MA 425 ASSIGNMENT SHEET Fall 2008

Text: Saff/Snider *Fundamentals of Complex Analysis*, Third Edition

This sheet will be updated as the semester proceeds, and I expect to give several quizzes/exams. The material is especially attractive, in that we have an opportunity to see our elementary calculus in a new light, and also to discover rather amazing ways to view it, all from introducing 'imaginary' numbers! There will be homework collected most days, and it would help the discussion if you would e-mail me in advance asking for discussion of particular problems.

Taking this course determined the mathematical interest for my next 50 (!) years.

It is important to come to **every** *class, and read the book at home.*

Some of the homework problems have answers/solutions in the back. There are far too many problems for us to penetrate a good percent in class, but there are lots of opportunities for you to work out problems on your own. I will be glad to write out solutions and post them unpon (reasonable) request.

In class I will do some extra problems, and they will also be considered as part of the basic course.

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Office Hours: MF 10:30-11 AM , W 2:30-3 PM and by appointment. Feel free to email me too.

1.1–2 *Introduction and later highlights.* Some of the main highlights can be introduced in the first lecture. $|z|, \bar{z},\ z = x + iy$: we almost always take x, y, u, v to be real and z, w to be *complex*. A complex number z can be written as $z = x + iy$ (with x, y real) (or $w = u + iv$) and the complex numbers form a *field*. We assume familiarity with the real numbers and elementary calculus, although you should get new insights on them.

Complex numbers can be identified with vectors in the plane.

Problems: p. 5: 3, 4, 5ac, 7b, 16bd, 30ad, 21, 30 [Note: 21 is simply using things you might know working with real numbers; problem 30 introduces us to the main loss when working with complex numbers: inequalities].

p. 12: 1, 5 (this can be done more easily after §1.4!), 7cdfg, 11, 16.

1.3 *Polar form.* $x + iy = r(\cos\theta + i\sin\theta)$. The'function' θ always causes problems, but if you calmly think about it you should be able to handle it–it is important that you understand it from the beginning! Rectangular coordinates are good for addition, polar coordinates good for multiplication (and logarithms, as we see soon)–which is why we endure them. $\arg z$ vs $\operatorname{Arg} z$, $\mathcal{A}rg\ z$.

Absolute value is far more important here than in calculus; it has the advantage in being defined by a formula: $|z|^2 = z\bar{z}$.

Problems: p. 22: 1b, 3, 5(all), 7b,d,h, 8, 10b, 19.

1.4 *Complex exponential.* Unquestionably the most important function:

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

unites algebra to trigonometry. DeMoivre's formula is an observation only, as are all the trig identities we (sometimes) see in high school.

Problems: p. 31: 4ac, 7 (don't forget this one!), 11, 12b, 20ab, 23a.

1.5 *Powers and Roots.* (This is a corollary to the previous section.) Notice the special use of the nth root ω_n for each positive integer n.

Problems: p. 37: 3, 4b, 6c, 7b, 10, 11, 16.

1.6 *Planar sets.* domain: open connected set. In calculus our functions are usually defined on $[a, b]$ or, more generally, on closed, bounded sets. As we will see in the next chapter, our functions will usually be defined on domains. Closed sets. Are there sets neither open nor closed?

Problems: p 42: 2-8, 11, 15-17, 19, 20.

1.7 *Stereographic projection.* In real calculus we have $\pm\infty$, but here there is only on ∞; the most 'natural' way to see it is using the Riemann sphere. We work out the relation between (x_1, x_2, x_3), the coordinates in $\mathbb{R}^3$ of the sphere, and the coordinates x, y in the plane, see p. 47. From the viewpoint of the sphere, lines and circles are the same thing: lines are circles passing through ∞, the north pole of the sphere. *chordal distance.*

Problems: p. 50: 1bc, 2, 5a-c, 6, 9.

2.1 *We start calculus.* $w = u + iv = f(x + iy)$. Graphs require two complex planes. Composition, just as in (real) calculus.

Problems: p. 56: 1adef, 4, 5, 7ab, 8ab, 10 (all), 13.

2.2 *Limits and continuity.* Still in MA 161! Are there any differences with what you had in high-school or as a freshman?

We also recall *differentiability,* which is part of the MA 261 syllabus. A real-valued function $u(x, y)$ is differentiable at (a, b) if there are constants A, B and functions $\varepsilon(x, y)$, $\eta(x, y)$ so that

$$u(x, y) - u(a, b) = A(x - a) + B(y - b) + \varepsilon(x, y) + \eta(x, y)$$

subject to

$$\frac{\varepsilon(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{\eta(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} \to 0$$

as $(x, y) \to (a, b)$.

Problems: p. 63: 1, 2, 4, 5, 7(all), 11 (all) —for some you 'plug in', for other you have to work a bit, 12 (important!), 14.

Differentiability problems: Show that if u is differentiable at (a, b) then the partial derivatives u_x, u_y exist at (a, b). Show that $A = u_x(a, b)$, $B = u_y(a, b)$. Next, find a function which is differentiable at $(0, 0)$ and whose partial derivatives exist *only* at that point. Show that if u is differentiable at (a, b) then u is continuous there.

2.3 *Analyticity.* Now we begin to diverge from MA 161. The definition of derivative should be familiar, but we will see that it has surprising consequences here. For example, a nice function such as $f(z) = \bar{z}$ has no derivative. But usual formulas carry through. Definition of *analytic, entire.* We will discuss problem 8 carefully in class.

Problems: p. 70: 2, 3, 4c, 6, 7(all), 9ac, 11a, c, d, g, h.

2.4 *Cauchy-Riemann.* In real variables it is hard to find a continuous function which has no derivative, but in this course, the Cauchy-Riemann equations give a simple way to produce many.

Problems: Here is an important one which brings back the issue of (real) differentiability. Prove directly from the definition that $f(z) = u(z) + iv(z)$ has a derivative at $z_0 = (a, b)$, if and only if u and v are differentiable at (a, b) and the Cauchy-Riemann equations hold (half of this is in the book on this section, but the proof is surprisingly easy). In how much of page 75 is the author talking about (real) differentiability without using the word?

Other problems: p. 77: 2, 5, 6 (we may do this one later a different way!), 8, 12, 15 (the Jacobian is often introduced in MA 261).

2.5 *Harmonic Functions.* "Real parts of analytic functions are the same thing as harmonic functions." Strictly speaking, this is *false*, but behind it is a truth you should be able to understand. Differences between local and global properties. Isothermal coordinates (relate to problem 8, sec. 2.3).

Problems:: p. 84: 3acef, 6, 9 (one way to do this is write $x = \log|z|, y = \arg z$ (makes sense locally) and use the ordinary laplace equation), 12 14 (not easy, but once you see it...), 17ab

2.6 *Harmonic functions as steady-state temperature* (optional). If we know the temperature at the boundary of every point of a nice domain, we can find it at each point inside by solving the *Dirichlet problem.*

Problems: p. 90: 1, 2

2. 7 *Julia/Mandelbrot* (optional). Let f be analytic and map $D \to D$ (here we take D to be the complex plane $\mathbb{C}$). Then we can take a point $z \in D$ and investigate the behavior of the sequence $\{z, f(z), f(f(z)) := f^2(z), f^3(z), \dots\}$, the *orbit* of z under the iterates of f. This is a simple dynamical system, and in the past 25 years has seen tremendous amount of activity (we usually don't see modern themes in an undergraduate course!). Terms: fixed point, basin of attraction, Jula set and filled-in Julia set (for polynomial functions f). We might discuss some computer pictures, depending on the interest of the class.

Problems: p. 95: 2 (think of $f(z) = 2z$ near $z = 0$ as a model), 3a, 10 (there is a good story behind that problem).

3.1 *Elementary functions.* We state the fundamental theorem of algebra (which is proved later), recall Taylor's theorem and partial fraction decomposition..

Problems: p. 108: 3ac, 5ab, 7, 11ac, 13b, 17 (after we learn about the logarithm, you might see an easier way to do this).

3.2 *Trigonometrial and related functions.* We see a connection between algebra and trigonometry. But e^z is periodic (with an imaginary period). Calculus rules are easy.

Problems: p. 115: 2, 3 (geometric), 5bdf, 9acd, 11, 17all, 18b (you should not use l'hopital rule, as we should mention in class).

3.3 *Logarithm.* Don't miss our discussion here, and read the book–this is often hard to understand, and the issues are exactly those we faced when talking about $\arg z = \Im(\log z)$. The high-school logarithm is a special case, but we have to be very careful about formulas such as

$$\log(zw) = \log z + \log w.$$

We also introduce Log $z, \mathcal{L}ogz$.

However, $log|z|$ is a 'function', and is harmonic for $z \neq 0$.

Problems: p. 123: 3, 4, 6, 7, 11, 14 (hard, be careful), 19.

3.4 §*2.6 reappears.* we see how important the logarithm is in solving the Dirichlet (ssteady-state temperature) problem.

Problems: p. 129: 2, 4, 5

3.5 *More trigonometry.* We see that clgebra and trigonetry continue to be the same thing, but to make this work, we need to use the logarithm. Also non-integer powers.

Problems: 1ae, 3ab, 4, 5, 8, 12, 15ab (these are not easy), 16 (you had this in elementary calculus, but may have forgotten...).

4.1 *Complex Integration.* This begins the heart of the course. We will be taking

$$\int_\gamma f(z)dz,$$

where γ is a *contour*. So here we discuss what we mean by a contour and it length.

Problems: p. 159: 1bd, 3 (for a circle, we have $x = \cos t, y = \sin t$, and you should modify that), 5, 8, 13abd

4.2 *Contour (line) integrals* We just change variables.

Problems: p. 170: 3abd, 5, 7, 11, 14ac

4.3 *Fundamental Theorem of Calculus* Here it means: independent of path, something that you knew but never realized!

Problems: p. 178: 2, 4 (give reason(s)), 5, 7, 10

4.4 *Cauchy's Theorem.* This is the most important result of the course, and one that one can appreciate many years later. We prove it differently than the book. The goal is to show THEOREM 9: if f is analytic in a simply-connected domain D and γ is a closed ontour (loop) in D, then

$$\int_\gamma f(z)\,dz = 0 --$$

so this course is making something of nothing!

Unfortunately, the proofs in this section (and in most books) require that the partial derivatives of f be continuous in D. We will follow the ideas of a famous proof due to E. Goursat in 1900 and this as a consequence of Green's theorem (stated on page 193). We will prove it under the assumption that γ is a rectangle and that f is analytic in each point of γ and its interior; to go from rectangles to general simply-connected domains is a technical issue I'd rather not do in this course.

In the book, the partial derivatives of the functions V_1 and V_2 are required to be continuous, and we are able to avoid that here: we just need that V_1 *and* V_2 *are differentiable in the sense introduced in §2.2 and that the integrand on the right side of Theorem 11 is continuous.* But these hypotheses are guaranteed since we are assuming that f is analytic at each point, so we have a complete proof making no extra assumtions.

Problems: p. 199: 6, 7a, 9 all, 10c, 13, 17.

4.5 *Cauchy's integral formula.* Now we can start evaluating integrals, even though we use on the results of the previous section, where the itnegral is zero. It works because $\int_{\{|z|=1\}} z^{-1} = 2\pi i$.

We find that analytic functions have derivatives of all orders. Thus, we can finally see that if u is harmonic in a neighborhood of a point z_0 then u is infinitely differentiable (make sure that you understand this).

Problems: p. 212: 4, 5,8, 11, 13. Number 16 is interesting for the more mathematically-inclined.

4.6 *Bounds, Maximum principle.* In real variables, if $|f(x)|$ is small, we can't say much about the size of $|f'|$; because of §4.5, the situation is completely different here.

Problems: p. 219: 2, 3, 4, 7, 8, 10, 16.

4.7 *More on harmonic functions.* We may show Poisson's formula, which solves the Dirichlet problem for a disk. But it is of perhaps more theoretical interest than for computation.

Problems: p. 225: 1, 2, 4, 8, 10(!), 13.

5.1 *Review of series.* (We are likely to go quickly here, but you should read it; you are likely to be examined on some of it. Joke: at the end of the baseball season they play the *World Sequence,* not the World Series.

The geometric series is the heart of the matter. We will mention the ratio test too. Absolute, uniform convergence. We might mention the integral test in class, but you need it for problem 13.

Problems: p. 239: 1c,d,f, 4, 5 [the converse is completely false!], 7b,c,d,f, 9, 11b,c,d, 14, 15b,c, 19, 20.

5.2 *Taylor series.* This is where we understand an important part of calculus. If f is a function, how can we tall whether f has a Taylor series about a point z_0 (or, in real variables, $x = x_0$) which converges to f; and for what region will this convergence hold? See Theorem 3, p. 243.

Just as in elementary calculus, the coefficient of $(z - z_0)^n$ is $f^{(n)}(z_0)/n!$, but because of Cauchy's formulas, this appears as a line integral invovling f, not its derivatives.

Problems: p. 249: 1c,f [of course this is the binomial formula–you might try, in the same way, $f(z) = (z - 1)^3$ at $z_0 = 0$ using the Taylor formula!], 3, 4, 5, 11, 13 [good problem].

5.3 *Power Series.* What is the difference between power series and Taulor series? Well, they start from two different points of view [explain!] but in the end theya re the same thing. What is remarkable is that *the uniform limit of a sequence of analytic functions is analytic.*. This has applications to power series solutions to ordinary differential equations. Study examples 1 and 2 in the chapter.

Problems: p. 258 1 [this shows that almost anything can happen on the circle of convergence], 2, 3adfg, 6, 10, 12, 15. *Extra:* Show that the matter of uniform limits does not work for functions $y = f(x),\ -1 \leq x \leq 1$ by finding a sequence of differentiable functions on $[-1, 1]$ which converge to a function which does not have a derivative at $x = 0$; by our work in class, the limit functions has to be continuous, however. [Hint: let the limit function be $y = |x|$.]

5.4 *theory of Convergence:* I may leave this for self-study. In the end, the convergence theory depends on what I can an 'article of faith' [which we math folks make into an 'axiom'] that the real number line has no holes. Note (2), which is a formula for the radius of convergence of a power series. But we know something better: R is the larges number for which the series $\sum a_n(z - z_0)^n$ is analytic in $\{|z - z_0| < R\}$. We meet $\liminf,\ \limsup$. (Unline lim, these always exist.)

Problems: p. 266: 1, 3bef, 5acd, 9, 10.

5.5 *Laurent series.* This is a representation for functions analytic in an annulus $\{\alpha < |z - z_0| < \beta\}$, and it is the sum of two power series, one convergent in $\{|z - z_0| < \beta\} <$ and one (in negative powers of $(z - z_0)$) which is holomorphic in $\{|z - z_0| > \alpha\}$. There is a formula for the coefficients, by they are not related to derivates at $z = z_0$; the function might not even make sense at $z = z_0$. we compute these series usually by algebraic manipulation; formula (1) is less used to compute a_j than to compute the integral on the right side.

Problems: p. 276: 1, 2, 5, 9.

5.6 *Singularities,* Removable, poles (order $m, 0 < m < \infty$), essential singularities. Near an (isolated) essential singularity an analytic function f comes arbitrarily close to every complex number (the example $e^{1/z}$ at $z = 0$ shows that the function might miss one or (if we count ∞) two values. That is, I did problem 14.

Problems: p. 284: 1 (all), 3ab, 4, 7, 12.

5.7 *Singularity at* ∞. We let $w = 1/z$ and consider the singularity at $w = 0$.

Problems: p. 290: 1a,e,h [h is tricky!], 3c, 5, 6, 7.

6.1 *Residues:* The theorem is super-simple at the stage, but we'll see how powerful it is for us.

Problems: p. 313: 1, 2, 3ceg.

6.2 *Trigonometric integrals:* Simple in principle, but factoring is usually awkward.

Problems: p. 317: 1, 4, 7, 11 (11 is tricky!).

6.3 *Improper read integrals on* $(-\infty, \infty)$: We go over the notion of improper integral, since this is something that, although not new to you, is often misunderstood [what is a *proper* integral? We should be clear on the distinction between principal value and integral; think of $\int_{-\infty}^{\infty} x\, dx$.

Problems: p. 325: 1, 2, 7, 9, 10.

6.4 *Improper integrals involving trigonometric functions:* In the first lecture I mentioned that we would do $\int_{-\infty}^{\infty} x^{-1} \sin x\, dx$. Make sure that you appreciate that it is not obvious that this integral should converge (both at 0 and ∞, although at 0 there is not much of a problem). I do not use Jordan's lemma, but instead use the fact that we can integrate on a rectangle rather than a circle; with a rectangle we han let y and x go to infinity independently.

Problems: p. 336: 1, 5, 6, 10 (even though the book calls some of these principal values, some are not!).

6.5 *Indented Contours:* Sometimes if the contour goes through the singularity, we get only a percentage of the residue. Here is where we do $\int (\sin x)/x\, dx$.

Problems: p. 344: 1ac, 4, 7, 10.

6.6 *Multiple-valued functions:* Keyhole contours.

Problems: p. 354: 1, 2 (why is this only for this rage of α?), 4, 8, 9a

6.7 *Argument principle, Rouché:* This is a more theoretical section of applications of the residue theorem; we give another proof of the fundamental theorem of algebra. The illustration on p. 361 is one ‘application’ of the theory.

Problems: p. 362: 1c-e; 3 [this is easy!], 7, 8, 10, 13, 18.

7.1 *Invariance of Laplace's equation:* We have already seen that soutions to steady-state temperature problems are harmonic functions. It is quite elementary (hardly worth an entire section!) to see that if U is harmonic on a domain D' and $\phi : D \to D'$ is conformal (analytic) then the function $u(z) = U(\phi(z))$ is harmonic. This makes is desirable to find mappings from domains D and D'; a conformal mapping is a one-one analytic mapping between domains; its inverse is also conformal (the title ‘conformal’ comes from the fact which we saw back in problem 8, p. 71: if f is analytic near z_0 and $f'(z_0) \neq 0$, then angles at z_0 are preserved.

Problems: p. 374: 1, 2a-c, 4.