MA 425 On Cauchy's theorem and Green's theorem

1. Introduction

No doubt the most important result in this course is Cauchy's theorem. There are many ways to formulate it, but the most simple, direct and useful is this: *Let f be analytic inside and on the simple closed curve γ. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Certainly the most natural way to prove it is by using Green's theorem, and we state the conclusion (the 'formula') of Green's theorem now, leaving a discussion of the 'appropriate' hypotheses for later. The formula reads: *D is a region bounded by a system of curves γ (oriented in the 'positive' direction with respect to D) and P and Q are functions defined on $D \cup \gamma$. Then*

$$\int_\gamma P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{1}$$

Certainly for (1) to hold, we need that P, Q have partial derivatives at each point, but there are examples to show that this is not enough. In any case, (1) leads to a trivial proof of Cauchy's theorem (this is only a formal proof, since we have not discussed if we are allowed to use (1); even so, I think it is impressive how 'simple' the proof becomes): $f = u + iv, dz = dx + idy$, and then

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) =$$
$$\int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx,$$

and so if we apply Green's theorem to each of these line integral and use the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$, we see the integrand in each double integral in (1) is zero. In that sense, Cauchy's theorem is an immediate consequence of Green's theorem, and in fact Green's theorem [as a special case of Stokes's theorem] is a fundamental result in mathematics and its applications – it is just the fundamental theorem of calculus in higher dimensions.

2. What is wrong?

There are two objections to the proof I just presented. One we do not worry about here– we have not carefully described what kind of curves we are allowing, and what we mean by the 'positive' direction of circuiting γ. This is really a problem of point-set topology or geometric measure theory, and this note offers no insight on that issue.

However the other objection relates to the hypotheses on P and Q needed to apply Green's theorem. Green's theorem is in all the calculus books, where it is always assumed that P and Q have *continuous partial derivatives*. When applied to our analytic function $f(z)$, it means that we are assuming that the partial derivatives u_x, u_y, v_x and v_y are continuous. Probably this is something that does not worry most students taking a first course, but the purpose of these notes is to show that we do not need that assumption; indeed Green's theorem holds when P and Q satisfy conditions which are immediately seen to be fulfilled when P and Q are the real and imaginary parts of the analytic function f. We will see that the conditions needed for P and Q fit *exactly* with what f being analytic means. We state Green's theorem in a revised form, where we consider only the case that R is a rectangle and γ is its boundary, ∂R:

Theorem 1$'$. *Let P and Q be* d*ifferentiable inside and on a rectangle R, with $\gamma = \partial R$, and suppose that*

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{dy} \tag{2}$$

is continuous. Then (1) holds.

NOTE. We will see in the next section that if P and Q are differentiable, then they have partial derivatives. One of the miracles of complex analysis is that the partial derivatives of a complex differentiable function are continuous, something that is certainly not true in the real calculus of several variables. However, at this stage of the development, we don't have enough information to show this. What we need here, is something just a little weaker: that the expression (2) be continuous. In principle, each of P_y and Q_x might be discontinuous, and yet the expression in (2) might still be continuous were the discontinuities of the two terms to cancel out. And it turns out that f being differentiable in the complex sense gives that the difference (2) is continuous.

So to be able to use this revised version of Green's theorem to derive Cauchy's theorem, we check three things: (*i*) what it means to be differentiable [this is shown in third-semester calculus, but often forgotten]; (*ii*) that u and v are differentiable at each point z at which $f'(z)$ exists, and that the expressions which appear in the double integrals are continuous. Finally (*iii*) we will prove Green's theorem under the hypothesis of Theorem 1$'$. Step (*iii*) in fact uses the same argument that E. Goursat introduced to give his famous 'elementary' proof of Cauchy's theorem, which appeared in volume 1 of the *Transactions of the American Mathematical Society.* (The other observations are not original either, but I am collecting them together for convenience.) Once we are allowed to use Green's theorem, Cauchy's theorem follows at once, as we saw in our 'simple' proof on the previous page.

3. Differentiability

We consider a real-valued function u in a domain D.

DEFINITION. The function $u(x, y)$ is *differentiable* at (x_0, y_0) if there are constants A and B so that

$$u(x, y) - u(x_0, y_0) = A(x - x_0) + B(y - y_0) + R(x, y),$$

where the 'remainder' R satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{3}$$

Note that $R(x_0, y_0) = 0$ when (3) is satisfied.

This is a standard definition in third-semester calculus. If we set y identically equal to y_0, and let $x \to x_0$, then $\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - x_0)^2} = |x - x_0|$, so (3) tells us that

$$\lim_{x\to x_0} \frac{u(x, y_0) - u(x, x_0) - A(x - x_0)}{|x - x_0|} = 0.$$

It is quite amazing how useful is this formulation. Since the limit on the right side is zero, we can multiply the left side by any factor of absolute value one without changing the equation (if the limit were 2, we couldn't do this — this is an illustration of the principle that in analysis one rarely proves that $f \to L$; rather we prove that $f - L \to 0$ or, better, $|f - L| \to 0$). So let's multiply this equation by $|x - x_0|/(x - x_0)$. Then we have at once

$$\lim_{x\to a} \frac{u(x, y_0) - u(x_0, y_0) - A(x - x_0)}{x - x_0} = 0:$$

thus $u_x(x_0, y_0)) = A$; similarly we see that $(\partial u/\partial y)(x_0, y_0) = B$.

So we now know that a differentiable function has partial derivatives. However, being differentiable is a much stronger property than only having partial derivatives, since it is a condition independent of how $(x, y) \to (x_0, y_0)$. For example, the function $u(x, y) =$

$xy/(x^2+y^2)$ has $u_x(0,0) = u_y(0,0) = 0$, but u is not even continuous at $(0,0)$ since $u(x,x) = 1/2$ (on the $45°$ line through the origin).

NOTE 1. In one-variable calculus, the statement that $A = f'(x_0)$ can be written as

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - A(x - x_0)}{|x - x_0|} = 0,$$

which is quite similar to (3), if far too awkward for us to introduce in an elementary course. But it shows that (3) is what you have been using for a long time.

NOTE 2. If you check many of the proofs of basic facts in a multi-variable calculus course, you will see that differentiability is often all that is needed, although books usually require that partial derivatives be continuous, since that hypothesis is easy to explain and check. Here is one example: if $u(x,y)$ is differentiable and $x = x(t), y = y(t)$ are differentiable [in one variable, that just means that x' and y' exist, but nothing about continuity], then the function $u(x(t), y(t))$ is a differentiable function of t, and the formula for the chain rule applies:

$$(u(x(t), y(t)))' = u_x x'(t) + u_y y'(t).$$

4. First blood

Let us prove a little theorem. The proof is not hard at all, and if you go through it, I hope you will see is that you are just rearranging equalities everywhere; the moral of the stary is that u and v differentiable (which sounds like an contrived definition) is as natural as $f = u + iv$ having a derivative.

Theorem 1. *Let $f = u + iv$ be defined in some neighborhood of $z_0 = (x_0, y_0)$. Then f' exists at z_0 if and only if u and v are differentiable at z_0 and at z_0 the partials of u and v satisfy $u_x = v_y,\ u_y = -v_x$ (Cauchy-Riemann).*

Proof. First let's assume that $f'(z_0) = A + iB$. We show that u and v are differentiable and satisfy Cauchy-Riemann. Let's write $f(z) - f(z_0)$ in terms of u and v. Then

$$\begin{aligned}
&f(z) - f(z_0) - (A+iB)(z - z_0) = (u+iv)(z) - (u+iv)(z_0) - (A+iB)(z - z_0)\\
&= (u+iv)(z) - (u+iv)(z_0) - (A+iB)((x - x_0) + i(y - y_0))\\
&= u(z) - u(z_0) - [A(x - x_0) - B(y - y_0)]\\
&\quad + i\{v(z) - v(z_0) - [B(x - x_0) + A(y - y_0)]\}.
\end{aligned}$$

We may divide by $z - z_0$ or $|z - z_0|$ as we wish. If we divide by $z - z_0$, the left side tends to 0 since $A + iB = f'(z_0)$. Dividing by $|z - z_0|$ preserves the real and imaginary parts of the numerator, and so both the real and imaginary parts also tend to 0. Now look at the last two lines of our last displayed computation. Since the real part has 0 as a limit, u must be differentiable at z_0, and similarly v must also be differentiable. Moreover, as we saw in §3, the numbers A and B which appear in the definition of differentiability are the partial derivatives. So we have

$$A = u_x(z_0) = v_y(z_0),\ B = v_x(z_0) = -u_y(z_0):$$

the Cauchy-Riemann equations hold.

Now we can go the other way almost by reading backwards. Let's assume that u and v are differentiable at z_0 and the partials of u and v satisfy the Cauchy-Riemann equations. Since $f = u + iv$ we may substitute for u and v their differentials, using R_1 and R_2 for the remainder term R in (3) to conclude that

$$\begin{aligned}
&f(z) - f(z_0) = (u+iv)(z) - (u+iv)(z_0) = (u(z) - u(z_0)) + i(v(z) - v(z_0))\\
&= A(x - x_0) + B(y - y_0) + R_1 + i[C(x - x_0) + D(y - y_0) + R_2]\\
&= A(x - x_0) + B(y - y_0) + i[-B(x - x_0) + A(y - y_0)] + R_1 + iR_2\\
&= (A+iB)(z - z_0) + R_1 + iR_2,
\end{aligned}$$

so on dividing by $z - z_0$ or $|z - z_0|$ as appropriate and recalling (3) for R_1 and R_2 we have that $f'(z_0) = A + iB$.

5. Proof of Theorem 1′ (useful form of Green's theorem)

We first need to know that Green's theorem holds if P (or Q) is a linear function: $P(x, y) = A + Bx + Cy$, with A, B, C constants. It is clear in this simple case that P has continuous partials, and so the standard proof of Green's theorem may be used with no guilt, but it is a useful exercise to check it directly. In fact, using elementary calculus, it is straightforward to check that

$$\int_\gamma (A + Bx)\, dx = 0,$$

(indeed, the integrand is the differential of the function $A + (1/2)Bx^2$), and so

$$\int_\gamma P\, dx = \int_\gamma Cy\, dx \tag{4}$$

(when we consider $\int_\gamma Q dy$, what will survive is $\int_\gamma \hat{B}x\, dx$ when $Q = \hat{A} + \hat{B}x + \hat{C}y$).

However, it is not hard to directly check that (1) holds in the situation (4), and I sketch the details when γ is the boundary of the rectangle with sides parallel to the coordinate axes and diagonal vertices $(0, 0)$ and (α, β), where $\alpha, \beta > 0$. Then on computing (4) only the terms which involve integration on the horizontal sides survive (on the others, $dx = 0$) and so

$$\int_\gamma Cy\, dx = C \cdot 0(\alpha - 0) - C \cdot \beta(0 - \alpha) = C\alpha\beta :$$

the first term on the right refers to the integral on the bottom of the rectangle, and the second is over the top. You will see that the last term is $C \iint dxdy$, as predicted by Green's theorem. (In fact, one doesn't need Green's theorem to see any of this; at Purdue it is done as the first application of integration.)

Thus, we may assume that (1) holds for linear functions.

We prove Theorem 1′ with R a rectangle, and write $R = R_0$. Let us call

$$\left| \int_\gamma P\, dx + Q\, dy - \iint_R (Q_x - P_y)\, dxdy \right| = \Delta_0. \tag{5}$$

If $\Delta_0 = 0$ there is nothing to prove, so let's assume that $\Delta_0 = h > 0$.

Here is Goursat's idea. Divide R into four similar rectangles, and look at the integrand in (5), and compute the same difference for each of these four, calling them Δ (we won't bother with subscripts for a moment). We can't have each each difference less than $h/4$, for if all four differences were less than that, we could add them and then the sum of these four discrepancies would be less than $h = \Delta_0$ (you should check that they add–certainly the double integrals add, and the line integrals do too, since inside R_0 any contribution from points that are on the boundary of two of the smaller rectangles cancel because the segments of these boundaries are travelled once in each direction). That means that there must be one smaller rectangle, each side of which is half that of R_0, for which the difference (which we call Δ_1) in the two terms is at least $h/4$, and we call that rectangle R_1.

Now we repeat this argument with R_1 and divide it into four similar rectangles; for one of them, which we call R_2, bounded by γ_2, we have that Δ_2, the expression exhibited in (5) is at least $h/4^2$.

We keep on this pattern, and for each positive integer n find rectangle R_n inside R_{n-1} whose boundary is γ_n with

$$\left| \int_{\gamma_n} P\, dy + Q\, dy - \iint_{R_n} (Q_x - P_y)\, dxdy \right| > h4^{-n}. \tag{6}$$

The *next* step is to use something math people see in topology, but which seems very 'obvious'. Certainly

$$R_0 \supset R_1 \supset R_2 \supset R_3 \ldots,$$

and the diameter of R_n is $D_0 2^{-n}$, where D_0 is the diameter of R_0. What we need is that

$$\cap_{n\geq 0} R_n = z_0, \tag{7}$$

where z_0 *is a point*. Since the diameter of R_n tends to zero, it should be clear that this intersection could not contain more than one point, and it is a basic fact about the plane (or euclidean space in general) that the intersection of any nested family of bounded closed non-degenerate rectangles is nonempty.

Of course P and Q are *differentiable!* at z_0, so near z_0 we have

$$\begin{aligned} P(z) &= P(z_0) + A(x-x_0) + B(y-y_0) + R = P_0 + Ax + By + R, \\ Q(z) &= Q(z_0) + \hat{A}(x-x_0) + \hat{B}(y-y_0) + R' = Q_0 + \hat{A}x + \hat{B}y + \hat{R}, \end{aligned} \tag{8}$$

where $A, \hat{A}, B, \hat{B}$ are the partial derivatives at $z_0 = (x_0, y_0)$ and

$$\lim_{z\to z_0} \left[\frac{|R(z)-R(z_0)|}{|z-z_0|} + \frac{|\hat{R}(z)-\hat{R}(z_0)|}{|z-z_0|} \right] = 0. \tag{9}$$

We use these expansions on the (small) rectangle R_n where n is large, and study the 'difference' Δ_n, using the notation from (5). At the very beginning of this chapter, we observed that (1) is true when P and Q are linear, and so on consulting (8) see that we need only consider the case that $P(z) = R(z), Q(z) = \hat{R}(z)$. We look at the line and double integrals which appear in (5) separately.

First, the length of γ_n is $d_n := c_0 2^{-n}$, where c_0 is the length of our original γ, and (9) gives that

$$|\int_{\gamma_n} R\,dx + \hat{R}\,dy| \leq (\varepsilon d_n)\cdot d_n = \varepsilon c_0 4^{-n}, \tag{10}$$

using the upper bound $\max_{\gamma_n}(|R(z)| + |\hat{R}(z)|)\cdot|\gamma_n|$, where ε may be taken as small as we wish provided n is large enough.

Now let's look at the double integral

$$\iint_{R_n} [Q_x - P_y]\,dxdy := \int_{R_n} (\hat{R})_y - (R)_x\,dxdy,$$

and observe that we have made no assumptions about either of the two terms inside the last integral *except the basic assumption that the expression in the integrand is continuous!*. And it's clever how this is used. We know that $(Q_x - P_y)(z_0) = 0$ since when we take $z = z_0$ in (9), we have that $R(z_0) = \hat{R}(z_0) = 0$. That means that if n is large, the integrand $(\hat{R})_x - R_y$ can also be made as small as we wish on all of R_n, since $(\hat{R}_x - R_y)(z_0) = 0$. In short, given $\varepsilon > 0$ we may choose n so large that $|(\hat{R})_y - R_x| < \varepsilon$ at each point of R_n; then

$$|\iint_{R_n} (\hat{R}_x - R_y)\,dxdy| \leq \varepsilon C_0 4^{-n},$$

with C_0 the area of R_0. Thus if n is so large that both this and (10) are valid, we find for some constant C^*

$$\Delta_n = \left| \int_{\gamma_n} P\,dx + Q\,dy - \iint_{R_n} (Q_x - P_y)\,dxdy \right| \leq C^* \varepsilon 4^{-n}, \tag{11}$$

which contradicts our assumption in (6) if ε is small compared to h.

This contradiction proves that our version of Green's theorem holds precisely under the hypotheses that are guaranteed by $f(z)$ having a derivative at each point of our rectangle R and its boundary.

(I hope you like the argument, I certainly do!)