10/10/2008 **MA 425** Answers

(20) 1. These questions require short answers. Always justify your answer with a reason.

(*a*). Can there be a function $f(z)$ analytic only on the real axis?

No – an analytic function must be defined on an open set

(*b*). If γ is a closed curve that does not pass through the origin, is

$$\int_\gamma z^{-4}\, dz = 0?$$

yes; The function $-\frac{1}{3}z^{-3}$ has as derivative

(*c*). If u and v are harmonic in a domain D, is $f = u + iv$ analytic?

No – $x + ix$ is not analytic

(*d*). If u is harmonic in $\Delta := \{|z| < 1\}$ is there an analytic function f with u the imaginary part of f.

yes. we have $v(z) - v(0) = \int_0^z u_x\,dy - u_y\,dx$ (independent of path)

(*e*). Let $D := \{1 < |z| < 2\}$ and $P(z)$ a nonconstant polynomial. Must there be an analytic function $F(z)$ in D with $F'(z) = P(z)$?

yes. We learn that in calculus

(20) 2. Let $z(t), 1 \leq t \leq 2$ be a smooth arc in the plane. What is the physical meaning of

$$\int_1^2 |z'(t)|\, dt?$$

the length of the arc $\{z(t)\}$

If $z(t) = x(t) + iy(t)$, write this integral in terms of $x'(t), y'(t)$.

$$\int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(20) 3. Prove using the definitions given in this course that

$$\sin^2 z = \frac{1 - \cos 2z}{2}.$$

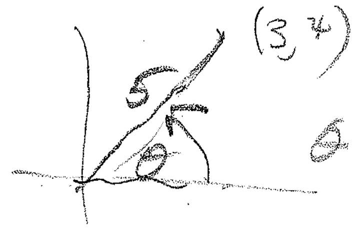
$$\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{e^{2iz} - 2 + e^{-2iz}}{-4}$$

$$= \frac{1}{2} - \frac{1}{2} \cos 2z$$

(15) 4. Find all solutions to the equation $e^z = 3 + 4i$.

$\log z = \log(3+4i)$



$\theta = \tan^{-1}\frac{4}{5}$

$\log(3+4i)$

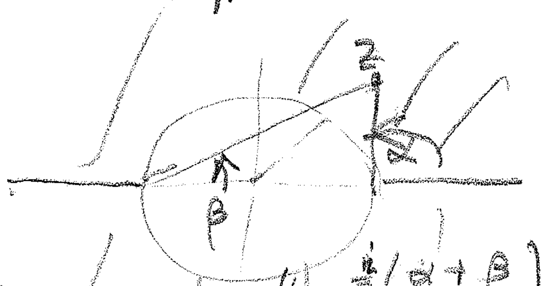
$= \log 5 + \tan^{-1}\frac{4}{5} + 2k\pi i,$

here $0 < \tan^{-1}\frac{4}{5} < \pi/2$, for example.

(20) 5. Show that there is a branch of the multiple-valued function

$$w = (z^2 - 1)^{1/2}$$

which is analytic in the region $\{1 < |z|\}$. How many possible branches are there?



$\alpha = \mathrm{Arg}(z-1)$

$\beta = \mathrm{Arg}(z+1) = \mathrm{Arg}(z-(-1))$

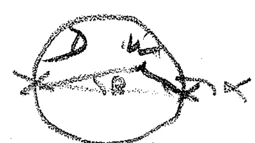
Method 1 $\quad w = |z^2-1|^{1/2} e^{\frac{i}{2}(\alpha+\beta)}$

if z moves in $D = \{|z|>1\}$, then at each time it returns to the same point, α increases by a multiple of 2π — but β changes by the same point. So if $k \cdot 2\pi i$ is the change of α, then $\alpha+\beta$ changes by $2k \cdot 2\pi i$, and $e^{\frac{1}{2}(\alpha+\beta)}$ doesn't change at all.

Method 2

$$z^2 - 1 = z^2(1 - 1/z^2)$$

Then, we have $\sqrt{z^2-1} = z\sqrt{1-1/z^2} = z\sqrt{1-u^2}$ where $|u|<1$.



In this case, α and β will not change as u moves in $D = \{|z|<1\}$

(15) 6. Let z_1, z_2, z_3 and w_1, w_2, w_3 be two ordered triples of complex numbers. Show that the triangle $T_1 : z_1, z_2, z_3$ is similar to $T_2 : w_1, w_2, w_3$ if

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

(Hint: a picture might help. Think of high-school geometry.)

SAS for similarity