SOLUTIONS

Name: ______________________ **Math 511** **July 9, 2008**

(*Work will be graded on the basis of clarity as well as accuracy.*)

(20) 1. True-False. Write T or F and give a reason in each case If the answer is F, it would be convincing to give an example.

(a) If A is a 5×3 matrix, $A^T A$ is singular. No. A^TA is 3×3 and can be invertible when A has rank 3.

(b) The determinant is a matrix A is unchanged if the rows of A are subject to elementary row operations (as in Chapter 1). No – we can multiply a row by a nonzero constant, or switch rows

(c) If x and y are orthogonal (and nonzero) then they are linearly independent.

True

(d)) If A is an $n \times n$ matrix, A always has at least one eigenvector.

True

(20) 2. FInd eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$\det|A - \lambda I| = \det\begin{pmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda = \lambda(\lambda - 3)$

$\lambda = 0$: eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 3$ $\quad A - 3I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$ eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(20) 3. Factor the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$$

into a product QR, recognizing that the first column is a unit vector.

1st column already a unit vector. So we want Q so second column is $\perp$ to first, and length 1.

We have,

$$q_2 = \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - \left[\begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - \sin\theta\cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} - \begin{pmatrix} \sin\theta\cos^2\theta \\ \cos\theta\sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta(1-\cos^2\theta) \\ -\cos\theta\sin^2\theta \end{pmatrix} = \begin{pmatrix} \sin^3\theta \\ \cos\theta\sin^2\theta \end{pmatrix}$$

We want a unit vector in the same direction, so can 'cancel' common factors. So $q_2' \parallel q_2'' = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \equiv q_2$

Going back to our main equation this means:

$$\begin{pmatrix} \sin\theta \\ 0 \end{pmatrix} = (\sin\theta\cos\theta)\, q_1 + q_2 \cdot \sin^2\theta$$

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & \sin\theta\cos\theta \\ 0 & \sin^2\theta \end{bmatrix}}_{R}$$

(15) 4. Find c so that $x = 0$ is a solution to the system

$$2x + 5y = c$$
$$3x + 4y = 2.$$

What is y in that situation?

Cramer; $x = \dfrac{\begin{vmatrix} c & 5 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}} = 0$ so $4c - 10 = 0$

$c = 5/2$

Then $y = \dfrac{\begin{vmatrix} 2 & 5/2 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}} = \dfrac{4 - \frac{15}{2}}{-7}$

(15) 5. Let A be a 3×3 upper triangular matrix with diagonal entries $1, 6, -1$. Explain why it can be diagonalized. How would you find the matrix S so that $AS = S\Lambda$, and what will the diagonal matrix Λ be?

3 distinct eigenvectors, say x_1, x_2, x_3

Then if S is the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

we have $AS = S\Lambda$, where $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{pmatrix}$