MA303 Differential Equations, Fall 2006
Second Exam
November 13, 2006

NAME: SECTION:

- You have one hour to complete your exam.
- Please show all your work neatly and indicate your final answers clearly.
- The exam is closed book and notes. Calculators are not allowed.

Problem	**Maximum**	**Score**
1	5	
2	5	
3	5	
4	5	
5	20	
6	20	
7	20	
8	20	
Total	100	

Part 1: No partial credit. Among the given choices for each problem, ONLY ONE is the correct answer. Either circle the correct choice or write it in the parenthesis. If you circle one choice but write down another in the parenthesis, the answer in the parenthesis will be considered your final answer. Each question is worth 5 points.

1. (**a**) The equilibrium point $(0,0)^t$ of the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & k \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is a spiral sink (that is, all solutions spiral towards the origin) if:

(a) $k > 1/5$.
(b) $k < 1/5$.
(c) $k > -1/5$.
(d) $k < -1/5$.
(e) $k = 1/5$.

2. (**c**) The solution of the initial value problem

$$X' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} X \qquad X(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

is

(a) $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t}$.

(b) $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-3t}$.

(c) $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$.

(d) $2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$.

(e) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$.

3. (**c**) The solution to the non-homogeneous system

$$X' = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} t \\ e^{-t} \end{bmatrix}$$

is:

(a) $c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$

(b) $c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$

(c) $c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$

(d) $c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$

(e) $c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \frac{1}{3} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$

4. (**See last page**) Consider the function

$$f(x) = \begin{cases} 2 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

The Fourier sine series of f of period 4π is:

(a) $\sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{nx}{2}.$

(b) $\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin \frac{nx}{2}.$

(c) $\sum_{n=1}^{\infty} \frac{8}{(2n-1)\pi} \sin \frac{nx}{2}.$

(d) $\sum_{n=1}^{\infty} \frac{8}{n\pi} \sin \frac{nx}{4}.$

(e) $\sum_{n=1}^{\infty} \frac{8}{(2n-1)\pi} \sin \frac{nx}{4}.$

Part 2. Partial credit section. Show all your work neatly and concisely, and indicate your final answer clearly. If you simply write down the final answer without appropriate intermediate steps, you may not get full credit for that problem.

5. (20 points)

(a) Solving the following system

$$X' = AX \qquad \text{where} \qquad A = \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix}$$

Solution:
The matrix A has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Hence the eigenvalues of A are $\lambda_{\pm} = -1 \pm 5i$. To write the general solution is sufficient to find one complex eigenvector. Consider $\lambda_+ = -1 + 5i$ the associated eigenvector v is found solving the system $(A - \lambda_+ I_2)v = 0$. That is we solve

$$\left[\begin{array}{cc|c} -5i & -5 & 0 \\ 5 & -5i & 0 \end{array}\right]$$

Hence, a basis for the eigenspace is $(1, -i)^t$. The two fundamental (real) solution are given by

$$\begin{aligned} x_1 &= e^{-t}\left(Re(v)cos5t - Imag(v)sin5t\right) \\ x_2 &= e^{-t}\left(Re(v)sin5t + Imag(v)cos5t\right) \end{aligned}$$

Note that $Re(v) = (1, 0)^t$ and $Imag(v) = (0, -1)^t$. Hence the general solution is given by

$$X(t) = c_1 x_1 + c_2 x_2 = e^{-t}\left(c_1 \begin{bmatrix} cos5t \\ sin5t \end{bmatrix} + c_2 \begin{bmatrix} sin5t \\ -cos5t \end{bmatrix}\right) \qquad (1)$$

The fundamental matrix is

$$\Psi(t) = e^{-t} \begin{bmatrix} cos5t & sin5t \\ sin5t & -cos5t \end{bmatrix}$$

(b) Sketch the phase portrait of the system.

(c) Solve the initial value problem $X' = AX$, $X(0) = \begin{bmatrix} -\pi \\ 2\pi \end{bmatrix}$.

solution:
From (1) it follows that

$$X(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\pi \\ 2\pi \end{bmatrix}$$

The solution to this system is $c_1 = -\pi$ and $c_2 = -2\pi$. Hence, the solution to the initial value problem is

$$X(t) = e^{-t} \left(-\pi \begin{bmatrix} cos5t \\ sin5t \end{bmatrix} - 2\pi \begin{bmatrix} sin5t \\ -cos5t \end{bmatrix} \right)$$

6. (20 points) Solve the following system

$$X' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} X = BX$$

Solution:
We first compute the eigenvalues of B and associated eigenvectors. The eigenvalues of B are found solving the equation

$$\det(B - \lambda I_3) = \det \left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} \right) = -\lambda \left((1-\lambda)^2 - 1 \right) = 0 \tag{2}$$

The eigenvalues (roots of (2)) are

$$\lambda_1 = 0 \text{ (double root)} \quad \lambda_2 = 2$$

The eigenvector associated to $\lambda_2 = 2$ is found by solving

$$(B - \lambda_2 I_3)v_2 = 0 \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array} \right]$$

Hence, $v_2 = c_2(1, 1, 1)^t$, where c_2 is a real number. Next we consider λ_1 and we solve

$$(B - \lambda_1 I_3)v_1 = Bv_1 = 0 \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array} \right]$$

Hence, eigenvectors associated to λ_1 have the form $v_1 = c_1(0, 0, 1)^t$, where c_1 is a constant. Since the space spanned by v_1 (eigenspace) is one dimensional we need to find the generalized eigenvector w associated to λ_1. The generalized eigenvector is found solving the system

$$Bw = v_1 \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array} \right]$$

Hence, we can choose $w = (-1, 1, 0)^t$. The general solution is given by

$$X(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

7. (20 points) Find eigenvalues and associated eigenfunctions of

$$X'' + \lambda X = 0; \qquad X(L) = 0 \qquad X'(0) = 0.$$

(You can assume that all the eigenvalues are real.)
Solution:

$\lambda < 0$. We let $\lambda = -\mu^2$. Thus, the characteristic equation is given by

$$r^2 - \mu^2 = 0$$

with the roots being $r = \pm\mu$. Hence, the general solution for the differential equation is

$$X(x) = c_1 e^{-\mu x} + c_2 e^{\mu x}$$

Therefore,

$$X' = -\mu c_1 e^{-\mu x} + \mu c_2 e^{\mu x}$$

Using the given boundary conditions we conclude that $c_1 = c_2 = 0$. Thus, the problem have only trivial solutions. Hence, there are no negative eigenvalues.

$\lambda = 0$. In this case the characteristic equation is $r^2 = 0$ and the roots are $r = 0$ (double). Hence, the general solution is $X(x) = c_1 x + c_2$. Using the boundary conditions $X(L) = 0$ and $X'(0) = 0$ we conclude that $c_1 = c_2 = 0$. Thus, the problem have only trivial solutions. Hence, $\lambda = 0$ is not an eigenvalue.

$\lambda > 0$. We Let $\lambda = \mu^2, \mu > 0$. Thus, the characteristic equation is given by

$$r^2 + \mu^2 = 0$$

with the roots being $r = \pm\mu i$. Hence, the general solution for the differential equation is

$$X(x) = c_1 cos(\mu x) + c_2 sin(\mu x)$$

Therefore,

$$X' = -\mu c_1 sin(\mu x) + \mu c_2 cos(\mu x)$$

Using $X'(0) = 0$ we obtain $c_2\mu = 0$. Since $\mu \neq 0$ it follows that $c_2 = 0$. Substituting $c_2 = 0$ into the expression of $X(x)$ and using $X(L) = 0$ yields

$$c_1 cos(\mu L) = 0 \quad \Leftrightarrow \quad c_1 = 0 \quad \text{or} \quad cos(\mu L) = 0$$

We reject $c_1 = 0$ since this will lead only to trivial solutions. We consider

$$cos(\mu L) = 0 \quad \Leftrightarrow \quad \mu = \frac{2n-1}{2L}, \quad n = 1, 2, \cdots$$

Therefore, the eigenvalues and the associated eigenfunctions are respectively

$$\lambda_n = \mu^2 = \left(\frac{2n-1}{2L}\right)^2 \qquad X_n = c_1 cos\left(\frac{2n-1}{2L}x\right), \quad n = 1, 2, \cdots$$

8. (20 points) Given

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x =, 0, \pi, 2\pi \\ -1 & \pi < x < 2\pi \end{cases} \qquad f(x + 4\pi) = f(x)$$

Find the Fourier series expansion of $f(x)$.

Solution:

To find the Fourier cosine expansion of f we extend f as an even function. Thus, the coefficients $b_n = 0$ and we only need to compute

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_0^L f(x) cos\frac{n\pi x}{L} dx$$

The last equality follows from the fact that $f(x)cos\frac{n\pi x}{L}$ is an even function. Since $f(x + 4\pi) = f(x)$ the period of f is 4π. Hence $L = 2\pi$ and a_n are computed by solving:

$$a_n = \frac{2}{2\pi}\left(\int_0^{\pi} cos\frac{nx}{2}dx + \int_{\pi}^{2\pi}(-1)cos\frac{nx}{2}dx\right) = \frac{4}{n\pi}sin\frac{n\pi}{2}$$

Hence, the Fourier cosine series expansion of f is:

$$S_n(x) = \sum_{n=0}^{\infty} \frac{4}{n\pi} sin\frac{n\pi}{2} cos\frac{nx}{2}$$

Solution to Q4:

To find the Fourier sine expansion of f we extend f as an odd function. Thus, the coefficients $a_n = 0$ and we only need to compute

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)sin\frac{n\pi x}{L}dx$$

The last equality follows from the fact that $f(x)sin\frac{n\pi x}{L}$ is an even function.

Since the period of f is 4π we have $L = 2\pi$ and b_n are computed by solving:

$$b_n = \frac{2}{2\pi}\int_{0}^{2\pi} sin\frac{nx}{2}dx = \frac{2}{2\pi}\int_{0}^{\pi} 2sin\frac{nx}{2}dx = -\frac{4}{n\pi}\left(cos\frac{n\pi}{2} - 1\right)$$

Since the value of $cos\frac{n\pi}{2}$ varies with the value of n, we discuss the following two cases:

1. n is odd. In this case we let $n = 2k - 1$, with $k = 1, 2, \cdots$. Note that when n is odd $cos\frac{n\pi}{2} = 0$. Hence,

$$b_{2k-1} = \frac{4}{(2k-1)\pi}$$

and

$$S_{2k-1}(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} sin\frac{(2k-1)x}{2}$$

2. n is even. We let $n = 2k$, with $k = 1, 2, \cdots$. Thus

$$b_{2k} = -\frac{2}{k\pi}\left(cos(k\pi) - 1\right) = -\frac{2}{k\pi}\left((-1)^k - 1\right)$$

and

$$S_{2k}(x) = \sum_{k=1}^{\infty} -\frac{2}{k\pi}\left((-1)^k - 1\right) sin(2kx)$$