Sept. 19, 2008 MA 425 ANSWERS

Exam 1

(18) 1. True–False. Remember that for an assertion to be true, it must be true in every circumstance; if there is a single counterexample, it is false. Be sure to justify your answer.

A. Arg z_1 + Arg z_2 = Arg $(z_1 z_2)$ False. Let $z_1 = z_2 = -1$. Then $\text{Arg } z_1 = \text{Arg } z_2 = \pi$, but $\text{Arg}(z_1 z_2) = \text{Arg } 1 = 0$

B. ~~A~~arg $(z_1 z_2)$ = ~~A~~arg z_1 + ~~A~~arg z_2 True. Each side has infinitely many values.

C. Arg $(z_1 + z_2)$ = Arg z_1 + Arg z_2 False; let $z_1 = z_2 = i$. Then $\text{Arg } z_1 = \pi/2$ but $\text{Arg}(z_1 + z_2) = \text{Arg}(2i) = \pi/2$.

(17) 2. Show directly that the function $f(z) = |z|^2$ has a derivative only at $z_0 = 0$.

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)\overline{(z+h)} - z\bar{z}}{h} = \frac{h\bar{z} + z\bar{h}}{h} = \bar{z} + z\frac{\bar{h}}{h}.$$

As $h \to 0$, $\frac{\bar{h}}{h}$ has absolute value 1 but no limit (write $h = te^{i\alpha}$, α fixed, and let $t \to 0$). So $z\bar{h}/h$ has no limit as $h \to 0$ unless $z = 0$. The other term, $\bar{z}$, is fine.

(10) 3. Complete the sentence: the function $u(x,y)(z = (x,y))$ is differentiable at $(1,2)$ if there are constants A and B and functions $\varepsilon(x,y), \eta(x,y)$ with:

$$u(x,y) = A(x-1) + B(y-2) + \varepsilon(x,y)(x-1) + \eta(x,y)(y-2),$$

where

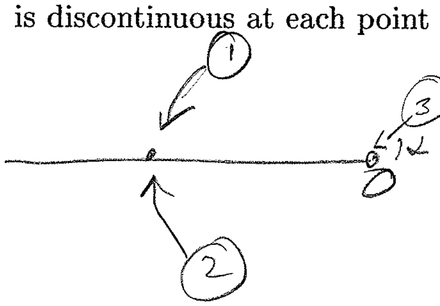
$$\frac{\varepsilon(x,y)}{\sqrt{(x-1)^2+(y-2)^2}} \to 0 \quad \text{and} \quad \frac{\eta(x,y)}{\sqrt{(x-1)^2+(y-2)^2}} \to 0$$

as $(x,y) \to (1,2)$

(15) 4. Show that the function

$$w = \text{Arg } z$$

is discontinuous at each point of the negative real axis. What happens at $z = 0$?



a) at $-t$, $t > 0$, we have $\text{Arg } z \to \pi$ as z moves on path 1, and $\text{Arg } z \to -\pi$ as z moves on path 2

b) If we take path 3 where $\text{Arg } z = \alpha$, we get α as the limit of $\text{Arg } z$ — here $-\pi < \alpha \leq \pi$.

(20) 5. Let $w = f(z)$ have a (complex) derivative at $z = i$. Prove that f is continuous at $z = i$.

Def of derivative at $z = i$

$$\frac{f(z) - f(i)}{(z - i)} \xrightarrow{z \to i} A, \quad A = f'(i)$$

Thus

$$\frac{f(z) - f(i)}{z - i} = A + \eta(z)$$

where $\eta(z) \to 0$ as $z \to i$, This means:

$$f(z) - f(i) = (A + \eta(z))(z - i).$$

Let $z \to i$, $\eta(z) \to 0$. Thus

$$|f(z) - f(i)| \le A\underbrace{|z - i|}_{\downarrow 0} + \underbrace{|\eta(z)|}_{\downarrow 0}\,\underbrace{|z - i|}_{\downarrow 0} \quad (\to 0 \text{ as } z \to i)$$

(20) 6. Let $w = f(z) = u + iv$.

(a) State the Cauchy–Riemann equations.

$u_x = v_y \qquad u_y = -v_x$

(b) Prove that if $f'(z)$ exists, then u and v satisfy the Cauchy–Riemann equations.

$f = u + iv \qquad \frac{f(z+h)-f(z)}{h} = \frac{u(z+h)-u(z)}{h} + i\,\frac{v(z+h)-v(z)}{h}$

a) let $h \to 0$, h real: $f'(z) = u_x + i v_x$ (A)

b) let $h \to 0$ $h = it$, $t \to 0$

$\frac{f(z+h)-f(z)}{h} = \frac{u(z+it)-u(z)}{it} + i\,\frac{v(z+it)-v(z)}{it}$

Let $t \to 0$. $f'(z) = -iu_y + v_y$. (B)

Equate (A), (B):

$u_x = v_y$, $v_x = -u_y$

(c) Let $f(z) = \cos x$ $(z = x + iy)$, where does f' exist? Where is f analytic?

$u(x,y) = \cos x \qquad v(x,y) = 0$

$u_x = -\sin x \qquad v_y = v_x = 0$

$u_y = 0$

$\therefore$ $f'(z)$ exists where $\sin x = 0$ $(z = x+iy)$

(network of lines)