(1) Linear equations / Gaussian elimination.

m, n integers. Solve as best possible the (linear) system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots = b_m$$

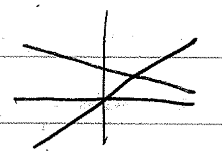
The $\{a_{ij}\}$ and $\{b_i\}$ are fixed. We call the system <u>homogeneous</u> if each $b_i = 0$ (there is then the <u>trivial</u> solution). When $n = 2, 3$ and maybe 4 we set $x_1 = x, x_2 = y, x_3 = z$ ...

Pictures helpful when $n = 2$ (and, less so, $n = 3$)

$n = 2$: $ax + a'y = b$ is a line (thru $\underline{O}$ if $b = 0$)

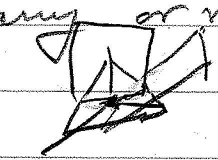
$n = 3$ a plane in 3 dimensions.

<u>Principle</u>; $n = 2$



"two lines <u>usually</u> meet in one point". But there can be infinitely many or none.

$n = 3$



3 planes "usually" meet at one point.

<u>Problems</u> Maybe there are really only two planes or one plane (so the solution might not exist, or could be a plane or line)

Ex
$$3x + 2y + 4z = 5$$
$$6x + 4y + 8z = b$$
$$9x + 6y + 12z = b'$$

* But if we change the coefficients 6, 4, 8 and 9, 6, 12 just a little, there will usually be a solution

<u>Def</u> Singular case: $m = n$ and 0 or ∞ly many solutions.

We can write our system as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

a single linear equation.

Gaussian elimination tries to get an <u>equivalent</u> system (same solutions) which is easier to solve. We may

1. interchange rows
2. multiply a row by a non zero constant
3. add $k\cdot$(row s) to row t.

All these steps are reversable.

Goal: to have ~~sys~~tem where leading element of each row which is non zero (on the left side) is 1, with the leading elements moving to the right as we go down rows. So ($n=4$) we have, say,

$$x_1 + 5x_2 + 7x_3 + 2x_4 = b_1$$
$$x_2 - kx_3 + \ell x_4 = b_2$$
$$x_4 = b_3$$

pivot elements

We can see it has a solution by <u>working up</u> (back substitution).

Algorithm. By 1, get $a_{11} \neq 0$, using 2 get $a_{11} = 1$, then use 3 to eliminate $a_{21}, a_{31}, \ldots, a_{m1}$ (changing $b_2, \ldots, b_m$). Then forget first equation and continue.

P 10 #'s 8, 9,

Principle: If x_1, x_2, x_3 are 3 vectors in 3-dimensional space, then the collection

$$u x_1 + v x_2 + w x_3$$

spans 3-dimensions ($x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$ is what you'd expect). But here

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

doesn't work. If we take $u = 2$, $v = -1$, the first two vectors already give $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. So we get nothing extra having $w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. What we have is instead the collection of "linear combinations"

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

So if we ask when is the system

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

solvable, we write

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \left(2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or

$$(u + 2w) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (v - w) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}:$$

$$\begin{bmatrix} u + v + w \\ u + 2v \\ v - w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}; \text{ we need } \boxed{b_1 = b_2 - b_3}$$

1-3 ~~4~~

p 15 # 10

$$2x + 3y + z = -8$$
$$4x + 7y + 5z = -20$$
$$-2y - 2z = 0$$

$$x + \tfrac{3}{2}y + \tfrac{1}{2}z = -4$$
$$(-4 \times \text{row}1) \quad 0 + y + 3z = 36$$
$$(+2 \times \text{row}2) \quad 0 + 4z = 72$$

$$x + \tfrac{3}{2}y + \tfrac{1}{2}z = -4$$
$$y + 3z = 36$$
$$z = 18$$

How could we "fudge" this to fail ?? When there is a row without a pivot.

Then there is not solution or infinitely many (identity)

Matrices, algebra.

Write original system as single matrix equation

$Ax = b$. $n \times 1$ matrix — column vector

coefficient matrix.

Basic operation: row times column vector [need same # of entries] — this is the inner product (see later)

Ax by rows (simpler) or columns (pp 20–21)

Ax is a sum of columns of A weighted by elements of x

Algebra: $+$ $\times$ (not commutative). But associative; 0 matrix, I: identity matrix.

Distributive (check $(EA)x = E(Ax)$).

E_{ij}: interchange

$E_i(c), c \neq 0$

$E_{ij}(c)$: ith row $\to$ i + j (row)

Elementary matrices (square, depend on # of rows)

Do the operation on E, mult on left by

For step 3 $-l$ (row j) from row i

$$E_{31}(-l) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$$

i j col j

We thus have the equation

$$E_k \cdots E_2 E_1 A = U \quad (\text{upper } \triangle\text{ular})$$

Ex p 26 # 5 $Ax = 0$ clearly. Let's find other solutions

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 3 & -6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & -3 & 3 \end{bmatrix}$$

(cont'd)

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $Ax = 0$ becomes

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_3 is a free variable. Then $x_2 = x_3$ and $x_1 = 2x_2$

p27 #12. Lower Δular means

$a_{ij} = 0$ if $j > i$ (check).

Let A, B be lower Δular. Show $C = AB$ is lower triangular. So let $j > i$, what is c_{ij}?

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \cdots + a_{im} b_{mj}$$

when $j > k$ $b_{kj} = 0$ and when $k > i$, $a_{ik} = 0$

Picture

$a_{ik} = 0$

i $\quad$ j $\quad$ $(j > i)$

$b_{kj} = 0$

p27 #16 $\underbrace{AB \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{1st col of } AB} = A \left(B \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) = A \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ \vdots \\ b_{m1} \end{pmatrix}$

$A \times$ 1st col of B.

<u>Note</u> $(AB)C = A(BC)$ but rarely is $AB = BA$
(sometimes only one will make sense!)