Densities of integers and hypercyclicity

1. Let positive integers $k_1, k_2, \ldots$ be given, and suppose we have sets $E_1, E_2, \ldots \subset \mathbb{C}$ with $\#(E_m) = k_m$. Then there exist $\delta_1, \delta_2, \ldots > 0$ and a map φ,

$$\varphi : \mathbb{Z}^+ \to \mathbb{C}$$

such that for each $a \in E_m$

(1) $\quad \underline{\text{dens}}\{n ; \varphi(n) = a \in E_m\} \geq \delta_m . \qquad (m = 1, 2, \ldots).$

2. Proof. Let $\{p_j\}$ be a sequence of primes with

(2) $$\sum \frac{1}{p_j} < \frac{1}{3}.$$

Label the elements of E_m as $a_1^m, \ldots, a_{k_m}^m$.

To start, we consider the set $\{l p_1, l = 1, 2, \ldots\}$, and correspond

$$l p_1 \xrightarrow{\varphi} a_l^1 ,$$

where on the right side we do indexing in E_1 mod k_1.

Thus

$$\underline{\text{dens}}\{n ; \varphi(n) \in E_1\} = 1/p_1 ,$$

and since there are k_1 elements in E_1, our assignment gives that

$$\underline{\text{dens}}\{n ; \varphi(n) = a, a \in E_1\} \geq 1/k_1 p_1 := \delta_1$$

Next, suppose we have $E_1, \ldots, E_{q-1}$ in the range of φ, and

$$\underline{\text{dens}}\{n ; \varphi(n) \in E_j\} > \delta_j' > 0 \qquad (1 \leq j \leq q-1)$$

and

$$\text{dens}\{n ; \varphi(n) = a, a \in E_j\} > \delta_j > 0 \qquad (1 \leq j \leq q-1),$$

where $\{n ; \varphi(n) \in E_j\} \subseteq \{l p_j , l \geq 1, l \not\equiv 0 (\text{mod}\, p_1, \text{mod}\, p_2, \ldots \text{mod}\, p_{j-1})\}$

We then consider the modified orbit

(3) $$\mathcal{O}'(p_q) = \{l p_q , l \geq 1, l \not\equiv 0 (\text{mod}(p_1 p_2 \cdots p_{q-1})\}.$$

We note that $\mathcal{O}'(p_q) \cap \mathcal{O}'(p_j) = \phi$ for $1 \leq j < q-1$, and

(4) $\quad \underline{\text{dens}}\{n \in l p_q \in \mathcal{O}'(p_q)\} \geq p_q^{-1}\{1 - p_1^{-1} - p_2^{-1} \cdots - p_{q-1}^{-1}\} \geq \frac{2}{3}\frac{1}{p_q} = \delta_q',$

as follows from (2). This gives at once that

(5) $\quad \underline{\text{dens}}\{l ; a \in \mathcal{O}'(p_q, a)\} \geq (2/3 k_q)(1/p_q) = \delta_q$

3. Remark A. Suppose we are given an increasing function $\varphi(n)$, where $\varphi \uparrow \infty$, but in principle arbitrarily slowly, and we impose the additional requirement that our map $n \xrightarrow{\Psi} \mathbb{C}$ satisfy

$$|\Psi(n)| < \varphi(n).$$

Then the previous facts remain valid. The only change is that a given $a \in E_m$ will only be introduced to the range of Ψ for $n > n_0(|a|)$. But that eliminates a bounded number of appearances of ~~each a~~ , and so changes none of the densities.

B. In 2 we only define Ψ for certain n; if n is not sent to $\cup E_m$, we can take $\Psi(n) = 0$.

4. Relation to frequent hyperbolicity. See the note [B, G]. They ask (and doubt) if, given $\varphi(x) \uparrow \infty$, there is a frequently hyperbolic function f with $M(r, f) \le \varphi(r) e^r r^{-1/2}$. As a specific test case they suggest (Problem 15) whether this is possible even for the simplest case that one wants to solve the problem for every $a \in \mathbb{C}$

$$(6) \qquad |f^{(n)}(0) - a| < \varepsilon,$$

where $\varepsilon > 0$ is fixed, and the inequality is to hold on an n-set of positive density. From sections 1–3, we show

5. Proposition Let $\varphi(x) \uparrow \infty$. Then the problem ( ) has a solution f of order 1, mean type, so that

$$(7) \qquad M(r, f) < Ce^r.$$

Remark The authors remark that the problem has no solution if $r^{1/2} M(r, f) < Ce^r$, and doubt (Problem 13) that such f exists as in ( ) to solve the problem. Our solution is modeled on the rate $r^{-1/2} e^r$, and need more investigation.

Proof Choose a network (ε_m, d_m), $\varepsilon_m \downarrow 0$, $d_m \uparrow \infty$, where the pair (ε_m, d_m) corresponds to producing a finite set E_m in $D_m : \{|z| < d_m\}$ so that any point of D_m is within distance ε_m of a point in E_m. If k_m is the cardinality of E_m, we see that k_m may be chosen; $k_m \sim (d_m/\varepsilon_m)^2$.

Let $\Psi : \mathbb{Z}^+ \xrightarrow{\text{onto}} \cup E_m$ be as in 1, where, following 3, we assume that $|\Psi(n)| < \varphi_1(n)$, where φ_1 will be specified in a moment. Then consider the entire function

$$(8) \qquad f(z) = \sum (n^{-1/2}/n!) \Psi(n) z^n \; (:= \sum a_n z^n);$$

with no loss of generality, we suppose $|\Psi(n)| < n^{1/2}$.

6. Estimates We see at once that

$$\log 1/|\alpha_n| = \log \Psi(n) - n + (n+\tfrac{1}{2})\log n,$$

so the classical formula

$$\rho = \limsup \frac{n \log n}{\log(1/|\alpha_n|)}$$

shows that f has order 1. Similarly, the exponential type τ is finite since (via Stirling)

$$e^{-1}\tau = e^{-1}\limsup n|\alpha_n|^{1/n} = \limsup n \cdot n^{-1/2n}(e^n n^{-(n+1/2)})^{1/n}$$
$$= \overline{\lim}\, e\,(1+o(1)),$$

so f has exponential type 1.

We would like more information on $M(r,f)$, see next section

We now claim: given any a, and $\varepsilon > 0$, there is a set $N(a)$ n having density $> \delta(|a|,\varepsilon)$ so that

(9) $$|f^{(n)}(0) - a| < \varepsilon \qquad (n \in N(a)).$$

Proof Choose m so large that $d_m > |a|+1$ and $\varepsilon_m < (1/2)\varepsilon$, then there is $a_m = a_m(a) \in E_m$ with

$$|a_m - a| < \varepsilon/2.$$

But we have taken the coefficients α_n in ( ) so that

(10) $$\text{dens}\,\{n;\ \alpha_n = a_m\} > \delta_m;$$

so the equation $f^n(0) = a_m$ has solutions for a set of density $> \delta_m$. This means that (6) holds

7 Bounds for $M(r,f)$ and sharpness.

1. Since we have agreed that $\Psi(n) < n^{1/2}$ à priori, we that the coefficients $\{\alpha_n\}$ of f satisfy

(11) $$|\alpha_n| \le n^{-1/2}(n!)^{-1}|\Psi(r)| \le 1/n!,$$

and so, since $|\sum \alpha_n z^n| \le \sum |z^n/n!| = e^{|z|}$, we have (7).

It is quite possible one can do better, and I suspect $\Psi(r) r^{-1} e^r$ should be sharp

2. We have shown that f is frequently hypercyclic for the special problem (6). But we should replace (6) by finding a more universal function f so that given a compact set K and $\delta, \varepsilon > 0$ the inequality

$$|f^{(n)}(z) - h(z)| < \varepsilon \qquad (z \in K)$$

holds for a set of n of positive lower density, h is any entire function