



Yes ($\sum (-1)^n \frac{(z-1)^n}{n}$ is an example)

(15) 2. Explain why the Taylor series to $f(z) = 1/(1-z)$ does not converge to f *uniformly* on $\{|z| < 1\}$.

$$\frac{1}{1-z} = \sum z^n, \quad |z| < 1.$$

And $\displaystyle\sum^{N} z^n = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{\text{error}}$

If N is fixed, we can have $\frac{z^{N+1}}{1-z}$ as large as we wish by, for example, taking z close to 1.

(15) 3. Let

$$F(z) = \int_{-1}^{1} \frac{|t|}{1 - zt^2}\, dt.$$

Show that F is analytic in some disk $\{|z| < R\}$ and determine R as best possible. Find $F'''(0)$. (corrected!)

Use power series

$$F(z) = \int_{-1}^{1} |t| \sum (zt^2)^n\, dt.$$

The coefficient of z^3 is $\displaystyle\int_{-1}^{1} |t| t^6\, dt$, which is $\dfrac{F'''(0)}{3!}$. But this integral is $\frac{1}{4}$!

(15) 4. Find the singular points of the function

$$f(z) = \frac{1}{e^z - 1} - \frac{1}{z},$$

and in particular find the residues at any poles. What is the nature of the singularity at $z = \infty$?

possible singularities at $z = 2n\pi i$, $z = \infty$.

At $z = 2n\pi i$, $e^z = 1 + (z - 2n\pi i) + \cdots$,

so

$$\frac{1}{e^z - 1} = \frac{1}{z - 2n\pi i}\,\frac{1}{(1 + \cdots\cdots)}.$$

So $z = 0$ is a removeable singularity, $z = 2\pi\pi i$ $(n \neq 0)$ is a simple pole.

$z = \infty$ is an "essential singlarity"* ! One way (easiest) ~~to see it~~ is that

$f \to -1$ as $z \to -\infty$ on real axis,
$f \to 0$ as $z \to +\infty$ " " "

so f has no limit. You can also check it by power series.

* not really; $z = \infty$ is not an isolated singlarity of $f(z)$ — there is no expansion of f in powers of $|z|$ for $R < |z| < \infty$ for any R!

(15) 5. Find

$$\int_\gamma e^{1/z} z^{-6}\, dz,$$

where $\gamma = \{|z| = 6\}$.

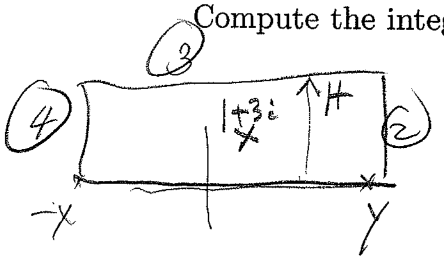
0 (there is no $\int \frac{a}{z}\, dz$ term)

(20) 6. Consider

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10}\, dx.$$

Is this only a principal value integral, or does it converge unconditionally?

Compute the integral/ p. v., showing all steps.



$z^2-2z+10$ has roots $z = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$

So by the residue theorem the integral is

$2\pi i \operatorname{Res}\left(\frac{ze^{iz}}{(z-(1-3i))\,(z-(1+3i))}\right)_{z=1+3i} = 2\pi i \frac{(1+3i)e^{i(1+3i)}}{6i} = \frac{\pi}{3}(1+3i)e^{-3}e^{i}$

$= \frac{\pi}{3} e^{-3}(1+3i)(\cos 1 + i \sin 1).$

Imag part is $\frac{\pi}{3}e^{-3}(\cos 1 - 3\sin 1)$.

Otherwise as in class – this is an integral, not just a p.v.

On (2) and (4), $\left|\frac{z}{z^2-2z+10}\right| \le \frac{C}{|z|}$ if $|z|$ large. So

$\left|\int_{2}\right| \le \frac{C}{Y}\int_0^H e^{-y}\,dy \le \frac{C}{Y} \to 0 \quad (Y \to \infty)$

Same for (4)

For (3):

$\left|\int_{3} \frac{ze^{iz}}{z^2-2z+10}\,dz\right| \le \frac{1}{H}\int_{-X}^{Y} e^{-H}\,dx \le \frac{(X+Y)}{H}e^{-H},$

So we take X, Y as desired, then let $H \to \infty$, and this $\to 0$ too.