MA 303 - Fall 2004 - Grigore Raul Tataru

Exam 2 - Solutions

Problem 1. The general solution of

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x} \qquad \text{is} \quad \vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

(a) Find the solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.
(b) Sketch the phase portrait and indicate the type and stability of $\vec{x} = 0$. Include in the phase portrait the solution in part (a) and solutions passing through $(0, 1)$, $(-1, -1)$ and $(1, 0)$.
(c) Find one particular solution and the general solution of

$$\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 3e^t \end{pmatrix}. \qquad (*)$$

Solution: (a) Need to find c_1 and c_2 such that:

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \Rightarrow \quad c_1 = c_2 = 1 \quad \Rightarrow \quad \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

(b)$\vec{x} = 0$ is a saddle point, unstable. (c) A fundamental matrix of the homogeneous equation is

$$\Psi(t) = \begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix}.$$

General solution of (*) is $\vec{x} = \Psi(t)\vec{u}$, where $\vec{u}$ is obtained by solving $\Psi(t)\vec{u}' = \vec{b}$:

$$\begin{pmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3e^t \end{pmatrix} \Rightarrow \begin{pmatrix} e^{-t} & 2e^{2t} \\ 0 & -3e^{2t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3e^t \end{pmatrix} \Rightarrow \begin{matrix} u_2' = -e^{-t} \\ u_1' = 2e^{2t} \end{matrix} \Rightarrow \begin{matrix} u_2 = e^{-t} + c_2 \\ u_1 = e^{2t} + c_1 \end{matrix}$$

Hence the general solution of (*) is:

$$\vec{x} = \Psi(t)\vec{u} = \vec{x} = u_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + u_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} = \vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t,$$

a particular solution being given by the last term above $\vec{x}_p = \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$.

Problem 2. Find the general real solution of

$$\vec{x}' = \begin{pmatrix} 0 & 0 & 0 \\ 15 & 11 & 2004 \\ 0 & 0 & 0 \end{pmatrix} \vec{x}.$$

Solution: Since $\det(A - \lambda I) = \lambda^2(11 - \lambda)$ the eigenvalues are $\lambda_1 = 11$, $\lambda_2 = \lambda_3 = 0$.
The eigenvector for $\lambda_1 = 11$ is:

$$\begin{pmatrix} -11 & 0 & 0 \\ 15 & 0 & 2004 \\ 0 & 0 & -11 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -11\xi_2 = 0 \\ -11\xi_3 = 0 \end{matrix} \quad \Rightarrow \quad \xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For the repeated eigenvalue $\lambda_2 = \lambda_3 = 0$ we will be able to find two linear independent eigenvectors:

$$\begin{pmatrix} 0 & 0 & 0 \\ 15 & 11 & 2004 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 15\xi_1 + 11\xi_2 + 2004\xi_3 = 0 \quad \Rightarrow \quad \xi^{(2)} = \begin{pmatrix} 11 \\ -15 \\ 0 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 2004 \\ 0 \\ -15 \end{pmatrix}.$$

The general solution is $\quad \vec{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{11t} + c_2 \begin{pmatrix} 11 \\ -15 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2004 \\ 0 \\ -15 \end{pmatrix}.$

Problem 3. Find the general real solution of

$$\vec{x}' = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 2 & -4 & 1 \end{pmatrix} \vec{x}.$$

Solution: The eigenvalues are:

$$\begin{vmatrix} -1-\lambda & 0 & 0 \\ -1 & -1-\lambda & 0 \\ 2 & -4 & 1-\lambda \end{vmatrix} = (-1-\lambda)^2(1-\lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 1,\ \lambda_2 = \lambda_3 = -1.$$

The eigenvector associated to the first eigenvalue $\lambda_1 = 1$ is

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -2 & 0 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} \xi_1 = 0 \\ \xi_2 = 0 \end{matrix} \quad \Rightarrow \quad \xi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \vec{x}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t.$$

We expect to find one or two eigenvectors associated to $\lambda_2 = \lambda_3 = -1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -\xi_1 = 0 \\ 2\xi_1 - 4\xi_2 + 2\xi_3 = 0 \end{matrix} \quad \Rightarrow \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-t}.$$

Note that in this case there do not exist 2 linear independent eigenvectors. A third solution will be of the form $\vec{x}^{(3)} = \xi^{(2)} te^{-t} + \vec{\eta} e^{-t}$, where $\vec{\eta}$ is a generalized eigenvector:

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & -4 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} -\eta_1 = 1 \\ 2\eta_1 - 4\eta_2 + 2\eta_3 = 2 \end{matrix} \quad \Rightarrow \quad \text{choose } \vec{\eta} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

The general solution is $\quad \vec{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} te^{-t} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} e^{-t} \right].$

Problem 4. Find the general real solution for

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

Solution: $\det(A - \lambda I) = \lambda^2 - 2\lambda + 5$ has complex conjugate solutions $1 \pm 2i$. To find the eigenvector associated to $1 + 2i$ solve:

$$\begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad -2i\xi_1 - \xi_2 = 0. \text{ Take } \vec{\xi} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

This gives one complex solution $\vec{x}^{(1)}$. Two real linear independent solutions can be obtained by collecting the real and imaginary part of $\vec{x}^{(1)}$:

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{t}(\cos 2t + i \sin 2t) = \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{t} + i \begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix} e^{t}.$$

The general solution is: $\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \sin 2t \\ -2\cos 2t \end{pmatrix} e^{t}$.

Problem 5. Find the eigenvalues and eigenfunctions of the following two-point boundary value problem (look only for nonnegative eigenvalues $\lambda \geq 0$):

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \quad y'(\frac{\pi}{3}) = 0.$$

Solution: For $\lambda = 0$ the roots of the characteristic equation $r^2 = 0$ are $r_1 = r_2 = 0$; the general solution is then given by $y = c_1 + c_2 x$ and the boundary conditions are satisfied if $c_2 = 0$, so the solutions are $y = c_1$ with c_1 arbitrary. Hence $\lambda_0 = 0$ is an eigenvalue and an eigenfunction for it is $y_0 = 1$.

For $\lambda > 0$ the roots of the characteristic equation $r^2 + \lambda = 0$ are $r_{1,2} = \pm i\sqrt{\lambda} = \pm i\mu$. The general solution and its derivative are given by:

$$y = c_1 \cos \mu x + c_2 \sin \mu x, \qquad y' = -c_1\mu \sin \mu x + c_2 \mu \cos \mu x.$$

The boundary condition $y'(0) = 0$ implies that $c_2 = 0$. The other boundary condition implies that $c_1 \mu \sin \frac{\mu\pi}{3} = 0$. One gets non-zero solutions $y = c_1 \cos \mu x$ with arbitrary c_1, when

$$\sin \frac{\mu\pi}{3} = 0 \quad \Leftrightarrow \quad \frac{\mu\pi}{3} = n\pi,\ n = 1, 2, 3 \ldots \quad \Leftrightarrow \quad \mu = 3n.$$

In conclusion the positive eigenvalues and the associated eigenfunctions are:

$$\lambda_n = (3n)^2 \qquad y_n(x) = \cos 3nx \qquad n = 1, 2, 3, \ldots$$

Problem 6. (a) Find the sine series of period 4 for

$$f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & 1 < x < 2 \end{cases}$$

(b) Sketch the graph of the function to which the series converges for $-2 < x < 6$.

Solution: The sine series of period 4 is

$$f(x) = \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{2},$$

where

$$b_m = \frac{2}{2} \int_0^2 f(x) \sin \frac{m\pi x}{2}\, dx = \int_0^1 \sin \frac{m\pi x}{2}\, dx = -\frac{2}{m\pi} \cos \frac{m\pi x}{2} \Big|_0^{\pi} = -\frac{2}{m\pi}\Big(\cos \frac{m\pi}{2} - 1\Big)$$

The sine series is

$$f(x) = \sum_{m=1}^{\infty} \frac{2}{m\pi}\Big(1 - \cos \frac{m\pi}{2}\Big) \sin \frac{m\pi x}{2}.$$

It converges to the odd extension of $f(x)$ to $(-2,2)$ extended afterward periodically (of period 4) everywhere:

$$\begin{cases} 0 & \text{on} \quad (-2,-1), \quad (1,3), \quad (5,6) \\ -\frac{1}{2} & \text{at} \quad -1,3 \\ -1 & \text{on} \quad (-1,0), \quad (3,4) \\ 0 & \text{at} \quad 0,4 \\ 1 & \text{on} \quad (0,1), \quad (4,5) \\ \frac{1}{2} & \text{at} \quad 1,5 \end{cases}.$$

Problem 7. Find the Fourier series for the following periodic function:

$$f(x) = \begin{cases} 1 & -\pi < x \leq 0 \\ 2 & 0 < x \leq \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

Write your answer as a series containing only nonzero terms.

Solution: The Fourier series is

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \big(a_m \cos\frac{m\pi x}{\pi} + b_m \sin\frac{m\pi x}{\pi}\big) \;=\; \frac{a_0}{2} + \sum_{m=1}^{\infty} \big(a_m \cos mx + b_m \sin mx\big),$$

where the coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \;=\; 3 \\ a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \;=\; \frac{1}{\pi}\int_{-\pi}^{0} \cos mx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\cos mx\,dx \;= \\ &= \frac{1}{m\pi}\sin mx\Big|_{-\pi}^{0} + \frac{2}{m\pi}\sin mx\Big|_{0}^{\pi} \;=\; 0 \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \;=\; \frac{1}{\pi}\int_{-\pi}^{0} \sin mx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2\sin mx\,dx \;= \\ &= -\frac{1}{m\pi}\cos mx\Big|_{-\pi}^{0} - \frac{1}{m\pi}2\cos mx\Big|_{0}^{\pi} \;=\; \frac{1}{m\pi}(1-\cos m\pi). \end{aligned}$$

Note that

$$\begin{aligned} b_{2n} &= 0 \\ b_{2n+1} &= \frac{2}{(2n+1)\pi}. \end{aligned}$$

The Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} b_{2n+1}\sin(2n+1)x \;=\; \frac{3}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin(2n+1)x.$$