RELATION BETWEEN PRINCIPAL VALUES AND CERTAIN IMPROPER INTEGRALS

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§1. Introduction. This note is motivated by my observing confusion among several MA 525 students about principle value integrals as opposed to ordinary integrals. Much of this, in principle (!) depends on notions that are usually taught in an elementary calculus class, but they are often presented without reference to context, and so are often forgotten years later. Hence these notes, which are keyed to the third edition of Saff-Snider "FUNDAMENTALS OF COMPLEX ANALYSIS" (herein referred to as [S]) and it is suggested that that source be consulted while reading these comments.

You might check your elementary calculus book about this. The basic integral that we study is $\int_a^b f(x)\,dx$, where f is continuous [or piece-wise continuous]. In this, a and b are both finite, with $a < b$. That is the classical Riemann integral, and as more complicated situations are encountered, this usual definition [we use Riemann sums when doing this at Purdue] is modified as needed.

In MA 525, we consider functions f which are continuous on the real axis (sometimes they are also allowed to have simple poles there) which are the restriction to the real axis of a function which is entire or meromorphic in the plane (meromorphic means that it is entire, except that it is allowed to also have poles). In Chapter 6 of [S], especially §§6.4, 6.5, the notion principal value (PV or pv) appears in several situations. But I think the background there often is not clear; I hope this is better.

We often have to compute things such as

$$\int_{-\infty}^{\infty} f(x)\,dx, \tag{1}$$

and the residue calculus gives one useful way to do this. The expression in (1) is defined as

$$\lim_{R\to\infty, S\to\infty} \int_{-R}^{S} f(x)\,dx, \tag{2}$$

while the approach to residues in most of [S] is to use instead the simpler

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{3}$$

By now the reason to like (3) should be clear: the real interval $[-R, R]$ can be naturally viewed as the base of a semicircle of radius R centered at 0 based on the real axis, and penetrating into one of the upper or lower halfplanes. And when dealing with integrals of rational functions (theme of §6.2) $P(x)/Q(x)$ where deg $Q \geq$ deg $P + 2$) the semi-circular contour works perfectly well.

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Here is a **Remark** that you should be sure you understand: If the limit in (2) exists, then we can find it using the simpler limit in (3). *But it important to understand that the examples are legion in which the limit (3) exists while (2) does not;* the simples examply is $f(x) = x$ (please see this for yourself or check with a colleague or me), and we can make many others, using ideas developed in these notes, for example.

What is the difference between (2) and (3)? For the first limit to exist, we need that given $\eta > 0$ (no matter how small) we may arrange that no matter how we choose $R' > R$ and $S' > S$, we will have

$$|\int_{-R'}^{-R} f(x)\,dx| + |\int_{S}^{S'} f(x)\,dx| < \eta, \tag{4}$$

provided R and S are sufficiently large. On the other hand, in (3) we need only that (with $R' > R$ chosen as we please and R sufficiently large) we have

$$\left|\left(\int_{-R'}^{-R} + \int_{R}^{R'}\right) f(x)\,dx\right| < \eta$$

(a sketch can be useful to create function for which this last property holds while (4) fails).

§2. A bit more theory. Here is the elementary calculus therem that I am assuming you have seen before : *if f is continuous on the real axis and for some $h > 0$ we have*

$$|f(x) < \frac{C}{|x|^{1+h}}$$

when x is sufficently large, then the improper integral (1) exists (and so the simpler process (3) may be used).

The book also considers principal values (or improper integrals) in the situation that f is allowed to have a simple pole on the real axis. If this occurs at $x = a$, instead of integrating through a we take

$$\lim_{\varepsilon \to 0} \int^{a-\varepsilon} + \int_{a+\varepsilon} f(x)\,dx, \tag{5}$$

where I don't indicate the lower or upper limits; if they are finite, they should be put there, but if they are, for example $\pm\infty$, the left limit in the first integral will be $-R$ and the upper limit in the second integral should be S, and we let $R, S \to \infty$ as in (2); this means that in general we would treat separately each finite simple pole, and in addition the improper behavior at $\pm\infty$; an example is illustrated on p. 343 of [S].

Much as we saw in §1, in order that

$$\int_{a-h}^{a+h} f(x)\,dx$$

exist as an improper integral [as opposed to existing merely as a principal value], where h is some fixed positive number, we must show that

$$\lim_{\varepsilon, \varepsilon' \to 0} \int_{a-h}^{a-\varepsilon'} + \int_{a+\varepsilon}^{a+h} f(x)\,dx \tag{6}$$

exist, here $\varepsilon, \varepsilon'$ are independent; the text does a good job explaining this on p. 337. And in your calculus it should have been shown that if, when $x - a$ is sufficiently small, we have

$$|f(x)| < \frac{C}{|x-a|^{1-k}}$$

for some constants C and $k > 0$, then the limit in (6) exists. Please reacquaint yourself with these facts.

§3. Examples. Let's consider some of the examples in the book, starting with the exercises on p. 336. I hope that studying these will make my points clearer; you can always check with me if not sure you understand.

• *No. 1.* Since $|\cos x| \le 1$ when x is real (but certainly not (!) when we consider $\cos z$ off the real axis) and the integrand is continuous, our test at the beginning of §2 shows that this integral exists—it is NOT just a principal value. But since it exists, one could find it by taking (as we observed above with (3) $\int_{-R}^{R} \cos x/(1+x^2)$, and then use a semi-circular contour, as the book does except that [since $|\cos z|$ is large as $z = x + iy$ with $|y| \to \infty$] we replace $\cos z$ with e^{iz} if our contour goes into the upper half plane (this is because $|e^{-z}| = e^{-y}$ and y is large on most of the semi-circle considered in [S]).

I have shown that for these problems, I prefer to work directly, by integrating

$$\frac{e^{iz}}{1+z^2}$$

about a rectangle with vertices $-R, S, S+iY, -R+iY$, where R and S are large but there is no relation between them, as opposed to what we confront in (3) or what is suggested in [S], but then once R and S are given, we allow Y to be very large so that the contribution to this integral around this rectangle is negligible except for that from $[-R, S]$ on the real axis.

• *No. 2.* Since we all know is that $|\sin x| \le 1$, our integrand in absolute value is comparable to $1/x$ when x is large, and that in **NOT** enough to allow us to use the test at the beginning of §2. That is why in [S] you are only asked to compute a principal value, but the method I use (and which I sketched in the paragraph right above this one) will show that this integral converges in the sense of (2). So here to see the real story you need to follow my path.

Observe! In Problem 2, if we ignore the factor $\sin x$ in the integrand (whose oscillatory behavior is responsible for the integral converging), we are in a situation unlike that of problem one: the rational function $z/(z^2 - 2z + 10) := P(z)/Q(z)$ only had deg $Q =$ deg $P + 1$, so we can't apply the test at the beginning of §2. However, the text at this stage only likes semi-circular contours, and so the authors are forced to only take a principal value (integrating on $[-R, R]$), and even there to get it to work and have the contribution from the semicircular arc tend to zero as $R \to \infty$ we need Jordan's lemma, on pp. 332-3.

• All these integrals on page 336 converge in the sense (2). In No. 8, when we look at this integrand on $(-\infty, \infty)$, it is an even function, and so we can get $\int_0^\infty$ by taking the usual semi-circular contour, but I don't think the book gives enough information for you to see that this integral converges as written. Studying Examples 2 and 3 on pp 334-5 can be insightful; these integrals converge not only as principal values, but one would need the rectangular method I have already mentioned.

§3. Finite points. In section 6.4 we take principal values about points on the real axis at which the function has a simple pole. In this case one always needs the symmetric limit in (5). The book's examples in this section are quite good, except that again many of these integrals converge as integrals, not just principal values. Let me comment on example 1, p 341. If we take the integral

$$\frac{e^{ix}}{x} = \frac{\cos x}{x} + i\frac{\sin x}{x},$$

we notice that these functions at $\pm\infty$ (in absolute value) are $1/|x|$, so the book is forced to its semicircle, whereas I use rectangles with base on the real axis on $[-R, S]$; both the text and I make a modification caused by the simple pole at the origin, going from $-\varepsilon$ to $+\varepsilon$ on the real axis by using a semicircle of radius ε which penetrates into the upper half plane.

Thus at 0, because there is a simple pole, we need the semicircle, but let's observe what happens as $\varepsilon \to 0$. Of course e^{iz}/z has a simple pole at the origin, and for that reason, $\int_{-\infty}^{\infty} x^{-1} \cos x \, dx$ *will be* a principla value because of the behavior at 0. But on the other hand, since $\sin x/x = 1$ when $x = 0$ (removable singularity), we know that $\int_{-\varepsilon}^{\varepsilon'} |x^{-1} \sin x| \, dx$ tends to 0 as $\varepsilon, \varepsilon' \to 0$: this means that $\int_0^{\infty} x^{-1} \sin x \, dx$ converges as $\lim_{R\to\infty} \int_0^R x^{-1} \sin x \, dx$.