**MA 303 - Grigore Raul Tataru**

# Exam 2 - Practice Problems

**Problem 1.** The general solution of

$$\vec{x}' = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \vec{x} \qquad \text{is} \quad \vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}.$$

(a) Find the solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) Sketch the phase portrait and indicate the type and stability of $\vec{x} = 0$. Include in the phase portrait the graph of the solution in part (a) and of solutions passing through the points $(0, 1)$ and $(3, 2)$. Also indicate for each solution the direction of increasing $t$.

(c) Find the general solution of

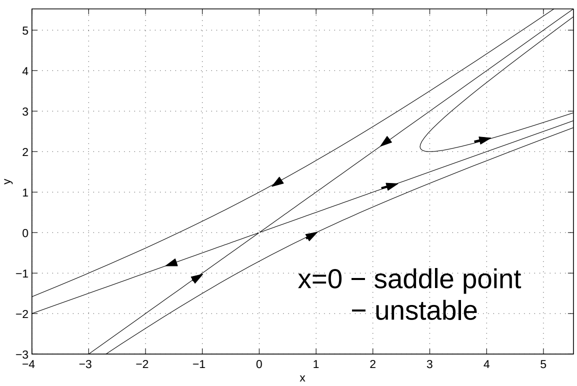
$$\vec{x}' = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{t} \\ e^{t} \end{pmatrix}. \qquad (*)$$

*Solution:* (a) Need to find $c_1$ and $c_2$ such that:

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1 + 2c_2 = 1,\ c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -1,\ c_2 = 1 \quad \Rightarrow$$

$$\vec{x} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

(b)




(c) A fundamental matrix of the homogeneous equation is

$$\Psi(t) = \begin{pmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t} \end{pmatrix}.$$

General solution of (*) is $\vec{x} = \Psi(t)\vec{u}$, where $\vec{u}$ is obtained by solving $\Psi(t)\vec{u} = \vec{b}$:

$$\begin{pmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{t} \\ e^{t} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} e^{-t} & 2e^{t} \\ 0 & -e^{t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{t} \\ -e^{t} \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} u_2' = 1 \\ u_1' = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} u_2 = t + c_2 \\ u_1 = c_1 \end{matrix}$$

Hence the general solution of (*) is:

$$\vec{x} = \Psi(t)\vec{u} = u_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + u_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{t}.$$

**Problem 2.** Find the general real solution of

$$\vec{x}' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \vec{x}.$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = -1$.

*Solution:* The eigenvector for $\lambda_1 = 0$ is:

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} -\xi_2 + \xi_3 = 0 \\ \xi_1 - \xi_2 = 0 \end{array} \quad \Rightarrow \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the repeated eigenvalue $\lambda_2 = \lambda_3 = -1$ we will be able to find two linear independent eigenvectors:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \xi_1 - \xi_2 + \xi_3 = 0 \quad \Rightarrow \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The general solution is $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}.$

**Problem 3.** Find the general real solution of

$$\vec{x}' = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} \vec{x}.$$

*Solution:* The eigenvalues are:

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 3-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 2,\ \lambda_2 = \lambda_3 = 3.$$

The eigenvector associated to the first eigenvalue $\lambda_1 = 2$ is

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} \xi_3 = 0 \\ \xi_2 = 0 \\ \xi_1\text{-arbitrary} \end{array} \quad \Rightarrow \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}.$$

We expect to find one or two eigenvectors associated to $\lambda_2 = \lambda_3 = 3$:

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} -\xi_1 + 2\xi_2 + \xi_3 = 0 \\ \xi_3 = 0 \end{array} \quad \Rightarrow \quad \xi^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{3t}.$$

Note that in this case there do not exist 2 linear independent eigenvectors. A third solution will be of the form $\vec{x}^{(3)} = \xi^{(2)} t e^{3t} + \vec{\eta} e^{3t}$, where $\vec{\eta}$ is a generalized eigenvector:

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} -\eta_3 = 1 \\ -\eta_1 + 2\eta_2 + \eta_3 = 2 \end{array} \quad \Rightarrow \quad \text{choose } \vec{\eta} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}.$$

The general solution is $\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_3 \left[ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} t e^{3t} + \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} e^{3t} \right].$

**Problem 4.** Find the general real solution for

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \vec{x}$$

*Solution:* $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$ has complex conjugate solutions $2 \pm i$. To find the eigenvector associated to $2 + i$ solve:

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad -i\xi_1 - \xi_2 = 0. \text{ Take } \vec{\xi} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

This gives one solution $\vec{x}^{(1)}$; another solution would be the conjugate of this one. However, we want two real linear independent solutions; these can be obtained by collecting the real and imaginary part of $\vec{x}^{(1)}$:

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(2+i)t} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{2t} + i \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} e^{2t}.$$

The general solution is then

$$\vec{x} = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} e^{2t}.$$

**Problem 5.** Find the eigenvalues and eigenfunctions of the following two-point boundary value problem (look only for positive eigenvalues):

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y(3) = 0.$$

*Solution:* We only consider the case $\lambda > 0$. The roots of the characteristic equation $r^2 + \lambda = 0$ are $r_{1,2} = \pm i\sqrt{\lambda} = \pm i\mu$. The general solution and its derivative are given by:

$$y = c_1 \cos \mu x + c_2 \sin \mu x, \qquad y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x.$$

The boundary condition $y'(0) = 0$ writes as $c_2\mu = 0 \Rightarrow c_2 = 0$. The other boundary condition implies that $c_1 \mu \cos 3\mu = 0$. One gets non-zero solutions, i.e. $y = c_1 \cos \mu x$ with arbitrary $c_1$, when

$$\cos 3\mu = 0 \quad \Leftrightarrow \quad 3\mu = \frac{(2n+1)\pi}{2},\ n = 0, 1, 2, \ldots \quad \Leftrightarrow \quad \mu = \frac{(2n+1)\pi}{6}$$

In conclusion the eigenvalues and the associated eigenfunctions are:

$$\lambda_n = \Big(\frac{(2n+1)\pi}{6}\Big)^2 \qquad y_n(x) = \cos \frac{(2n+1)\pi x}{6} \qquad n = 0, 1, 2, 3, \ldots$$

**Problem 6.** Find the Fourier series of the following periodic function:

$$f(x) = \begin{cases} 2 & -2 < x \le 0 \\ -2 & 0 < x \le 2 \end{cases} \qquad f(x+4) = f(x)$$

Sketch the graph of the function to which the series converges for $-6 < x < 6$.(mark clearly the value at points of discontinuity)
*Solution:* The Fourier series is

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \big(a_m \cos \frac{m\pi x}{2} + b_m \sin \frac{m\pi x}{2}\big),$$

where the coefficients are

$$a_m = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{m\pi x}{2}\,dx \;=\; 0 \quad \text{(by direct computation or noticing that the integrand is an odd function )}$$

$$\begin{aligned} b_m &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{m\pi x}{2}\,dx = 2\frac{1}{2}\int_{0}^{2} f(x)\sin\frac{m\pi x}{2}\,dx = \quad \text{(even function)} \\ &= \int_0^2 -2\sin\frac{m\pi x}{2}\,dx = \frac{4}{m\pi}(\cos m\pi - 1) \end{aligned}$$

Note that $b_{2n} = 0$ and $b_{2n+1} = -\frac{8}{(2n+1)\pi}$. The Fourier series is:

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{2} = -\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2}.$$

The graph required is as for $f(x)$ except at the points of discontinuity $0, \pm 2, \pm 4$ where the value is 0, the average of the left and right limits at those points.

**Problem 7.** Find the sine series of period $4\pi$ for the following function and sketch the graph of the function to which the series converges on the interval $(-6\pi, 6\pi)$:

$$f(x) = \begin{cases} 2 & 0 < x \le \pi \\ -1 & \pi < x \le 2\pi \end{cases}.$$

*Solution:* The sine series of period $4\pi$ is

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\frac{mx}{2},$$

where

$$\begin{aligned} b_m &= \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin\frac{mx}{2}\,dx = \frac{1}{\pi}\Big(\int_0^{\pi} 2\sin\frac{mx}{2}\,dx + \int_{\pi}^{2\pi}(-1)\sin\frac{mx}{2}\,dx\Big) = \\ &= \frac{1}{\pi}\Big(-2\frac{2}{m}\cos\frac{mx}{2}\Big|_0^{\pi} + \frac{2}{m}\cos\frac{mx}{2}\Big|_{\pi}^{2\pi}\Big) = \frac{2}{m\pi}\Big(-3\cos\frac{m\pi}{2} + 2 + \cos m\pi\Big) \end{aligned}$$

The sine series is

$$f(x) = \sum_{m=1}^{\infty}\frac{2}{m\pi}\Big(-3\cos\frac{m\pi}{2} + 2 + (-1)^m\Big)\sin\frac{mx}{2}.$$

It converges to the odd extension of $f(x)$ to $(-2\pi, 2\pi)$ extended afterward periodically (of period $4\pi$) everywhere:

$$\begin{cases} 1 & \text{on} \quad (-6\pi,-5\pi), \quad (-2\pi,-\pi), \quad (2\pi,3\pi) \\ -\frac{1}{2} & \text{at} \quad -5\pi, -\pi, 3\pi \\ -2 & \text{on} \quad (-5\pi,-4\pi), \quad (-\pi,0), \quad (3\pi,4\pi) \\ 0 & \text{at} \quad -4\pi,-2\pi,0,2\pi,4\pi \\ 2 & \text{on} \quad (-4\pi,-3\pi), \quad (0,\pi), \quad (4\pi,5\pi) \\ \frac{1}{2} & \text{at} \quad -3\pi,\pi,5\pi \\ -1 & \text{on} \quad (-3\pi,-2\pi), \quad (\pi,2\pi), \quad (5\pi,6\pi) \end{cases}.$$